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Preface

Dear Conference Participant,

Welcome to the International Hybrid Conference on Mathematical Development and Applications (ICOMAA-23) we organized the sixth. First of all, I would like to start my words by reminding one of G. H. Hardy's words:

"Mathematics, more than any other art or science, is a young man's game."

This phrase he expressed in his book “A Mathematician's Apology” is quite meaningful. Because Newton discovered his biggest ideas, fluxions and the law of gravitation, when he was just 24 years old. He found the 'elliptic orbit' at 37 years old. Also, Galois (at twenty-one), Abel(twenty-seven), Ramanujan (thirty-three), and Riemann(at forty) had passed away in their youth.

That's why we thought we should continue this series of conferences that brings together exciting and productive young mathematicians. So, we aim to bring together scientists and young researchers from all over the world and their work on the fields of mathematics and applications of mathematics, to exchange ideas, to collaborate and to add new ideas to mathematics in a discussion environment. With this interaction, functional analysis, approach theory, differential equations and partial differential equations and the results of applications in the field of Mathematics are discussed with our valuable academics, and in mathematical developments both science and young researchers are opened. We are happy to host many prominent experts from different countries who will present the state-of-the-art in real analysis, complex analysis, harmonic and non-harmonic analysis, operator theory and spectral analysis, applied analysis.

I would like to express my gratitude to those who see and appreciate our efforts and innovative steps that we have made to improve our conference every year, to our dear invited speakers and to all our participants. I owe a debt of gratitude to the Scientific committee, organizing committee, local organizing committee and for their efforts throughout this conference series.

The conference brings together about 203 participants and 9 invited speakers from 22 countries (Azerbaijan, India, Algeria, Bangladesh, Georgia, Greece, India, Iran, Iraq, Italy, Kazakhstan, Kosovo, Malaysia, Mexico, Morocco, Pakistan, Poland, Saudi Arabia, Turkey, United Arab Emirates, Uzbekistan, Yemen).

It is also an aim of the conference to encourage opportunities for collaboration and networking between senior academics and graduate students to advance their new perspective. Additional emphasis on ICOMAA-23 applies to other areas of science, such as natural sciences, economics, computer science, and various engineering sciences, as well as applications in related fields

The conference program represents the efforts of many people. I would like to express my gratitude to all membership the scientific committee, external reviewers, sponsors and, honorary committee for their continued support to the ICOMAA. I also thank the invited speakers for presenting their talks on current researches. Also, the success of ICOMAA depends on the effort and talent of researchers in mathematics and its applications that have writtenand submitted papers on a variety of topics. So, I would like to sincerely thank all participants of ICOMAA-2023for contributing to this great meeting in many different ways. I believe and hope that each of you will get the maximum benefit from the conference.

Prof. Dr. Yusuf ZEREN

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SOME PROPERTIES OF GRILL TOPOLOGICAL SPACES VIA $\mathcal{G}_{\omega}^{\alpha}O(X)$

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Abstract:
The main idea of this work is to introduce and investigate a new class of open sets in grill topological spaces, namely open $\mathcal{G}_{\omega}^{\alpha}-$ sets, which is considered as a strong form of the class of $\mathcal{G}_{\omega}^{\alpha}-$open sets and it is induced topology by the collection of $\mathcal{G}_{\omega}^{\alpha}-$open sets. Next, we study the separation axioms in the collection of $\mathcal{G}_{\omega}^{\alpha}-$open sets.

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1 Introduction

Some classes of weak or strong forms of open sets in topological spaces are structured, investigated, and introduced as important studies in the general topology. In 1963, [19] Levine introduced the class of semi-open sets in topological spaces as a weak form of the class of open sets. In 1965, Njastad [9] introduced the class of $\alpha-$open sets in topological spaces as a class stronger than semi-open sets and weaker than the class of open sets. In 1982, Hdeib [6] introduced the class of $\omega-$open sets as weaker than the class of open sets in topological spaces. In 1982, [8] Mashhour et al., introduced the class of pre-open sets, and the weak form of the class of $\alpha-$open sets in topological spaces. In 1983, [2] Abd El-Monsef et al., introduced a weak form of pre-open sets which is called $\beta-$open sets. In 2009, [10] T. Noiri et al., introduced the classes of $\beta-\omega-$open sets and $\alpha-\omega-$open sets. In 2016, [11] Rajasekaran et al., introduced the classes of semi$-\omega-$open sets, which are new generalized classes of $\omega-$open sets in topological spaces, such as $\alpha-\omega-$open sets weeker than the class of $\alpha-$open sets, $\omega-$open sets and stronger than the class of semi-$\omega-$open sets. The concept of grill on a topological space, [3] given by Choquet. Roy and Mukherjee, [12], are introduced in 2007, on a typical topology induced by a grill. On the concept of a grill topological space, Hatir, and Jafari, [5] defined and investigated the notions in this part such as $\mathcal{G}$-pre-open set. Al-Omari and Noiri, [1] introduced the notions of $\mathcal{G}$-semi-open sets, $\mathcal{G}\alpha-$open sets such as the class of $\mathcal{G}\alpha-$open sets, which are week forms of the class of open sets in the topological spaces (X,τ). Also the class of $\mathcal{G}\alpha-$open sets strong of the class of $\alpha-$open sets in the topological spaces and the class of $\mathcal{G}$-semi-open sets in the grill topological spaces. In 2020, [17] they introduced the class of $\mathcal{G}^{\omega}-$open sets in grill topological spaces as a weaker than the class of $\omega-$open sets, and stronger than the class of $\beta-\omega$-open sets in the topological spaces.

This work consists of five sections, which are organized as follows:
In the Preliminaries, we recalled some of the basic facts and definitions about topological spaces and grill topological spaces, which will be used throughout this work.
In the third section, we introduced the concepts of the open $\mathcal{G}_{\omega}^{\alpha}-$ sets and their relationships with the other known concepts of openness. We next give the notions of the closure operator of open $\mathcal{G}_{\omega}^{\alpha}-$ sets.

In the fourth section the separation axioms are investigated and introduced by the collection of $\mathcal{G}_{\omega}^{\alpha}-$open sets.

2 Preliminaries

For a topological space (X,τ) and $A\subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

Theorem 1. [7] For a topological space (X,τ) and $A,B\subseteq X$. If B is an open set in (X,τ), then $Cl(A)\cap B\subseteq Cl(A\cap B)$.

Theorem 2. [7] For a topological space (X,τ),

1. $Cl(X-A)=X-Int(A)$ for all $A\subseteq X$.
2. $Int(X-A)=X-Cl(A)$ for all $A\subseteq X$.

Definition 1. [9] A subset A in a topological space (X, T) is called: An $\alpha-$open set if $A \subseteq Int(Cl(Int(A)))$. The complement of an $\alpha-$open set is called an $\alpha-$closed set. The set of all $\alpha-$closed sets in (X, τ) is denoted by $\alpha C(X)$ and the set of all $\alpha-$open sets in (X, τ) is denoted by $\alpha O(X)$.

Definition 2. [18] A subset A of a topological space (X, τ) is called a regular open (simply $r-$open) set if $A = Int(Cl(A))$. The complement of an $r-$open set is called a regular closed (simply an $r-$closed) set.

Theorem 3. [18] A subset A of a topological space (X, τ) is an $r-$closed set if and only if $A = Cl(Int(A))$.

Definition 3. [6] A subset A in a topological space (X, T) is called an $\omega-$open set if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. The complement of $\omega-$open set is called an $\omega-$closed set. The set of all $\omega-$closed sets in (X, τ) is denoted by $\omega C(X, \tau)$ and the set of all $\omega-$open sets in (X, τ) is denoted by $\omega O(X)$. The $\omega-$interior operator of $A$is defined as the union of all $\omega-$open sets which is a contained in A and denotes $Int_\omega(A)$, the $\omega-$closure operator of A is defined as the intersection of all $\omega-$closed sets which contain A and denotes $Cl_\omega(A)$.

Theorem 4. [6] For a topological space (X, τ), every open set is an $\omega-$open set.

Definition 4. [3] A non-null collection $\mathcal{G}$ of subsets of a topological spaces (X, τ) is said to be a grill on X if $\mathcal{G}$ satisfies the following conditions:
(i) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$
(ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 5. [13] Let X be a nonempty set and $\emptyset \neq A \subseteq X$. Then the collection $[A] = \{B \subseteq X : A \cap B \neq \emptyset\}$ is a grill on X and it is called the principal grill on X generated by A, (easily $\mathcal{G}_{[A]}$) on X.

For a grill topological space $(X, \tau, \mathcal{G})$, the operator $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ from the power set $\mathcal{P}(X)$ of X to $\mathcal{P}(X)$ was defined in [12] in the following manner : For any $A \in \mathcal{P}(X)$,

$$\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for each open neighborhood } U \text{ of } x\}.$$

This operator is called the operator associated with the grill $\mathcal{G}$ and the topology τ.
Then the operator $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in \mathcal{P}(X)$, was also shown in [12] to be a Kuratowski closure operator. So for a grill topological space $(X, \tau, \mathcal{G})$ there exists a topology $\tau_\mathcal{G}$ on X is defined by

$$\tau_\mathcal{G} = \{U \subseteq X : \Psi(X - U) = X - U\},$$

where $\tau \subseteq \tau_\mathcal{G}$ and for any $A \subseteq X$, $\Psi(A) = Cl_\mathcal{G}(A)$ such that $Cl_\mathcal{G}(A)$ denotes the set of all $\mathcal{G}-$closure points of A. A point $x \in X$ is called a $\mathcal{G}-$closure point of A if for every open set U in $(X, \tau_\mathcal{G})$ containing x, $U \cap A \neq \emptyset$. A point $x \in A$ is called a $\mathcal{G}-$interior point of A if there is an open set U in $(X, \tau_\mathcal{G})$ such that $x \in U \subseteq A$. The set of all $\mathcal{G}-$interior points of A is denoted by $int_\mathcal{G}(A)$.

Theorem 5. [12] Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then for $A, B \subseteq X$, the following properties hold:

1. $A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$;
2. $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$;
3. $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$;
4. If $U \in \tau$ then $U \cap \Phi(A) \subseteq \Phi(U \cap A)$.

Theorem 6. [12] Every closed set in (X, τ), is a closed set in $(X, \mathcal{G}, \tau)$.

Definition 6. [17] A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}^\omega-$open set if $A \subseteq Cl(Int_\omega(\Psi(A)))$. The complement of $\mathcal{G}^\omega-$open set is called a $\mathcal{G}^\omega-$closed set.

Theorem 7. [17] Every $\mathcal{G}^\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is a $\beta_\omega-$open set in the topological space (X, τ).

Definition 7. [14] A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}^\alpha_\omega-$open set if $A \subseteq Int(\Psi(Int_\omega(A)))$. The complement of $\mathcal{G}^\alpha_\omega-$open set is called a $\mathcal{G}^\alpha_\omega-$closed set. The set of all $\mathcal{G}^\alpha_\omega-$open sets in $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}^\alpha_\omega O(X)$ and the set of all $\mathcal{G}^\alpha_\omega-$closed sets in $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}^\alpha_\omega C(X)$.

Theorem 8. [14] For any grill topological space $(X, \tau, \mathcal{G})$ with a countable set X. Then every $\mathcal{G}^\alpha_\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is an $\omega-$open set.

Theorem 9. [14] Every $\mathcal{G}^\alpha_\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}^\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$.

Theorem 10. [14] Every $\mathcal{G}\alpha$-open set in a grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}^\alpha_\omega-$open set in the grill topological space $(X, \tau, \mathcal{G})$.

Theorem 11. [14] Let A_μ be any $\mathcal{G}^\alpha_\omega-$open set in a grill topological space $(X, \tau, \mathcal{G})$, for each $\mu \in I$. Then $\cup_{\mu \in I} A_\mu$ is a $\mathcal{G}^\alpha_\omega-$open set in the grill topological space $(X, \tau, \mathcal{G})$, where I is an index set.

Definition 8. [15] A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ of a grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, σ) is called a $\mathcal{G}^\alpha_\omega-$continuous function if $f^{-1}(A)$ is a $\mathcal{G}^\alpha_\omega-$open set in $(X, \tau, \mathcal{G})$ for every open set A in (Y, σ).

Definition 9. [15] Let $(X,\tau,\mathcal{G})$ be a grill topological space and (Y,σ) be a topological space. Then the function $f:(X,\tau,\mathcal{G})\to(Y,\sigma)$ is called:

- A $\mathcal{G}_\omega^\alpha-$closed function if $f(U)$ is a closed set in (Y,σ) for every $\mathcal{G}_\omega^\alpha-$closed set U in $(X,\tau,\mathcal{G})$.
- A $\mathcal{G}_\omega^\alpha-$open function if $f(U)$ is an open set in (Y,σ) for every $\mathcal{G}_\omega^\alpha-$open set U in $(X,\tau,\mathcal{G})$.

Definition 10. [4] A topological space (X,τ) is called:

1. A T_0-space if for two points $x\neq y\in X$, there is an open set G in (X,τ) such that $x\in G$ and $y\notin G$.
2. A T_1-space if for two points $x\neq y\in X$, there are two open sets G and U in (X,τ) such that $x\in G$, $y\notin G$, $y\in U$ and $x\notin U$.
3. A T_2-space or Hausdorff space if for two points $x\neq y\in X$, there are two open sets G and U in (X,τ) such that $x\in G$, $y\in U$ and $U\cap G=\emptyset$.
4. A regular space if for each closet set F in (X,τ) and each $x\notin F$, there are two open sets G and U in (X,τ) such that $F\subseteq G$, $x\in U$ and $U\cap G=\emptyset$. A topological space (X,τ) is called A T_3-space if it is a regular space and T_1-space.
5. A normal space if for each two disjoint closed sets F and M in (X,τ), there are two open sets G and U in (X,τ) such that $F\subseteq G$, $M\subseteq U$ and $U\cap G=\emptyset$. A topological space (X,τ) is called a T_4-space if it is a normal space and T_1-space.

Theorem 12. [18] A topological space (X,τ) is a T_1-space if and only if $\{x\}$ is a closed set in (X,τ) for all $x\in X$.

Theorem 13. [18] A topological space (X,τ) is a regular space if and only if for each $x\in X$ and for each open set N in (X,τ) containing x, there is an open set M in (X,τ) containing x such that $Cl(M)\subseteq N$.

Theorem 14. [16] Let $(X,\tau,\mathcal{G})$ be a grill topological space and $A\subseteq X$. Then A is not $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$ if and only if $A\nsubseteq H\subseteq\Psi(Int_\omega(A))$, for each open set H in (X,τ).

Corollary 1. [16] Let $(X,\tau,\mathcal{G})$ be a grill topological space, A and $B\subseteq X$. Then $A\cap B$ is a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$ if and only if there is an open set H in (X,τ) such that

$$A\cap B\subseteq H\subseteq\Psi(Int_\omega(A\cap B)).$$

Theorem 15. [16] Let $(X,\tau,\mathcal{G})$ be a grill topological space, $A\subseteq X$. If A is $\mathcal{G}_\omega^\alpha-$open set, then $\Psi(A)=\Psi(Int_\omega(A))$, in $(X,\tau,\mathcal{G})$.

Theorem 16. [16] Let $(X,\tau,\mathcal{G})$ be a grill topological space. If (X,τ) is a door space, then every $\mathcal{G}_\omega^\alpha-$open set in the grill topological space $(X,\tau,\mathcal{G})$ is an open set in (X,τ).

Theorem 17. [16] Let $(X,\tau,\mathcal{G})$ be a grill topological space, $A\subseteq X$. Then A is $\mathcal{G}_\omega^\alpha-$open set if and only if there is an open set H in (X,τ) such that $A\subseteq H\subseteq\Psi(Int_\omega(A))$.

From all the previous relationships in the background studied, we have the following figure.

3 $\mathcal{G}_\omega^\alpha-$Induced space

n the section, we introduced the concepts of an open $\mathcal{G}_\omega^\alpha-$ sets and their relationships with the other known concepts of openness. We next give the notions of the closure operator of an open $\mathcal{G}_\omega^\alpha-$ sets.

3.1 Open $\mathcal{G}_\omega^\alpha-$sets

Definition 11. For a set of all $\mathcal{G}_\omega^\alpha-$open sets $\mathcal{G}_\omega^\alpha O(X)$ in a grill topological space $(X,\tau,\mathcal{G})$, and $A\subseteq X$. A set A is called an open $\mathcal{G}_\omega^\alpha-$set, if for each $\mathcal{G}_\omega^\alpha-$open set B there exists an open set H in a topological space (X,τ) such that, $A\cap B\subseteq H\subseteq\Psi(Int_\omega(A\cap B))$. The complement of A is called a closed $\mathcal{G}_\omega^\alpha-$set in the topological space $(X,\tau,\mathcal{G})$. The set of all open $\mathcal{G}_\omega^\alpha-$set is denoted by $O\mathcal{G}_\omega^\alpha(X)$, and the set of all closed $\mathcal{G}_\omega^\alpha-$set is denoted by $C\mathcal{G}_\omega^\alpha(X)$.

Theorem 18. Let $(X,\tau,\mathcal{G})$ be a grill topological space, and $A\subseteq X$. Then A is an open $\mathcal{G}_\omega^\alpha-$set if and only if $A\cap B\in\mathcal{G}_\omega^\alpha O(X)$ for each $\mathcal{G}_\omega^\alpha-$open set B in the grill topological space $(X,\tau,\mathcal{G})$.

Proof: Suppose that A is an open $\mathcal{G}_\omega^\alpha-$set in $(X,\tau,\mathcal{G})$. Let B be a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$, $H\in(X,\tau)$. Since $A\in O\mathcal{G}_\omega^\alpha(X)$, by definition of open $\mathcal{G}_\omega^\alpha-$set, Corollary (1) and for each $\mathcal{G}_\omega^\alpha-$open set B, we have $A\cap B\in\mathcal{G}_\omega^\alpha O(X)$.
Conversely, since $A\cap B\in\mathcal{G}_\omega^\alpha O(X)$ for each $\mathcal{G}_\omega^\alpha$ − open set B in the grill topological space $(X,\tau,\mathcal{G})$, by Corollary (1), there is an open set H in a topological space (X,τ) such that

$$A\cap B\subseteq H\subseteq\Psi(Int_\omega(A\cap B)),\ \text{for each } \mathcal{G}_\omega^\alpha-\text{open set } B.$$

Hence A is an open $\mathcal{G}_\omega^\alpha-$set in the grill topological space $(X,\tau,\mathcal{G})$. □

Corollary 2. Let $(X,\tau,\mathcal{G})$ be a grill topological space, $A\subseteq X$. A set A is not $\mathcal{G}_\omega^\alpha-$set in a grill topological space $(X,\tau,\mathcal{G})$ if and only if there is a $\mathcal{G}_\omega^\alpha-$open set B such that $A\cap B\notin\mathcal{G}_\omega^\alpha O(X)$.

Proof: Let A not be to an open $\mathcal{G}_\omega^\alpha-$set and B be a $\mathcal{G}_\omega^\alpha-$open set in a grill topological space $(X,\tau,\mathcal{G})$. Suppose that $A\cap B\in\mathcal{G}_\omega^\alpha O(X)$ for each $\mathcal{G}_\omega^\alpha-$open set B. Then, by Theorem (18), A is an open $\mathcal{G}_\omega^\alpha-$set. This is a contradiction by hypothesis. Hence, $A\cap B\notin\mathcal{G}_\omega^\alpha O(X)$.

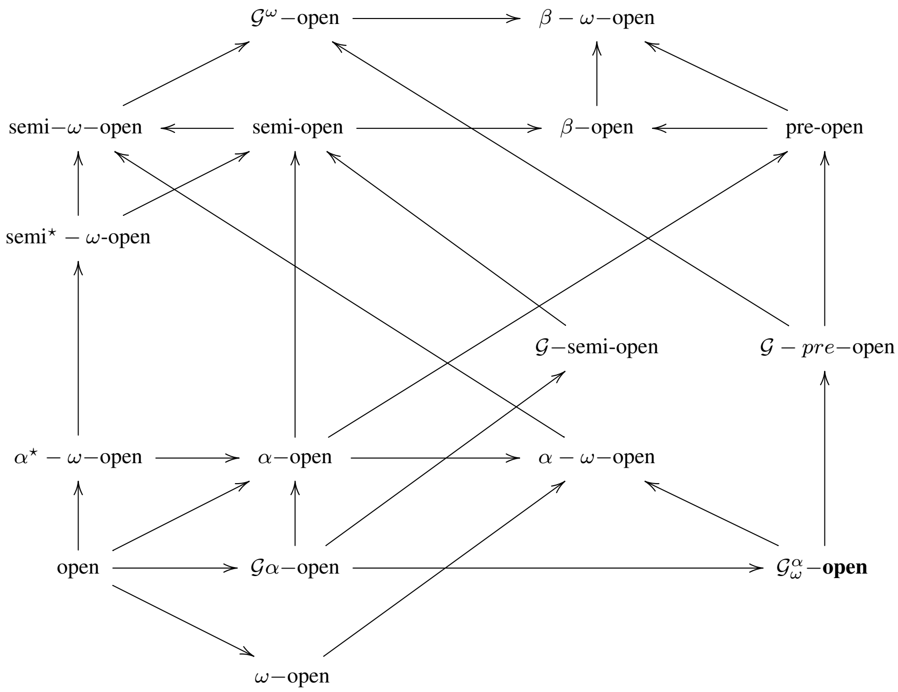

Fig. 1: Relation for open sets

Conversely, since $A\cap B\notin \mathcal{G}^\alpha_\omega O(X)$, by Theorem (14) , $we get A\cap B\nsubseteq H\subseteq \Psi(Int_\omega(A\cap B))$, for each open set H in a topological space (X,τ). Hence, by Definition (18), we have A is not an open $\mathcal{G}^\alpha_\omega-$set in the grill topological space $(X,\tau,\mathcal{G})$. □

Theorem 19. Let $(X,\tau,\mathcal{G})$ be a grill topological space and $A\subset X$. If A is not a $\mathcal{G}^\alpha_\omega-$open set, then A is not an open $\mathcal{G}^\alpha_\omega-$set in the grill topological space $(X,\tau,\mathcal{G})$.

Proof: Let $A\subset X$ and $A\notin \mathcal{G}^\alpha_\omega O(X)$. By Theorem (14), we get

$$A=A\cap X\nsubseteq H\subseteq \Psi(Int_\omega(A))=\Psi(Int_\omega(A\cap X)),$$

for each open set H in (X,τ). So there is a $\mathcal{G}^\alpha_\omega$ − open set X such that $A\cap X\notin \mathcal{G}^\alpha_\omega O(X)$. Hence A is not an open $\mathcal{G}^\alpha_\omega-$set in the topological space $(X,\tau,\mathcal{G})$. □

Corollary 3. Every open $\mathcal{G}^\alpha_\omega-$set is a $\mathcal{G}^\alpha_\omega-$open set in the grill topological space $(X,\tau,\mathcal{G})$.

Proof: Let $A\subset X$and A be an open $\mathcal{G}^\alpha_\omega-$set in $(X,\tau,\mathcal{G})$. Suppose that $A\notin \mathcal{G}^\alpha_\omega O(X)$. Since by Theorem (19), we have A is not an open $\mathcal{G}^\alpha_\omega-$set in $(X,\tau,\mathcal{G})$, by hypothesis, which is a contradiction. Hence every open $\mathcal{G}^\alpha_\omega-$set is a $\mathcal{G}^\alpha_\omega-$open set in the grill topological space $(X,\tau,\mathcal{G})$. □

The converse of Corollary (3) need not be true.

Example 1. Let $(X,\tau,\mathcal{G})$ be a grill topological space on the set of $X=\{1,2,3,4\}$ with $\tau=\{\emptyset,\{1,2,3\},X\}$ and $\mathcal{G}=\mathcal{P}(X)-\{\emptyset\}$. The set $A=\{1,2,4\}$ is a $\mathcal{G}^\alpha_\omega-$open set, which is not an open $\mathcal{G}^\alpha_\omega-$set in the grill topological space $(X,\tau,\mathcal{G})$.

Theorem 20. Every open set in a topological space (X,τ), is an open $\mathcal{G}^\alpha_\omega-$set in the grill topological space $(X,\tau,\mathcal{G})$.

Proof: Let A be any open set in (X,τ) and $B\in \mathcal{G}^\alpha_\omega O(X)$. Since by Theorem (**??**), $A\cap B$ is a $\mathcal{G}^\alpha_\omega-$open set, we get every open set in a topological space (X,τ) is an open $\mathcal{G}^\alpha_\omega-$set in the grill topological space $(X,\tau,\mathcal{G})$. □

The converse of Theorem (20) need not be true.

Example 2. Let $(X,\tau,\mathcal{G})$ be a grill topological space on the set of $X=\{1,2,3,4\}$ with $\tau=\{\emptyset,\{1,2,3\},X\}$ and $\mathcal{G}=\mathcal{P}(X)-\{\emptyset\}$. Then the set $A=\{1,2\}$ is an open $\mathcal{G}^\alpha_\omega-$set in the grill topological space $(X,\tau,\mathcal{G})$, but it is not an open set in a topological space (X,τ).

Remark 1. Let $(X,\tau,\mathcal{G})$ be a grill topological space, A and B be two $\mathcal{G}^\alpha_\omega-$open sets. If $A\cap B\notin \mathcal{G}^\alpha_\omega O(X)$, then A and B are not $\mathcal{G}^\alpha_\omega-$sets in the grill topological space $(X,\tau,\mathcal{G})$.

Proof: Since $A \cap B \notin \mathcal{G}^{\alpha}_{\omega}O(X)$, by Theorem (2) for a set A there is at least a $\mathcal{G}^{\alpha}_{\omega}-$open B such that $A \cap B \notin \mathcal{G}^{\alpha}_{\omega}O(X)$. Similarly, for a set B, there is at least a $\mathcal{G}^{\alpha}_{\omega}-$open A such $B \cap A \notin \mathcal{G}^{\alpha}_{\omega}O(X)$. Hence $A, B \notin O\mathcal{G}^{\alpha}_{\omega}(X)$. □

Theorem 21. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then A is an open $\mathcal{G}^{\alpha}_{\omega}-$set in a grill topological space $(X, \tau, \mathcal{G})$ if and only if for each $\mathcal{G}^{\alpha}_{\omega} -$ open set B, ${}_{\mathcal{G}^{\alpha}_{\omega}}Cl(X - (A \cap B)) = X - (A \cap B)$.

Proof: Suppose that A is an open $\mathcal{G}^{\alpha}_{\omega}-$set in $(X, \tau, \mathcal{G})$. By Theorem (2) $A \cap B \in O\mathcal{G}^{\alpha}_{\omega}(X)$ for each $\mathcal{G}^{\alpha}_{\omega}-$open set B, then

$$X - (A \cap B) = X - {}_{\mathcal{G}^{\alpha}_{\omega}}Int(A \cap B) = {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(X - (A \cap B))$$

. Conversely, it is similar to the above argument. □

Theorem 22. For the set of all $\mathcal{G}^{\alpha}_{\omega}-$ open set $\mathcal{G}^{\alpha}_{\omega}O(X)$ in a grill topological space $(X, \tau, \mathcal{G})$, there is a topology on X equivalent the set of all open $\mathcal{G}^{\alpha}_{\omega}-$sets $O\mathcal{G}^{\alpha}_{\omega}(X)$ defined by

$$\tau_{\mathcal{G}^{\alpha}_{\omega}} = \{A \subseteq X : {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(X - (A \cap B)) = (X - (A \cap B)) \text{ for each } \mathcal{G}^{\alpha}_{\omega} - \text{open set B.}\}$$

Proof:

1. If $A = X$. Then for each a $\mathcal{G}^{\alpha}_{\omega} -$ open set B.

$${}_{\mathcal{G}^{\alpha}_{\omega}}Cl(X - (X \cap B)) = X - (X \cap B) = {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(X - B) = X - B,$$

Also if $A = \emptyset$. Then for each $\mathcal{G}^{\alpha}_{\omega} -$ open set B in $(X, \tau, \mathcal{G})$.

$${}_{\mathcal{G}^{\alpha}_{\omega}}Cl(X - (\emptyset \cap B)) = X - (\emptyset \cap B) = {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(X - \emptyset) = X - \emptyset.$$

Hence $X, \emptyset \in \tau_{\mathcal{G}^{\alpha}_{\omega}}$.
2. Let $A_1, A_2 \in \tau_{\mathcal{G}^{\alpha}_{\omega}}$, this mean that $A_1, A_2 \in O\mathcal{G}^{\alpha}_{\omega}(X)$ by Theorem (21). Suppose that $A_1 \cap A_2 \notin O\mathcal{G}^{\alpha}_{\omega}(X)$. Now by Corollary (3) $A_1, A_2 \in \mathcal{G}^{\alpha}_{\omega}O(X)$, then $A_1, A_2 \notin O\mathcal{G}^{\alpha}_{\omega}(X)$, by Remark (1). Therefore this is a contradiction. Hence, $A_1 \cap A_2 \in \tau_{\mathcal{G}^{\alpha}_{\omega}}$.
3. Let $A_i \in \tau_{\mathcal{G}^{\alpha}_{\omega}}$ for each $i \in \Delta$. Then by Theorem (21) $A_i \in O\mathcal{G}^{\alpha}_{\omega}(X)$ for each $i \in \Delta$. So $A_i \cap B \subseteq A_i \in \mathcal{G}^{\alpha}_{\omega}O(X)$ by Theorem (18). Now for each $i \in \Delta$ and for each a $\mathcal{G}^{\alpha}_{\omega} -$ open set B, we get $\cup_{i\in\Delta}(Ai \cap B) \subseteq \cup_{i\in\Delta}(Ai) \in \mathcal{G}^{\alpha}_{\omega}O(X)$ by Theorem (11). Since

$$(\cup_{i\in\Delta}Ai) \cap B = \cup_{i\in\Delta}(Ai \cap B) \in \mathcal{G}^{\alpha}_{\omega}O(X).$$

for each $B \in \mathcal{G}^{\alpha}_{\omega}O(X)$. Hence $\cup_{i\in\Delta}A_i \in \tau_{\mathcal{G}^{\alpha}_{\omega}}$, for each $i \in \Delta$.

From 1, 2, 3 the collection $\tau_{\mathcal{G}^{\alpha}_{\omega}}$ is a topology on X. □

Remark 2. The triple $(X, \tau, \mathcal{G}^{\alpha}_{\omega})$, (easily$(X, \tau, \tau_{\mathcal{G}^{\alpha}_{\omega}})$) is called a $\mathcal{G}^{\alpha}_{\omega}-$topological space, if $O\mathcal{G}^{\alpha}_{\omega}(X)$ (easily $\tau_{\mathcal{G}^{\alpha}_{\omega}}$) is a topology on X.

Remark 3. The concepts of openness in $(X, \tau_{\mathcal{G}^{\alpha}_{\omega}})$ and openness in $(X, \tau_{\mathcal{G}})$ are independent.

Example 3. Let $(X, \tau, \mathcal{G})$ be a grill topological space on the set $X = \{1, 2, 3, 4\}$ with $\tau = \{\emptyset, \{2, 1\}, \{2\}, X\}$. If $\mathcal{G} = \mathcal{G}_{[\{4,1\}]}$, then

$$\tau_{\mathcal{G}} = \{\emptyset, \{2, 1\}, \{2\}, \{1\}, \{1, 3, 4\}, \{1, 4\}, \{1, 2, 4\}, X\},$$

$$\tau_{\mathcal{G}^{\alpha}_{\omega}} = \mathcal{G}^{\alpha}_{\omega}O(X) = O\mathcal{G}^{\alpha}_{\omega}(X) = \{\emptyset, \{2, 1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}, X\}.$$

Now the set $\{1, 2, 3\} \in O\mathcal{G}^{\alpha}_{\omega}$, but $\{1, 2, 3\} \notin \tau_{\mathcal{G}}$. Also the set $\{1, 3, 4\} \in \tau_{\mathcal{G}}$, but it is not open in the $\mathcal{G}^{\alpha}_{\omega}-$topological space $(X, \tau, \tau_{\mathcal{G}^{\alpha}_{\omega}})$.

3.2 $\mathcal{G}^{\alpha}_{\omega}T-$***space***

Definition 12. A grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}^{\alpha}_{\omega}T-$space if the intersection of any two arbitrary $\mathcal{G}^{\alpha}_{\omega}-$open sets A and B is $\mathcal{G}^{\alpha}_{\omega}-$open set in the grill topological space $(X, \tau, \mathcal{G})$.

Theorem 23. Let $(X, \tau, \mathcal{G})$ be the grill topological space, and A be the subset of X. If $Cl(A) \subseteq {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(A)$, then the grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}^{\alpha}_{\omega}T-$space.

Proof: Let A be any subset of X in the grill topological space $(X, \tau, \mathcal{G})$ and $Cl(A) \subseteq {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(A)$. Since ${}_{\mathcal{G}^{\alpha}_{\omega}}Cl(A) \subseteq Cl(A)$ and by hypothesis $Cl(A) \subseteq {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(A)$, we get $Cl(A) = {}_{\mathcal{G}^{\alpha}_{\omega}}Cl(A)$, for any subset A of X.
Now, let G and H be two arbitrary $\mathcal{G}^{\alpha}_{\omega}-$open sets. Then G^c and H^c are $\mathcal{G}^{\alpha}_{\omega}-$closed sets. So

$${}_{\mathcal{G}^{\alpha}_{\omega}}Cl(G^c \cup H^c) = Cl(G^c \cup H^c)$$

is a closed set in a topological space (X, τ), also ${}_{\mathcal{G}^{\alpha}_{\omega}}Cl(G^c \cup H^c)$ is a $\mathcal{G}^{\alpha}_{\omega}-$closed set in the grill topological space $(X, \tau, \mathcal{G})$. Therefore $X - (G^c \cup H^c) = G \cap H$ is a $\mathcal{G}^{\alpha}_{\omega}-$open set. Hence the grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}^{\alpha}_{\omega}T-$space. □

Theorem 24. Let $(X, \tau, \mathcal{G})$ be a grill topological space and (X, τ) be a door space. Then the grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space.

Proof: Let (X, τ) be a door space, A and B be two $\mathcal{G}_\omega^\alpha-$open sets in the grill topological space $(X, \tau, \mathcal{G})$. Since by Theorem (16)

$$_{\mathcal{G}_\omega^\alpha}Int(A) = Int(A) = A,\ _{\mathcal{G}_\omega^\alpha}Int(B) = Int(B) = B,$$

then $A \cap B = Int(A \cap B) = {}_{\mathcal{G}_\omega^\alpha}Int(A \cap B) \in \mathcal{G}_\omega^\alpha O(X)$. Hence the grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space. □

Theorem 25. Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space, if every $\mathcal{G}_\omega^\alpha-$open set is a closed set in $(X, \tau, \mathcal{G})$.

Proof: Let A, B be two sets which are both $\mathcal{G}_\omega^\alpha-$open sets and closed sets in the grill topological space $(X, \tau, \mathcal{G})$. Since $A = \Psi(A), B = \Psi(B)$, we get

$$A \subseteq Int(A) = Int(\Psi(A)), B \subseteq Int(B) = Int(\Psi(B)).$$

So

$$A = Int(A) = Int(\Psi(A)), B = Int(B) = Int(\Psi(B)).$$

Therefore

$$A \cap B = Int(A \cap B) = {}_{\mathcal{G}_\omega^\alpha}Int(A \cap B) \in \mathcal{G}_\omega^\alpha O(X).$$

Hence the grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space. □

Theorem 26. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space if and only if the finite union of $\mathcal{G}_\omega^\alpha-$closed sets in $(X, \tau, \mathcal{G})$, is a $\mathcal{G}_\omega^\alpha-$closed set.

Proof: Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space. Let B_i be arbitrary $\mathcal{G}_\omega^\alpha-$open set, $i = 1, 2..., n \in \mathbb{N}$, where the set of natural numbers $\mathbb{N}$. Since by hypothesis

$$\cap_i^n(B_i) \in \mathcal{G}_\omega^\alpha O(X), X - B_i \in \mathcal{G}_\omega^\alpha C(X),$$

then

$$X - (\cap_i^n(B_i)) = \cup_i^n(X - B_i).$$

is a $\mathcal{G}_\omega^\alpha-$closed set. Hence the finite union of $\mathcal{G}_\omega^\alpha-$closed sets in $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$closed set. Conversely, similar to the above argument. □

Theorem 27. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space if and only if $\mathcal{G}_\omega^\alpha O(X) = O\mathcal{G}_\omega^\alpha(X)$.

Proof: Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space. Let A, B be two arbitrary $\mathcal{G}_\omega^\alpha-$open sets, Since $A \cap B \in \mathcal{G}_\omega^\alpha O(X)$, for any $A, B \in \mathcal{G}_\omega^\alpha O(X)$ there is an open set H in (X, τ) such that $A \cap B \subseteq H \subseteq \Psi(Int_\omega(A \cap B))$. Therefore $A, B \in O\mathcal{G}_\omega^\alpha(X)$ by Definition (11). Hence $\mathcal{G}_\omega^\alpha O(X) \subseteq O\mathcal{G}_\omega^\alpha(X)$. It is well known that $O\mathcal{G}_\omega^\alpha(X) \subseteq \mathcal{G}_\omega^\alpha O(X)$. Therefore, we obtain that $O\mathcal{G}_\omega^\alpha(X) = \mathcal{G}_\omega^\alpha O(X)$.
Conversely, let A, B be an arbitrary two $\mathcal{G}_\omega^\alpha-$open sets. Since A and $B \in O\mathcal{G}_\omega^\alpha(X)$, by Theorem (18). So by hypothesis $A \cap B \in \mathcal{G}_\omega^\alpha O(X)$. Hence a grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space. □

Theorem 28. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space if and only if the set of all $\mathcal{G}_\omega^\alpha-$open set $\mathcal{G}_\omega^\alpha O(X)$ is a topology on X.

Proof: Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space. Now

1. $X, \emptyset \in \mathcal{G}_\omega^\alpha O(X)$.
2. Let A and B be two $\mathcal{G}_\omega^\alpha-$open sets. We have by hypothesis, $(A \cap B) \in \mathcal{G}_\omega^\alpha O(X)$.
3. Let $A_i \in \mathcal{G}_\omega^\alpha O(X)$ for each $i \in \Delta$. Then $(\cup_{i \in \Delta} Ai) \in \mathcal{G}_\omega^\alpha O(X)$, by Theorem (11). From 1, 2, 3, the collection $\mathcal{G}_\omega^\alpha O(X)$ is a topology on X.

Conversely, let A and B be two arbitrary $\mathcal{G}_\omega^\alpha-$open sets . Since $\mathcal{G}_\omega^\alpha O(X)$ is a topology on X, then $(A \cap B) \in \mathcal{G}_\omega^\alpha O(X)$. Hence $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T-$space. □

3.3 $\mathcal{G}_\omega^\alpha-$***Induced Operators***

Definition 13. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $x \in X,\ A \subseteq X$. The set A is called a $\mathcal{G}_\omega^\alpha-$neighborhood ((easily. $\mathcal{G}_\omega^\alpha - nhd$)) of x in the grill topological space $(X, \tau, \mathcal{G})$ if there exists a $\mathcal{G}_\omega^\alpha-$open set B containing x such that $x \in B \subseteq A$. The set of all $\mathcal{G}_\omega^\alpha nhd$ of x is denoted ${}_{\mathcal{G}_\omega^\alpha}N_x$. The set of all ${}_{\mathcal{G}_\omega^\alpha}N_x$. is denoted ${}_{\mathcal{G}_\omega^\alpha}N_X$, where ${}_{\mathcal{G}_\omega^\alpha}N_x = \{A \subseteq X : A \text{ is } \mathcal{G}_\omega^\alpha - nhd \text{ of } x\}$ and ${}_{\mathcal{G}_\omega^\alpha}N_X = \{{}_{\mathcal{G}_\omega^\alpha}N_x : x \in X\}$

Theorem 29. Let $(X, \tau, \mathcal{G})$ be a grill topological space , $x,\ y \in X,\ A$ and $B \subseteq X$. Then the following hold:

1. For each $x \in X,\ {}_{\mathcal{G}_\omega^\alpha}N_x \neq \emptyset$.
2. If $A \in {}_{\mathcal{G}_\omega^\alpha}N_x$ then $x \in A$.
3. If $A \in {}_{\mathcal{G}_\omega^\alpha}N_x,\ A \subseteq B$ then $B \in {}_{\mathcal{G}_\omega^\alpha}N_x$.
4. If $A \in {}_{\mathcal{G}_\omega^\alpha}N_x,$ then $B \cup A \in {}_{\mathcal{G}_\omega^\alpha}N_x$.

Proof:

1. For each $x \in X$, ${}_{\mathcal{G}_\omega^\alpha}N_x$ contains $\mathcal{G}_\omega^\alpha-$open set X which is containing x.
2. Since $A \in {}_{\mathcal{G}_\omega^\alpha}N_x$, we get there is a $\mathcal{G}_\omega^\alpha-$open set B containing x such that $x \in B \subseteq A$.
3. Since $A \in {}_{\mathcal{G}_\omega^\alpha}N_x$, and $A \subseteq B$, we obtain that there is a $\mathcal{G}_\omega^\alpha-$open set H containing x such that $H \subseteq A \subseteq B$. Therefor $B \in {}_{\mathcal{G}_\omega^\alpha}N_x$.
4. Since $A \in {}_{\mathcal{G}_\omega^\alpha}N_x$, and $A \subseteq B \cup A$, we have $B \cup A \in {}_{\mathcal{G}_\omega^\alpha}N_x$.

□

Remark 4. Let $(X, \tau, \tau_{\mathcal{G}_\omega^\alpha})$ be a $\mathcal{G}_\omega^\alpha-$topological space. If for x in X and for each $\mathcal{G}_\omega^\alpha-$open set B_x containing x, then $X \cap B_x$ is a $\mathcal{G}_\omega^\alpha - nbh$ of x in a grill topological space $(X, \tau, \mathcal{G})$.

Remark 5. Let $(X, \tau, \tau_{\mathcal{G}_\omega^\alpha})$ be a $\mathcal{G}_\omega^\alpha-$topological space. If for x in X and for each $\mathcal{G}_\omega^\alpha-$open set B_x containing x, then $\emptyset \cap B_x$ is not $\mathcal{G}_\omega^\alpha - nbh$ of x in a grill topological space $(X, \tau, \mathcal{G})$.

Theorem 30. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then A is a $\mathcal{G}_\omega^\alpha-$open set if and only if A is a $\mathcal{G}_\omega^\alpha - nbh$ of it is points.

Proof: Let x be any point in X and A be a $\mathcal{G}_\omega^\alpha-$open set containing x in $(X, \tau, \mathcal{G})$. Since by Definition (13), we have for any A is a $\mathcal{G}_\omega^\alpha - nbh$ of x. Therefor A is a $\mathcal{G}_\omega^\alpha - nbh$ of it is points in $(X, \tau, \mathcal{G})$.
Conversely, Since A is a $\mathcal{G}_\omega^\alpha - nbh$ of it is points, we get by Definition (13) a $\mathcal{G}_\omega^\alpha-$open set A_x contains x, such that $x \in A_x \subseteq A$. Therefor $\cup_{x\in A} A_x = A$. By Theorem(11), we have A is a $\mathcal{G}_\omega^\alpha-$open in $(X, \tau, \mathcal{G})$.

□

Definition 14. Let ${}_{\mathcal{G}_\omega^\alpha}N_x$ be a set of all $\mathcal{G}_\omega^\alpha - nbh$ of a point $x \in X$ in the grill topological space $(X, \tau, \mathcal{G})$ and $(X, \tau, \tau_{\mathcal{G}_\omega^\alpha})$ be a $\mathcal{G}_\omega^\alpha-$topological space.
1. The closure point operator of a subset A of X in $(X, \tau, \tau_{\mathcal{G}_\omega^\alpha})$ is denoted by $\Upsilon(A)$ and defined by

$$\Upsilon(A) = \{x \in X : \exists\, \mathcal{G}_\omega^\alpha - \text{open set } B_x \text{ containing } x \text{ such that } A^c \cap B_x \notin {}_{\mathcal{G}_\omega^\alpha}N_x\}.$$

Theorem 31. Let $(X, \tau, \tau_{\mathcal{G}_\omega^\alpha})$ be a $\mathcal{G}_\omega^\alpha-$topological space, $(X, \tau, \mathcal{G})$ be a grill topological space, A and $B \subseteq X$. Then the following hold:

1. $\Upsilon(A) = \emptyset$ *if and only if* $A = \emptyset$.
2. $\Upsilon(X) = X$.
3. $A \subseteq \Upsilon(A)$
4. $A \subseteq B$ *then* $\Upsilon(A) \subseteq \Upsilon(B)$.
5. $\Upsilon(A \cap B) \subseteq \Upsilon(A) \cap \Upsilon(B)$.
6. $\Upsilon(A \cup B) = \Upsilon(A) \cup \Upsilon(B)$.
7. $\Upsilon(A) = \Upsilon(\Upsilon(A))$
8. ${}_{\mathcal{G}_\omega^\alpha}Cl(A) \subseteq \Upsilon(A)$.
9. $\Upsilon({}_{\mathcal{G}_\omega^\alpha}Cl(A)) = \Upsilon(A)$.
10. ${}_{\mathcal{G}_\omega^\alpha}Cl(\Upsilon(A)) = \Upsilon(A)$.

Proof:

1. Since $\emptyset \subseteq A$ for every subset A of X, then $\emptyset \subseteq \Upsilon(\emptyset)$. Conversely, Since $x \in \Upsilon(\emptyset)$, we get there is an $\mathcal{G}_\omega^\alpha-$open set B_x containing a point x such that $X \cap B_x \notin \mathcal{G}_\omega^\alpha - nhd$ of x in $(X, \tau, \mathcal{G})$, then by Remark (4) = $\emptyset$. Hence $\Upsilon(\emptyset) \subseteq \emptyset$. Therefore $\emptyset = \Upsilon(\emptyset)$.
2. Since for every subset of X, we have $\Upsilon(X) \subseteq X$. Let x be any point in X. Since for $x \in X \exists\, \mathcal{G}_\omega^\alpha-$open set B_x containing x such that $X - X = \emptyset \cap B_x \notin \mathcal{G}_\omega^\alpha - nhd$ of x in $(X, \tau, \mathcal{G})$, then by Definition 14, $x \in \Upsilon(X)$. Hence $X \subseteq X$. Therefore $X = \Upsilon(X)$.
3. Let x be any point in X. Since $x \notin (X - A) \cap B_x$ for $\mathcal{G}_\omega^\alpha-$open set B_x containing x, by Definition 14, we get $x \in \Upsilon(A)$. Hence $A \subseteq \Upsilon(A)$.
4. Let x be any point in X and $A \subseteq G \subseteq X$. Suppose that $x \in \Upsilon(A)$. Since $X - G \subseteq (X - A)$ and $(X - A) \cap B_x \notin \mathcal{G}_\omega^\alpha - nbh$ of x, by part (3) of Theorem (29), we get $(X - G) \cap B_x \notin \mathcal{G}_\omega^\alpha - nbh$ of x. Therefore $x \in \Upsilon(G)$. Hence $\Upsilon(A) \subseteq \Upsilon(G)$.
5. It is clear that $\Upsilon(A \cap B) \subseteq \Upsilon(A) \cap \Upsilon(B)$, by part(4).
6. • It is clear that $\Upsilon(A) \cup \Upsilon(B) \subseteq \Upsilon(A \cup B)$, by part(4).
• Let $x \notin \Upsilon(A) \cup \Upsilon(B)$. Since $x \in X - (\Upsilon(A) \cup \Upsilon(B))$, then $x \in (\Upsilon(A))^c \cap (\Upsilon(B))^c$. So $x \in (\Upsilon(A))^c$; $x \in (\Upsilon(B))^c$. Therefore there exist $\mathcal{G}_\omega^\alpha-$ sets G, H containing x such that $G \cap B_x \subseteq A^c \cap B_x, H \cap B_x \subseteq B^c \cap B_x$, G and H are $\mathcal{G}_\omega^\alpha - nbh$ of x for each $\mathcal{G}_\omega^\alpha-$open set B_x. Since $x \in (G \cap H) \cap B_x \subseteq (A^c \cap B^c) \cap B_x$ and $(G \cap H) \cap B_x$ is $\mathcal{G}_\omega^\alpha - nbh$ of it is points, by Definition (14), we have $x \notin \Upsilon(A \cup B)$.
Hence $\Upsilon(A \cup B) = \Upsilon(A) \cup \Upsilon(B)$.
7. • It is clear that $\Upsilon(A) \subseteq \Upsilon(\Upsilon(A))$, by part(3).
• Let $x \in \Upsilon(\Upsilon(A))$, then $x \in \Upsilon(A)$ by Definition (14) and Theorem (**??**). Therefore $\Upsilon(\Upsilon(A)) \subseteq \Upsilon(A)$
Hence $\Upsilon(A) = \Upsilon(\Upsilon(A))$
8. Let $x \in {}_{\mathcal{G}_\omega^\alpha}Cl(A)$, then $x \in A$ or $x \in A^c$. So $x \in A^c \cap B_x \neq \emptyset$. Now if $x \notin \Upsilon(A)$, then by Theorem (**??**) and Definition (13), there is a $\mathcal{G}_\omega^\alpha-$open set $G \subseteq A^c$ containing x. Therefore $x \notin {}_{\mathcal{G}_\omega^\alpha}Cl(A)$. That is contradiction. Hence $x \in \Upsilon(A)$ and ${}_{\mathcal{G}_\omega^\alpha}Cl(A) \subseteq \Upsilon(A)$.
9. • It is clear that $\Upsilon(A) \subseteq \Upsilon({}_{\mathcal{G}_\omega^\alpha}Cl(A))$, by part(3).
• Let $x \in \Upsilon({}_{\mathcal{G}_\omega^\alpha}Cl(A))$, then $\Upsilon({}_{\mathcal{G}_\omega^\alpha}Cl(A)) \subseteq \Upsilon(\Upsilon(A)) \subseteq \Upsilon(A)$, by parts (8) and (7).
Hence $\Upsilon({}_{\mathcal{G}_\omega^\alpha}Cl(A)) = \Upsilon(A)$.
10. • It is clear that $\Upsilon(A) \subseteq {}_{\mathcal{G}_\omega^\alpha}Cl(\Upsilon(A))$, by $\mathcal{G}_\omega^\alpha-$closure operator.
• Let $x \in {}_{\mathcal{G}_\omega^\alpha}Cl(\Upsilon(A))$, then $x \in \Upsilon({}_{\mathcal{G}_\omega^\alpha}Cl(\Upsilon(A)) \subseteq \Upsilon(\Upsilon(\Upsilon(A))) \subseteq \Upsilon(\Upsilon(A)) \subseteq \Upsilon(A)$, by parts (8) and (7). Therefore $\Upsilon({}_{\mathcal{G}_\omega^\alpha}Cl(\Upsilon(A)) \subseteq \Upsilon(A)$.
Hence ${}_{\mathcal{G}_\omega^\alpha}Cl(\Upsilon(A)) = \Upsilon(A)$.

□

Corollary 4. Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then $\Upsilon(A)$ is $\mathcal{G}_\omega^\alpha-$closed set for any subset A of X in $(X, \tau, \mathcal{G})$.

Proof: Let A be a subset of X. Since ${}_{\mathcal{G}_\omega^\alpha}Cl(\Upsilon(A)) = \Upsilon(A)$, by last theorem, then $\Upsilon(A)$ is $\mathcal{G}_\omega^\alpha-$closed set. □

4 $\mathcal{G}_\omega^\alpha-$Separation axioms.

In the section the concepts of the separation axioms are investigated and introduced by the collection of $\mathcal{G}_\omega^\alpha-$open sets, as $\mathcal{G}_\omega^\alpha T_1-$space, $\mathcal{G}_\omega^\alpha T_2-$space, $\mathcal{G}_\omega^\alpha T_3-$space and $\mathcal{G}_\omega^\alpha T_4-$space. We give their relationships with the other known concepts of separation axioms.

4.1 $\mathcal{G}_\omega^\alpha T_1-$space and $\mathcal{G}_\omega^\alpha T_2-$space

Definition 15. A grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}_\omega^\alpha T_1-$space if for each two element $x \neq y \in X$ there are two $\mathcal{G}_\omega^\alpha-$open sets H and , G such that $x \in H, y \notin G,\ y \in H$ and $x \notin G$.

Definition 16. A grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}_\omega^\alpha T_2-$space if for each two element $x \neq y \in X$ there are two $\mathcal{G}_\omega^\alpha-$open sets H and G such that $x \in H, y \in G$ and $G \cap H = \emptyset$.

Theorem 32. For grill topological space $(X, \tau, \mathcal{G})$, every $\mathcal{G}_\omega^\alpha T_2-$space is a $\mathcal{G}_\omega^\alpha T_1-$space.

Proof: Let $(X, \tau, \mathcal{G})$ be a $\mathcal{G}_\omega^\alpha T_2-$space. Then for each two element $x \neq y \in X$ there are two $\mathcal{G}_\omega^\alpha$ − open sets H, G such that $x \in H, y \in G$ and $G \cap H = \emptyset$. Thus there are two sets $\mathcal{G}_\omega^\alpha$ − open H, G such that $x \in H, y \notin H$ and $y \in G, x \notin G$, for each two element $x \neq y \in X$. Therefore by Definition (15), $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_1-$space. □

The converse of the Theorem (32) need not be true.

Example 4. Let $(X, \tau, \mathcal{G})$ be a grill topological space. If τ is the co- finite topology τ_{cof}, withe maximal $\mathcal{G} = \{X\}$. Then, $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_1-$space which is not $\mathcal{G}_\omega^\alpha T_2-$space.

Theorem 33. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_2-$ space if and only if for each $x \neq y \in X$, there is a $\mathcal{G}_\omega^\alpha-$open set B in $(X, \tau, \mathcal{G})$ containing x such that $y \notin {}_{\mathcal{G}_\omega^\alpha}Cl(B)$.

Proof: Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_2-$ space. Let $x \neq y \in X$. Then there are two $\mathcal{G}_\omega^\alpha-$open sets G and U in $(X, \tau, \mathcal{G})$ such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Take $H = G$. Then, H is a $\mathcal{G}_\omega^\alpha-$open set in $(X, \tau, \mathcal{G})$ containing x and so $y \notin H \subseteq {}_{\mathcal{G}_\omega^\alpha}Cl(H) \subseteq X - U$.
Conversely, let $x \neq y \in X$ be any points in $(X, \tau, \mathcal{G})$. And by the hypothesis, there is a $\mathcal{G}_\omega^\alpha-$open set H in $(X, \tau, \mathcal{G})$ containing x such that $y \notin {}_{\mathcal{G}_\omega^\alpha}Cl(H)$. Then, $X - {}_{\mathcal{G}_\omega^\alpha}Cl(H)$ is a $\mathcal{G}_\omega^\alpha-$open sets in $(X, \tau, \mathcal{G})$ containing y such that $x \in H, y \in X - {}_{\mathcal{G}_\omega^\alpha}Cl(H)$ and $H \cap [X - {}_{\mathcal{G}_\omega^\alpha}Cl(H)] = \emptyset$. Hence $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_2-$ space. □

Theorem 34. Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ be a $\mathcal{G}_\omega^\alpha-$continuous injective function from a grill topological space$(X, \tau, \mathcal{G})$ to a topological space (Y, σ) and (Y, σ) be a T_1-space. Then, $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_1-$space.

Proof: Let $x \neq y \in X$ be any points in X and (Y, σ) be a T_1-space . Since f is injective, we have $f(x) \neq f(y) \in (Y, \sigma)$, also there are two open sets B and H in (Y, σ) such that

$$f(x) \in B, f(y) \in H,\ f(x) \notin H \text{ and } f(y) \notin B.$$

Then, we obtain:

$$x \in f^{-1}(B), y \in f^{-1}(H), x \notin f^{-1}(H) \text{ and } y \notin f^{-1}(B).$$

Since B and H are open sets in (Y, σ) and f is $\mathcal{G}_\omega^\alpha-$continuous, we get $f^{-1}(H)$ and $f^{-1}(B)$ are $\mathcal{G}_\omega^\alpha-$open sets in $(X, \tau, \mathcal{G})$. Hence $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_1-$space. □

Theorem 35. Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ be a $\mathcal{G}_\omega^\alpha-$continuous injective function from a grill topological space$(X, \tau, \mathcal{G})$ to a topological space (Y, σ) and (Y, σ) be a T_2-space. Then, $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_2-$space.

Proof: Let $x \neq y \in X$ be any points in X and (Y, σ) be a T_2-space . Since f is injective, we have $f(x) \neq f(y) \in (Y, \sigma)$, also there are two open sets B and H in (Y, σ) such that $f(x) \in B$, $f(y) \in H$ and $H \cap B = \emptyset$. So

$$x \in f^{-1}(B), y \in f^{-1}(H) \text{ and } f^{-1}(A \cap B) = f^{-1}(B) \cap f^{-1}(H) = \emptyset.$$

Since B and H are open sets in (Y, σ) and f is $\mathcal{G}_\omega^\alpha-$continuous, we get: $f^{-1}(H), f^{-1}(B) \in \mathcal{G}_\omega^\alpha O(X)$ in $(X, \tau, \mathcal{G})$. Hence $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_2-$space. □

4.2 $\mathcal{G}_\omega^\alpha T_3-$space and $\mathcal{G}_\omega^\alpha T_4-$space

Definition 17. A grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}_\omega^\alpha-$regular space ($\mathcal{G}_\omega^\alpha r-$space) if for each $x \in X$ and each closed set A in (X, τ) not containing x there are two $\mathcal{G}_\omega^\alpha$ − open sets H and G such that $x \in H, A \subseteq G$ and $G \cap H = \emptyset$. If the grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$regular space and (X, τ) is a T_1-space, then $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}_\omega^\alpha T_3-$space.

Definition 18. A grill topological space $(X,\tau,\mathcal{G})$ is called a $\mathcal{G}_\omega^\alpha-$normal space if for each two disjoint closed sets A,B in (X,τ) there are two $\mathcal{G}_\omega^\alpha$ $-$ open sets H and G such that $A\subseteq G, B\subseteq H$ and $G\cap H=\emptyset$. If the grill topological space $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$normal space and (X,τ) is a T_1-space, then $(X,\tau,\mathcal{G})$ is called a $\mathcal{G}_\omega^\alpha T_4-$space.

Theorem 36. If(X,τ) is a regular space, then $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha r-$space for each grill $\mathcal{G}$.

Proof: Let x be any point $\in X$, and (X,τ) be a regular-space . By hypothesis in (X,τ), for each $x\in X$ and for each a closed set F not containing x there are two open sets H and G such that $x\in H, F\subseteq G$ and $G\cap H=\emptyset$. Since H and G are $\mathcal{G}_\omega^\alpha-$open sets in the grill topological space $(X,\tau,\mathcal{G})$, for each $x\in X$ and each closed set F in (X,τ) not containing x there are two $\mathcal{G}_\omega^\alpha$ $-$ open sets H and G such that $x\in H, F\subseteq G$ and $G\cap H=\emptyset$. Therefore by Definition (17) $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha r-$space. □

Theorem 37. Every normal space is a $\mathcal{G}_\omega^\alpha-$normal space.

Proof: Similar to the proof of the above theorem. □

Theorem 38. Every T_i-space is a $\mathcal{G}_\omega^\alpha T_i-$space, $i=2,3$ and 4.

Proof: Let $(X,\tau,\mathcal{G})$ be a T_i-space, $i=1,2,3$ and 4. Since every open set in (X,τ) is a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$,by part one of Definition 10 and Definition 16, we have every T_1-space is a $\mathcal{G}_\omega^\alpha T_2-$space, by part two of Definition 10 and Definition 16, we have every T_2-space is a $\mathcal{G}_\omega^\alpha T_2-$space. By part four of Definition 10 and Definition 17, we get every T_3-space is a $\mathcal{G}_\omega^\alpha T-$space. And also by part five of Definition 10, and Definition 18, we have every T_4-space is a $\mathcal{G}_\omega^\alpha T_4-$space. □

Theorem 39. A grill topological space $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha r-$ space if and only if for each $x\in X$ and for each open set A in (X,τ) containing x, there is a $\mathcal{G}_\omega^\alpha-$open set B in $(X,\tau,\mathcal{G})$ containing x such that ${}_{\mathcal{G}_\omega^\alpha}Cl(B)\subseteq A$.

Proof: Suppose that $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha r-$regular space. Let x be any point in X and A be any open set in (X,τ) containing x. Since $X-A$ is a closed set in (X,τ) and $x\notin(X-A)$. By hypothesis, there are two $\mathcal{G}_\omega^\alpha-$open sets G and B in $(X,\tau,\mathcal{G})$ such that $(X-A)\subseteq G$, $x\in B$ and $B\cap G=\emptyset$. Now $x\in B\in\mathcal{G}_\omega^\alpha O(X)$ in the grill topological space $(X,\tau,\mathcal{G})$ containing x. Then $B\subseteq(X-G)$, that is

$${}_{\mathcal{G}_\omega^\alpha}Cl(B)\subseteq {}_{\mathcal{G}_\omega^\alpha}Cl(X-G)\subseteq(X-G)\subseteq A.$$

Conversely, let x be any point in X and F be any closed set in (X,τ) non containing x. Then $x\in(X-F)$ and $(X-F)$ is an open set in (X,τ) containing x. By the hypothesis, for the open set $(X-F)$ there is a $\mathcal{G}_\omega^\alpha-$open set B in $(X,\tau,\mathcal{G})$ containing x such that ${}_{\mathcal{G}_\omega^\alpha}Cl(B)\subseteq(X-F)$. Then $F\subseteq X-{}_{\mathcal{G}_\omega^\alpha}Cl(B)$ and $X-{}_{\mathcal{G}_\omega^\alpha}Cl(B)$ is a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$. Since B is a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$ containing x, we have $B\cap[X-{}_{\mathcal{G}_\omega^\alpha}Cl(B)]=\emptyset$. Then $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$regular space. □

Theorem 40. Every $\mathcal{G}_\omega^\alpha T_3-$ space is a $\mathcal{G}_\omega^\alpha T_2-$ space.

Proof: Let $(X,\tau,\mathcal{G})$ be a $\mathcal{G}_\omega^\alpha T_3$ space and $x\neq y\in X$ be any points in X. Since (X,τ) is a T_1-space, by Theorem (12), $\{x\}$ is a closed set in (X,τ) and $y\notin\{x\}$. Since $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha r-$regular space, there are two $\mathcal{G}_\omega^\alpha-$open sets G and U in $(X,\tau,\mathcal{G})$ such that $x\in\{x\}\subseteq G$, $y\in U$ and $U\cap G=\emptyset$. Hence $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_2$ space. □

Theorem 41. Every $\mathcal{G}_\omega^\alpha T_4-$ space is a $\mathcal{G}_\omega^\alpha T_3$ space.

Proof: Let $(X,\tau,\mathcal{G})$ be a $\mathcal{G}_\omega^\alpha T_4$ space. Let F be any closed set in (X,τ) and $x\notin F$ be any point in X. Since (X,τ) is a T_1-space, then by Theorem (12), $\{x\}$ is a closed set in (X,τ) and $F\cap\{x\}=\emptyset$. Since $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$normal space, there are two $\mathcal{G}_\omega^\alpha-$open sets G and U in $(X,\tau,\mathcal{G})$ such that $x\in\{x\}\subseteq G$, $F\subseteq U$ and $U\cap G=\emptyset$. Hence $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_3$ space. □

Theorem 42. A grill topological space $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$normal space if and only if for each the closed set F in (X,τ) and for each the open set G in (X,τ) containing F, there is a $\mathcal{G}_\omega^\alpha-$open set H in $(X,\tau,\mathcal{G})$ containing F such that ${}_{\mathcal{G}_\omega^\alpha}Cl(H)\subseteq G$.

Proof: Suppose that $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$normal space. Let F be any closed set in (X,τ) and G be any open set in (X,τ) containing F. Then $X-G$ is a closed set in (X,τ) and $F\cap(X-G)=\emptyset$. Since $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$normal space, there are two $\mathcal{G}_\omega^\alpha-$open sets H and U in $(X,\tau,\mathcal{G})$ such that $(X-G)\subseteq U$, $F\subseteq H$ and $U\cap H=\emptyset$. Take $V=H$ is a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$ containing F. Then $V=H\subseteq(X-U)$, this implies,

$${}_{\mathcal{G}_\omega^\alpha}Cl(V)\subseteq {}_{\mathcal{G}_\omega^\alpha}Cl(X-U)\subseteq(X-U)\subseteq G.$$

Conversely, let F and H be any two closed sets in (X,τ) such that $F\cap H=\emptyset$. Then $H\subseteq(X-F)$ and $X-F$ is an open set in (X,τ) containing closed set H. By the hypothesis, there is a $\mathcal{G}_\omega^\alpha-$open set V in (X,τ) containing H such that ${}_{\mathcal{G}_\omega^\alpha}Cl(V)\subseteq(X-F)$. Then $F\subseteq X-{}_{\mathcal{G}_\omega^\alpha}Cl(V)$ and $X-{}_{\mathcal{G}_\omega^\alpha}Cl(V)$ is a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$. Since V is a $\mathcal{G}_\omega^\alpha-$open set in $(X,\tau,\mathcal{G})$ containing H and $V\cap(X-{}_{\mathcal{G}_\omega^\alpha}Cl(V))=\emptyset$, we have $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$normal space. □

Theorem 43. Let $f:(X,\tau,\mathcal{G})\to(Y,\sigma)$ be $\mathcal{G}_\omega^\alpha-$continuous injective function. If (Y,σ) is a regular space and f is a $\mathcal{G}_\omega^\alpha-$closed function , then the grill topological space $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$regular space.

Proof: Suppose that (Y,σ) is a regular space, f is a $\mathcal{G}_\omega^\alpha-$closed function, $x\in X$ and A is any open set in (X,τ) containing x. Then $X-A$ is a closed set in (X,τ) and $x\notin(X-A)$. Take $F=(X-A)$. By hypothesis, $f(F)$ is a closed set in (Y,σ) not containing $f(x)$, and there are two open sets H and B in (Y,σ) such that $f(F)\subseteq H$, $f(x)\in B$ and $H\cap B=\emptyset$. Now since f is $\mathcal{G}_\omega^\alpha-$continuous injective, we get: $f^{-1}(H)$ and $f^{-1}(B)$ are $\mathcal{G}_\omega^\alpha-$open sets in $(X,\tau,\mathcal{G})$. Also

$$F\subseteq f^{-1}(H), x\in f^{-1}(B)$$

and

$$f^{-1}(H)\cap f^{-1}(B)=f^{-1}(H\cap B)=f^{-1}(\emptyset)=\emptyset.$$

Hence by Definition (17), $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$regular space. □

Theorem 44. Let $f:(X,\tau,\mathcal{G})\to(Y,\sigma)$ be $\mathcal{G}_\omega^\alpha-$continuous injective function. If (Y,σ) is a regular space and f is a $\mathcal{G}_\omega^\alpha-$open function , then the grill topological space $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$regular space.

Proof: The proof is similar to that of the above theorem. □

Theorem 45. Let $f:(X,\tau,\mathcal{G})\to(Y,\sigma)$ be $\mathcal{G}_\omega^\alpha-$continuous injective function from the grill topological space $(X,\tau,\mathcal{G})$ to a regular space (Y,σ). If f is a $\mathcal{G}_\omega^\alpha-$closed function and (X,τ) is a T_1-space, then $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_3-$space.

Proof: Since (Y,σ) is a regular space and f is a $\mathcal{G}_\omega^\alpha-$closed function, we have $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha-$regular space by Theorem (43). Since (X,τ) is a T_1-space, we get $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_3-$space, by Definition (17). □

Theorem 46. Let $f:(X,\tau,\mathcal{G})\to(Y,\sigma)$ be $\mathcal{G}_\omega^\alpha-$continuous injective function from the grill topological space $(X,\tau,\mathcal{G})$ to a normal space (Y,σ). If f is a $\mathcal{G}_\omega^\alpha-$closed and (X,τ) is a T_1-space, then $(X,\tau,\mathcal{G})$ is a $\mathcal{G}_\omega^\alpha T_4-$space.

Proof: The proof is similar to that of Theorem (45). □

5 CONCLUSIONS

From this work, we have the following conclusions:

• On openness properties.
1. For a grill topological space $(X,\tau,\mathcal{G})$ the concept of openness of open $\mathcal{G}_\omega^\alpha-$ set is a strong form of the concept of openness of open $\mathcal{G}_\omega^\alpha-$set, but it is an independent form of openness of a topology $\tau_\mathcal{G}$.
2. The concept of openness of open $\mathcal{G}_\omega^\alpha-$ set is week form of the concept of openness of open set in (X,τ).
• On $\mathcal{G}_\omega^\alpha-$ space induced property.
1. The set of all open $\mathcal{G}_\omega^\alpha-$ set $O\mathcal{G}_\omega^\alpha(X)$ is form topology on a set X.
2. The concept of $\mathcal{G}_\omega^\alpha T-$space is strong form of $\mathcal{G}^\omega-$induced space.
• On separation axioms properties.
1. The concept of $\mathcal{G}_\omega^\alpha T_i-$space, is strong form of the concept of $\mathcal{G}_\omega^\alpha T_{i+1}-$space, $i=1,2,3$.
2. The concept of $\mathcal{G}_\omega^\alpha T_i-$space, is week form of the concept of T_i-space, $i=1,2,3,4$.

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Cones generated by a generalized fractional maximal function

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Abstract: The paper considers the space of generalized fractional-maximal function, constructed on the basis of a rearrangement-invariant space. Two types of cones generated by a nonincreasing rearrangement of a generalized fractional-maximal function and equipped with positive homogeneous functionals are constructed. The question of embedding the space of generalized fractional-maximal function in a rearrangement-invariant space is investigated. This question reduces to the embedding of the considered cone in the corresponding rearrangement-invariant spaces. In addition, conditions for covering a cone generated by generalized fractional-maximal function by the cone generated by generalized Riesz potentials are given. Cones from non-increasing rearrangements of generalized potentials were previously considered in the works of M. Goldman, E. Bakhtygareeva, G. Karshygina and others.

Keywords: covering of cones, cones generated by generalized fractional-maximal function, non-increasing rearrangements of functions, rearrangement-invariant spaces.

1 Introduction

In this work introduced two types of cones of non-negative monotonically non-increasing functions on the positive semiaxis generated by generalized fractional maximal functions and equipped with corresponding positively homogeneous functionals. We give the conditions on the function Φ, under which there are pointwise mutual covering of these cones.

In the work of Hakim D.I., Nakai E., Savano Y. [1], Mustafaev R., Bilgicli N. [2], Kuchukaslan A. [3], Gogatishvili A. [4] a generalized fractional-maximal functions of another type were defined, a particular case of which is the classical fractional-maximal function.

It is known that the maximal function is a very important operator in the theory of functions. With their help, many important issues of the theory of function and harmonic analysis are solved. The generalized fractional-maximal functions are also closely related to the generalized Riesz potentials, considered in the works of Goldman M.L. [5-7] (see also [8-10]).

The study of various properties of operators using a generalized fractional-maximal function is sometimes easier than the study of such operators using a generalized potential.

In this paper, we aim to determine the cones of non-negative measurable functions generated by a generalized fractional-maximal function and to investigate the properties of such cones.

2 Definitions, notation and auxiliary statements

Let (S, Σ, μ) be space with a measure. Here is Σ is σ-algebra of subsets of the set S, on which is determined a non-negative σ- finite, $\sigma-$ additive measure μ. By $L_0 = L_0(S, \Sigma, \mu)$ denotes the set of μ-measurable real-valued functions $f : S \to R$, and by L_0^+ a subset of the set L_0 consisting of non-negative functions:

$$L_0^+ = \{f \in L_0 : f \geq 0\}.$$

By $L_0^+(0, \infty; \downarrow)$ we denote the set of all non-increasing functions belonging to L_0^+.

Definition 1. *[11] A mapping $\rho : L_0^+ \to [0, \infty]$ is called a functional norm (short: FN), if the next conditions are met for all $f, g, f_n \in L_0^+$, $n \in N$:*

$(P1)$ $\rho(f) = 0 \Rightarrow f = 0$, $\mu-$ *almost everywhere (briefly: $\mu-$ a.e.);*
$\rho(\alpha f) = \alpha\rho(f), \alpha \geq 0; \rho(f + g) \leq \rho(f) + \rho(g)$ *(properties of the norm);*
$(P2)$ $f \leq g$, *($\mu-$ a.e.)* $\Rightarrow \rho(f) \leq \rho(g)$ *(monotony of the norm);*
$(P3)$ $f_n \uparrow f \Rightarrow \rho(f_n) \to \rho(f)(n \to \infty)$ *(the Fatou property);*
$(P4)$ $0 < \mu(\sigma) < \infty \Rightarrow \int_\sigma f d\mu \leq c_\sigma \rho(f), f \in L_0^+$. *(Local integrability);*
$(P5)$ $0 < \mu(\sigma) < \infty \Rightarrow \rho(\chi_\sigma) < \infty$ *(finiteness of the FN for characteristic functions (χ_σ) of sets of finite measure).*
Here $f_n \uparrow f$ means that $f_n \leq f_{n+1}$, $\lim\limits_{n\to\infty} f_n = f$ ($\mu-$ a.e.)

Definition 2. *Let ρ be a functional norm. The set of functions $X = X(\rho)$ from L_0, for which $\rho(|f|) < \infty$ is called a Banach function space (briefly: BFS), generated by the FN ρ. For $f \in X$ we assume*

$$\|f\|_X = \rho(|f|).$$

Let $L_0 = L_0(R^n)$ be the set of all Lebesgue measurable functions $f : R^n \to C$; $\dot{L}_0 = \dot{L}_0(R^n)$ be the set of functions $f \in L_0$, for which the non-increasing rearrangement of the f^* is not identical to infinity. Non-increasing rearrangement f^* is defined by the equality:

$$f^*(t) = inf\{y \in [0;\infty) : \ \ \lambda_f(y) \le t\}, \ \ t \in R_+ = (0;\infty),$$

where

$$\lambda_f(y) = \mu_n \left\{x \in R^n : \ \ |f(x)| > y\right\}, \quad y \in [0,\infty)$$

is the Lebesgue distribution function. It is known that f^* is a non-negative, non-increasing and right-continuous function on $\mathbb{R}_+$; f^* is equimeasurable with $|f|$, i.e.

$$\mu_1 \left\{t \in \mathbb{R}_+ : \ \ f^*(t) > y\right\} = \mu_n \left\{x \in \mathbb{R}^n : \ \ |f(x)| > y\right\},$$

here μ is the Lebesgue measure (on $\mathbb{R}^n$ or on $\mathbb{R}_+$, respectively, see [1]).

Let $f^{\#} : \mathbb{R}^n \to \mathbb{R}^n$ denote a symmetric rearrangement of f, i.e. a radially symmetric non-negative non-increasing right continuous function (as a function of $r = |x|$, $x \in \mathbb{R}^n$) that is equimeasurable with f. That is

$$f^{\#}(r) = f^*(v_n r^n); \ f^*(t) = f^{\#}\left((\frac{t}{v_n})^{\frac{1}{n}} \right), \ \ r, t \in \mathbb{R}_+,$$

here v_n is the volume of the n-dimensional unit ball.

The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau; \ \ t \in R_+.$$

It is clear that f^{**} is an a non-increasing function on R_+.

Really, let $t_1 \le t_2$, then

$$f^{**}(t_2) = \frac{1}{t_2}\int_0^{t_2} f^*(\tau)d\tau = \frac{1}{t_2}\int_0^{t_1} f^*(\tau)d\tau + \frac{1}{t_2}\int_{t_1}^{t_2} f^*(\tau)d\tau \le \frac{1}{t_2}\int_0^{t_1} f^*(\tau)d\tau + f^*(t_1)\cdot\frac{t_2 - t_1}{t_2}.$$

Hence, we have

$$f^{**}(t_2) \le \frac{1}{t_2}\int_0^{t_1} f^*(\tau)d\tau + \frac{t_2 - t_1}{t_2 t_1}\int_0^{t_1} f^*(\tau)d\tau \le \left(\frac{1}{t_2} + \frac{t_2 - t_1}{t_2 t_1}\right)\int_0^{t_1} f^*(\tau)d\tau = \frac{1}{t_1}\int_0^{t_1} f^*(\tau)d\tau = f^{**}(t_1)$$

Definition 3. *A functional norm ρ is said to be rearrangement-invariant if*

$$f^* \le g^* \Rightarrow \rho(f) \le \rho(g).$$

Banach function space $X = X(\rho)$, generated by a rearrangement invariant functional norm ρ will be called a rearrangement invariant space (in short: RIS).

Example 1. *Let $S = R^n$, $\mu \equiv \mu_n$ be the Lebesgue measure in R^n, $1 \le p \le \infty$; $u \in L_0(R^n)$, $0 < u < \infty$, (μ-a.e.); $u \in L_p^{loc}(R^n)$, $\frac{1}{u} \in L_{p'}^{loc}(R^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$.*

The space $X = L_{p,u}(R^n)$ with a norm $f_X = f_{L_{p,u}}$ i.e.:

$$\|f\|_X = \left(\int_{R^n} |fu|^p d\mu \right)^{\frac{1}{p}}, \ 1 \le p < \infty; \ \|f\|_X = \|fu\|_{L_\infty}, \ p = \infty$$

is a BFS. Associated space:

$$X^{'} = L_{p',\frac{1}{u}}(R^n).$$

Everywhere in this work, we denote rearrangement invariant space (in short: RIS) by $E = E(R^n)$, and by $E^{'} = E^{'}(R^n)$ the associated rearrangement-invariant space and $\tilde{E} = \tilde{E}(R_+)$, $\tilde{E}^{'} = \tilde{E}^{'}(R_+)$ their Luxembourg representation, i.e. such RIS that

$$\|f\|_E = \|f^*\|_{\tilde{E}}, \ \ \|g\|_{E'} = \|g^*\|_{\tilde{E}'}, \tag{1}$$

Let Ω_0 be a set of all nonnegative, finite on R_+, decreasing and right continuous functions:

$$\Omega_0 = \{g : R_+ \to [0;\infty); \ \ g\downarrow, \ \ g(t+0) = g(t), \ \ t \in R_+\}.$$

Definition 4. *A function $f : R_+ \to R_+$ is called quasi-decreasing and is denoted by $f \downarrow$ (quasi-increasing and is denoted by $f \uparrow$) if there exists $C > 1$, such that*

$$f(t_2) < Cf(t_1) \;\; if t_1 < t_2.$$

$$(f(t_1) < Cf(t_2) \;\; if t_1 < t_2)$$

Throughout this work we will denote by C, C_1, C_2 positive constants, generally speaking, different in different places.
By the notation $f(x) \cong g(x)$ we mean that there are constants $C_1 > 0, C_2 > 0$ such that

$$C_1 f(t) \leq g(t) \leq C_2 f(t), \;\; t \in \mathbb{R}_+.$$

Definition 5. *Let $n \in \mathbb{N}$ and $R \in (0; \infty]$. We say that a function $\Phi : (0; R) \to R_+$ belongs to the class $A_n(R)$ if:*
(1) Φ is non-increasing and continuous on $(0; R)$;
(2) the function $\Phi(r)r^n$ is quasi-increasing on $(0, R)$.
For example, $\Phi(t) = t^{-\alpha} \in A_n(\infty), \;\; 0 < \alpha < n$.

Definition 6. *[12] Let $n \in \mathbb{N}$ and $R \in (0; \infty]$. A function $\Phi : (0; R) \to R_+$ belongs to the class $B_n(R)$ if the following conditions hold:*
(1) Φ is non-increasing and continuous on $(0; R)$;
(2) there exists $C > 0$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1} d\rho \leq C\Phi(r)r^n, \;\; r \in (0, R). \tag{2}$$

For example,

$$\Phi(\rho) = \rho^{\alpha - n} \in B_n(\infty) \; (0 < \alpha < n); \quad \Phi(\rho) = ln\frac{eR}{\rho} \in B_n(R), R \in R_+.$$

For $\Phi \in B_n(R)$ the following estimate also holds

$$\int_0^r \Phi(\rho)\rho^{n-1} d\rho \geq n^{-1}\Phi(r)r^n, \; r \in (0, R).$$

Therefore

$$\int_0^r \Phi(\rho)\rho^{n-1} d\rho \cong \Phi(r)r^n, \; r \in (0, R), \tag{3}$$

$$\Phi \in B_n(R) \Rightarrow \{0 \leq \Phi \downarrow; \; \Phi(r)r^n. \uparrow, \; r \in (0, R)\}. \tag{4}$$

Definition 7. *Let $\Phi \in A_n(\infty)$. The generalized fractional-maximal function $M_\Phi f$ is defined for the function $f \in L^1_{loc}(\mathbb{R}^n)$ by*

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int_{B(x,r)} |f(y)| dy,$$

where $B(x, r)$ is a ball with the center at the point x and radius r. That is, consider the operator M_Φ: $L^1_{loc}(R^n) \to \dot{L}_0(R^n)$.

In the case $\Phi(r) = r^{\alpha - n}$, $\alpha \in (0; n)$ we obtain the classical fractional maximal function $M_\alpha f$:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)| dy.$$

We denote by $M^\Phi_E = M^\Phi_E(R^n)$ the set of the functions u, for which there is a function $f \in E(R^n)$ such that

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M^\Phi_E} = inf\{\|f\|_E : f \in E(R^n), \;\; M_\Phi f = u\} \tag{5}$$

such a space M^Φ_E will be called an space of generalized fractional-maximal function.

Note that in the works of Goldman M.L., Bakhtigareeva E.G [4-5], the generalized Riesz potential was considered using the convolution operator:

$$A : E_1(R^n) \to \dot{L}_0(R^n),$$

$$Af(x) = (G * f)(x) = 2\pi^{-n/2} \int\limits_{R^n} G(x-y)f(y)dy,$$

where the kernel $G(x)$ satisfies the conditions:

$$G(x) \cong \Phi(|x|), \;\; x \in R^n \tag{6}$$

$$\Phi \in B_n(\infty); \;\; \exists c \in R_+.$$

The kernel of the classical Riesz potential has the form

$$G(x) = |x|^{\alpha-n}, \;\; \alpha \in (0; n).$$

Note that, unlike the operator A the operator M_Φ is not linear.

Definition 8. *Define $\Im_T = \{K(T)\}$ for $T \in (0,\infty]$ as a set of cones considering from measurable non-negative functions on $(0,T)$, equipped with positive homogeneous functionals $\rho_{KM(T)} : K(T) \to [0,\infty)$ with properties:*
(1) $h \in K(T), \;\; \alpha \geq 0 \Rightarrow \alpha h \in K(T), \quad \rho_{K(T)}(\alpha h) = \alpha \rho_{K(T)}(h);$
(2) $\rho_{K(T)}(h) = 0 \Rightarrow h = 0$ *almost everywhere on* $(0,T)$.

Definition 9. *[5] Let $K(T), M(T) \in \Im_T$. The cone $K(T)$ covers the cone $M(T)$ (notation: $M(T) \prec K(T)$) if there exist $C_0 = C_0(T) \in R_+$, and $C_1 = C_1(T) \in [0,\infty)$ with $C_1(\infty) = 0$ such that for each $h_1 \in M(T)$ there is $h_2 \in K(T)$ satisfying*

$$\rho_{K(T)}(h_2) \leq C_0 \rho_{M(T)}(h_1), \;\; h_1(t) \leq h_2(t) + C_1 \rho_{M(T)}(h_1), \;\; t \in (0,T).$$

The equivalence of the cones means mutual covering:

$$M(T) \approx K(T) \Leftrightarrow M(T) \prec K(T) \prec M(T).$$

Let E is rearrangement-invariant space (briefly: RIS). We consider the following two cones of decreasing rearrangements of generalized fractional maximal function equipped with homogeneous functionals, respectively:

$$K_1 \equiv KM_E^\Phi := \{h \in L^+(\mathbb{R}_+) : h(t) = u^*(t), \; t \in \mathbb{R}_+, \; u \in M_E^\Phi\},$$

$$\rho_{K_1}(h) = \inf\{\|u\|_{M_E^\Phi} : u \in M_E^\Phi; \; u^*(t) = h(t), \; t \in \mathbb{R}_+\}; \tag{7}$$

$$K_2 \equiv K\widetilde{M}_E^\Phi := \{h : h(t) = u^{**}(t), \; t \in \mathbb{R}_+, \; u \in M_E^\Phi\},$$

$$\rho_{K_2}(h) = \inf\{\|u\|_{M_E^\Phi} : u \in M_E^\Phi; \; u^{**}(t) = h(t), \; t \in \mathbb{R}_+\}. \tag{8}$$

This means that the cones K_1 and K_2 consist of non-increasing rearrangements of generalized fractional maximal functions.

Note that in the works of Goldman M.L. [5], Bokayev N.A., Goldman M.L., Karshygina G.Zh. [9-10] cones generated by generalized potentials are considered. They study the space of potentials $H_E^G \equiv H_E^G(\mathbb{R}^n)$ in n-dimensional Euclidean space:

$$H_E^G(\mathbb{R}^n) = \{u = G * f : f \in E(\mathbb{R}^n)\},$$

where $E(\mathbb{R}^n)$ is an rearrangement invariant space (RIS).

$$\|u\|_{H_E^G} = \inf\{\|f\|_E : f \in E(\mathbb{R}^n); G * f = u\},$$

$$M(T) \equiv KM_E^G(T) = \{h(t) = u^*(t), t \in (0;T), u \in H_E^G\},$$

$$\rho_{M(T)}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^*(t) = h(t), t \in (0;T)\};$$

$$\widetilde{M}(T) \equiv K\widetilde{M}_E^G(T) = \{h(t) = u^{**}(t), t \in (0;T) : u \in H_E^G\},$$

$$\rho_{\widetilde{M}}(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G; u^{**}(t) = h(t)), t \in (0;T)\}.$$

In the following Theorem 1 [13] gives the estimate for a non-increasing rearrangement of a generalized fractional maximal function $(M_\Phi f)$ by non-increasing rearrangement of the function f.

Theorem 1. *Let $\Phi \in A_n(\infty)$. Then there exist a positive constant C, depending from $n \in \mathbb{N}$ such that*

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \ \ t \in (0, \infty),$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$.

In the following theorem we give the compares of the cone generated by a generalized fractional-maximal function and the cone generated by the generalized Riesz potential.

Theorem 2. *Let $\Phi \in B_n(\infty)$ and kernel $G(x)$ satisfies the condition (6). Then cone generated by the generalized potential covers the cone generated by the generalized maximal function, i.e. $KM^\Phi_E \prec KM^G_E$.*

Lemma 1. *The following covering takes place*

$$K_1 \prec K_2.$$

Theorem 3. *Let $\Phi \in B_n(\infty)$. The embedding*

$$M^\Phi_E(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n) \tag{9}$$

is equivalence to the next embedding

$$K_1 M^\Phi_E(\mathbb{R}_+) \mapsto \widetilde{X}(\mathbb{R}_+) \tag{10}$$

3 Conclusion

In this paper, we considered the space of generalized fractional maximal functions and investigated the various cones generated by nonincreasing rearrangement of generalized fractional maximal function. Equivalent descriptions of such cones and conditions for their mutual covering are given. Then these cones are used to construct a criterion for embedding the space of generalized fractional maximal functions in the rearrangement invariant spaces (RIS).

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Locally Co-Coherent Modules

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Abstract:
Locally coherent modules play a central role in algebraic geometry, as they provide a framework for studying the structure of varieties and schemes. Their dual counterparts, locally co-coherent modules, are less well-known, but they are nonetheless important in a variety of mathematical contexts. In the seminal paper [19], the authors introduced the notion of a locally coherent module and studied its properties. In this work, we introduce a dual notion, which we call a locally co-coherent module, and investigate its fundamental characteristics. Furthermore, we provide a comprehensive and rigorous study of locally co-coherent modules. We begin by introducing the definition and basic properties of these modules. We then examine their relationship to locally coherent modules and other algebraic objects. Finally, we discuss some of the applications of locally co-coherent modules in other areas of mathematics.

Keywords: finitely cogenerated module, finitely copresented module, co-coherent module, and locally co-coherent module.
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1 Introduction

The theoretical foundations and notation employed in the current investigation are informed by seminal contributions delineated in references [1], [2], [3], [4], [5], [6], [7], [8], [9], and [10]. Specifically, these sources provide foundational descriptions and formalizations of the key concepts studied herein, including coherent functors [2], locally coherent modules [3], Cohen-Macaulay modules [4], and the relationship between coherent functors and Gorenstein categories [10]. Moreover, they establish structural typologies, categorical frameworks, and mathematical properties germane to properly contextualizing the study's analytical objectives and modeling approach. By drawing upon these scholarly works, the requisite terminology, structural postulates, and problem conceptualizations are delineated for systematically interrogating the impact of graph operations on relational transformations and attendant complexity shifts. In summation, the cited literature furnishes the theoretical apparatus and notational conventions underscoring the methodology and interpretation of results within the present investigation. Firas and Karim [11] delineate properties of local modules, an important construct in the study of coherent structures. Meanwhile, Nam, Tri, and Dong [12] examine properties of generalized local cohomology modules with respect to ideal pairs, shedding light on their categorical properties and behaviors. By drawing upon the formalizations and examinations of such algebraic notions presented in these sources, the requisite terminology, structural postulates, and analytical objectives employed herein are properly defined and contextualized. This prior work therefore establishes the theoretical foundations and notational conventions underpinning the methodology and interpretation of findings within the present research.

Throughout this paper, R means a ring with an identity element and all modules are unital R-modules. In [19] R-MOD denote a category of unital right R-modules and $\sigma[M]$ is a subcategory of R-MOD and its objects are submodules of M-cogenerated is studied. Similarly to 'finitely presented', 'finitely copresented' also depends on the category referred to ($\sigma[M]$, R-MOD) see [19].

The notion of locally coherent modules was introduced and studied in [19], such that it is defined as the following : Let M be an R-module. A module $N \in \sigma[M]$ is called a coherent module in $\sigma[M]$ if it is finitely generated and every finitely generated submodule of N is a finitely presented in $\sigma[M]$. If all finitely generated submodules of a module $N \in \sigma[M]$ are coherents, then N in $\sigma[M]$ is called a locally coherent module.

In this paper, we introduce and study the dual notion of the locally coherent module which is called a locally co-coherent module in $\sigma[M]$ and is defined as the flowing: Let M be an R-module. A module $N \in \sigma[M]$ is called a co-coherent module if it is finitely cogenerated and every finitely cogenerated factor module of N is finitely copresented in $\sigma[M]$. If all finitely cogenerated factors modules of module $N \in \sigma[M]$ are co-coherents, then N is called a locally co-coherent module in $\sigma[M]$.

In (Lemma 1.) gives characterization of locally co-coherent module $\in \sigma[M]$ such that A module T is called locally co-coherent module in $\sigma[M]$ if and only if it is finitely cogenerated and every finitely cogenerated factor module of T is finitely copresented in $\sigma[M]$).

In (Proposition 1.) explain that every finitely cogenerated submodule of a locally co-coherent module is locally co-coherent in $\sigma[M]$.

In (Theorem 1.) We study some properties and behavior of the notion of locally co-coherent module $\in \sigma[M]$ on short exact sequences such that if R is a ring and let $0 \to X \to K \to L \to 0$ be a short exact sequence of modules, then we have:

- If K is locally co-coherent and L is finitely cogenerated, then X is locally co-coherent in $\sigma[M]$.
- If X and L are locally co-coherents, then K is locally co-coherent in $\sigma[M]$.
- If $K = X \bigoplus L$, then K is locally co-coherent in $\sigma[M]$ if and only if X and L are locally co-coherent in $\sigma[M]$.
- If K is locally co-coherent in $\sigma[M]$ and N, H are finitely cogenerated submodules of K, then $N \bigcap H$ is finitely cogenerated.

See also the duality of this theorem in [17] As a conseqance of (Theorem 1.) we get (Corollary 2.) such that if $N_1, N_2......, N_n$ are submodules of N in $\sigma[M]$, then $\bigoplus_{i=1}^{n} N_i$ is locally co-coherent module if and only if $N_1, N_2......, N_n$ are locally co-coherent submodules of N in $\sigma[M]$.

Also in (Proposition 2.) if $f : L \longrightarrow N$ is a homomorphism between locally co-coherent modules L, N in $\sigma[M]$, then we proved that $Kerf, Imf, andCokef$ are locally co-coherent modules.

In (Proposition 3.) Let X to be locally co-coherent and Y, Z to be finitely cogenerated factor modules of X. If

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ f\Big\downarrow & & \Big\downarrow f' \\ Z & \xrightarrow{g'} & P \end{array}$$

is a pushout diagram, then we proved that P is finitely cogenerated.

In section 5 we study Properties such that if M be an R-module, U finitely copresented module in $\sigma[M]$ and $N \in \sigma[M]$. If every submodule of N is U-cogenerated, then the following is proved:

N is locally co-coherent in $\sigma[M]$ if and only if for every $f \in Hom(N, U^k), k \in \mathbb{N}$, the submodule $Kerf$ is finitely cogenerated (Imf is finitely copresented) if and only if (1) for any $f \in Hom(N, U)$, the submodule $Kerf$ is finitely cogenerated and (2) the intersection of any two finitely cogenerated submodules of N is finitely cogenerated.

In section 6 we study Characterizations of locally co-coherent modules in $R - MOD$ where For an R-module N the following is proved: N is locally co-coherent in $\sigma[M]$ if and only if for every $f \in Hom(N, U^k), k \in \mathbb{N}$, the submodule $Kerf$ is finitely cogenerated (Imf is finitely copresented); if and only if (i) for any $f \in Hom(N, U)$, the submodule $Kerf$ is finitely cogenerated, and (ii) the intersection of any two finitely cogenerated submodules of N is finitely cogenerated.

Recall some important definitions which are basic in this work. An R-module M is called finitely generated, if for any family $(M_i)_{i\in I}$ of submodules of M with $\sum_{i\in I} M_i = 0$, there is a finite subset J of I such that $\sum_{j\in J} M_j = 0$ (see [13, 18, 19]).

As in the classical case, finitely presented module M is defined as a module that is finitely generated such that, for every short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$, if L is finitely generated, then K is also finitely generated (see [18, 19]).

Dually and similarly, for a ring R, an R-module M is called finitely cogenerated if for every family $\{M_i\}_{i\in I}$ of submodules of M with $\bigcap_{i\in I} M_i = 0$, there is a finite subset $J \subset I$ such that $\bigcap_{i\in J} N_i = 0$.

A module M is said to be finitely copresented if it is finitely cogenerated and for every short exact sequence $0 \longrightarrow M \longrightarrow L \longrightarrow K \longrightarrow 0$, with L is finitely cogenerated, then also K is finitely cogenerated (see [19], pages 248-249).

2 Locally co-coherent modules

Definition 1. *Let M be an R-module. A module $N \in \sigma[M]$ is called co-coherent module if it is finitely cogenerated and every finitely cogenerated factor module of N is finitely copresented in $\sigma[M]$. If all finitely cogenerated factors modules of the module $N \in \sigma[M]$ are co-coherents, then N in $\sigma[M]$ is called a locally co-coherent module.*

The following result gives a characterization of a locally co-coherent modules.

Lemma 1. *Let M be an R-module. A module $T \in \sigma[M]$ then the following tow conditions are equivalents:*

1. *T is locally co-coherent module in $\sigma[M]$.*
2. *T is finitely cogenerated module in $\sigma[M]$ and every finitely cogenerated factor module of T is finitely copresented in $\sigma[M]$ that is meaning For every short exact sequence $0 \longrightarrow N = T/H \longrightarrow L \longrightarrow K \longrightarrow 0$, in $\sigma[M]$ with L is finitely cogenerated, then K is finitely cogenerated.*

Proof: $(1) \Rightarrow (2)$ Suppose that T is locally co-coherent, then it is finitely cogenerated and every factor module of T is also finitely cogenerated and there exists an exact sequence $0 \longrightarrow N = T/H \longrightarrow L \longrightarrow K \longrightarrow 0$ with, L is finitely cogenerated, then K is finitely cogenerated in $\sigma[M]$) where H is a submodule of T.
$(2) \Rightarrow (1)$ is seen dually to the proof of 25.1 in[19]: Suppose that T is finitely cogenerated module in $\sigma[M]$ and let $0 \longrightarrow N \longrightarrow B \longrightarrow c \longrightarrow 0$ be a short exact sequence with B is finitely cogenerated we obtain with a pushout the commutative exact diagram :

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & N = T/H & \longrightarrow & L & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & D & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & C & = & C & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

If B is finitely cogenerated, then D and C are finitely cogenerated, then $N = T/H$ is finitely copresented in $\sigma[M]$ and T is a locally co-coherent module in $\sigma[M]$. □

Proposition 1. *Every finitely cogenerated submodule of a locally co-coherent module is locally co-coherent in $\sigma[M]$*

Proof: : Let N be a locally co-coherent module, then it is finitely cogenerated, and let L be a submodule of N so L is finitely cogenerated and N/L is finitely copresented (because N locally co-coherent module in $\sigma[M]$) and therefore it is finitely cogenerated. let K be a submodule of L

so $L \diagup K$ is finitely cogenerated and it is a submodule of $N \diagup L$ so $L \diagup K$ is finitely copresented module and hence L is locally co-coherent module. □

Now we study in the following theorem some properties and behavior of a locally co-coherent module in $\sigma[M]$ on a short exact sequence; see more [16, 17, 19]

Theorem 1. *Let R be a ring and let M be a R-module. Let $0 \longrightarrow X \longrightarrow K \longrightarrow L \longrightarrow 0$ be a short exact sequence of modules in $\sigma[M]$. Then we have the following.*

1. *If K is locally co-coherent and L is finitely cogenerated then X is locally co-coherent in $\sigma[M]$.*
2. *If X and L are locally co-coherent if and only if K is locally co-coherent in $\sigma[M]$.*
3. *If $K = X \bigoplus L$, then K is locally co-coherent in $\sigma[M]$ if and only if X and L are locally co-coherent in $\sigma[M]$.*
4. *If K is locally co-coherent in $\sigma[M]$ and N, H are finitely cogenerated submodules of K, then $N \bigcap H$ is finitely cogenerated.*

Proof: :(1) Suppose that K is locally co-coherent and L is finitely co-generated in $\sigma[M]$. Let $X \longrightarrow Y$ be (epic, i.e., homomorphism surjective) and Y be finitely cogenerated. Forming a pushout, we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & X & \rightarrow & K & & L & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \rightarrow & D & \rightarrow & L & \longrightarrow & 0 \end{array}$$

we have Y and L are finitely cogenerated, then D is finitely cogenerated and – by assumption finitely copresented and from $30.2, (1)$ in [19] Y is also finitely copresented, hence X is locally co-coherent module in $\sigma[M]$.

(2) Suppose that X and L are locally co-coherents in $\sigma[M]$. Let $K \longrightarrow Z$ be epic and Z finitely cogenerated. By forming a pushout, we get the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & X & \rightarrow & K & \rightarrow & L & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & Y & \rightarrow & Z & \rightarrow & H & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Here K is finitely copresented and H is finitely cogenerated, hence finitely copresented and Z is also finitely copresented, K is locally co-coherent module in $\sigma[M]$.

(3) This follows immediately from (2) in 1 and From 1.

(4) Under the given assumptions, $K = N + L$ is locally co-coherent in $\sigma[M]$, then K/N and K/L are co-coherents and from 2 $K/N \oplus K/L$, then there is an exact sequence $0 \longrightarrow K/N \oplus K/L \longrightarrow K \longrightarrow N \cap L \longrightarrow 0$ and hence $N \cap L$ has to be finitely cogenerated in $\sigma[M]$. □

Corollary 1. *Let M be an R-module. A module $N \in \sigma[M]$ and Let $N_1, N_2 ......, N_n$ are submodules of N in $\sigma[M]$, then $\bigoplus_{i=1}^{n} N_i$ is locally co-coherent module if and only if $N_1, N_2 ......, N_n$ are locally co-coherent submodules of N in $\sigma[M]$.*

Proof: Let $N_1, N_2 ......, N_n$ be locally co-coherent submodules of N in $\sigma[M]$. We have a short exact sequence

$$0 \rightarrow N_n \rightarrow \bigoplus_{i=1}^{n} N_i \longrightarrow \bigoplus_{i=1}^{n-1} N_i \longrightarrow 0$$

and by induction if $n = 2$, then we get

$$0 \rightarrow N_2 \rightarrow \bigoplus_{i=1}^{2} N_i \longrightarrow N_1 \longrightarrow 0$$

from 1 (4) the asseration is true. Now we suppose that $N_1, N_2 ......, N_n$ are locally co-coherents if and only if $\bigoplus_{i=1}^{n} N_i$ is locally co-coherent module and we prove it when n+1. The short exact

$$0 \rightarrow N_{n+1} \rightarrow \bigoplus_{i=1}^{n+1} N_i \longrightarrow N_1 \longrightarrow 0$$

and from 1 (2) implies that M_{n+1} is locally co-coherent module (because N_1 is locally co-coherent). We have also

$$0 \rightarrow N_{n+1} \rightarrow \bigoplus_{i=1}^{n+1} N_i \longrightarrow \bigoplus_{i=1}^{n} N_i \longrightarrow 0$$

, then from 1 (3) $\bigoplus_{i=1}^{n+1} N_i$ is locally co-coherent module and it follows that $N_1, N_2 ......, N_n$ are locally co-coherents if and only if $\bigoplus_{i=1}^{n} N_i$ is locally co-coherent module for every n in $\sigma[M]$. □

Corollary 2. *Let M be an R-module. A module $N \in \sigma[M]$ and $N_1, N_2 ......, N_n$ are modules. If $N_1, N_2 ......, N_n$ are locally co-coherent modules in $\sigma[M]$, then $\bigcap_{i=1}^{n} N_i$ is locally co-coherent modules in $\sigma[M]$.*

Proof: : Where $\bigcap_{i=1}^{n} N_i$ is a submodule of N_i for $i = 1, 2, ..., n$ which are locally co-coherent modules in $\sigma[M]$, then by 1 $\bigcap_{i=1}^{n} N_i$ is locally co-coherent modules in $\sigma[M]$. □

Proposition 2. *If $f : L \longrightarrow N$ is a homormorphism between locally co-coherent modules L, N in $\sigma[M]$, then $Kerf, Imf$ and $Cokef$ are also locally co-coherent modules.*

Proof: : Since f is homomorphism between locally co-coherent modules L, N implies that $kerf$ is a submodule of L and Imf also a submodule of N, then, $Kerf$ and Imf are finitely cogenerated modules in $\sigma[M]$ and by 1 implies that $Kerf$ and Imf are locally co-coherent modules in $\sigma[M]$. And we know that $cokerf = N/Imf$ so there is a short exact $0 \longrightarrow Imf \longrightarrow N \longrightarrow cokerf = N/Imf \longrightarrow 0$, then $cokerf$ is finitely cogenerated module and it is finitey copresented, hence by 1 $cokerf$ is locally co-coherent modules in $\sigma[M]$. □

Proposition 3. *Assume X to be locally co-coherent and Y, Z are finitely co-generated factor modules of X. If*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & P \end{array}$$

is a pushout diagram, then P is finitely cogenerated.

Proof: : The given diagram can be extended to the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & K & \to & X & \to & Y & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L & \to & Z & \to & P & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

and X is locally co-coherent and Y is finitely cogenerated, then K is locally co-coherent by (1) from 1, and hence L is finitely copresented. Therefore, P is finitely cogenerated. □

2.1 Properties of locally co-coherent M in $\sigma[M]$

Theorem 2. *Assume the R-module M to be locally co-coherent in $\sigma[M]$. Then*

1. *Every module in $\sigma[M]$ is finitely cogenerated by co-coherent modules.*
2. *Every finitely copresented module is co-coherent in $\sigma[M]$.*

Proof: : (1) By 26.1, in [19] as a dially $M^{\mathbb{N}}$ is locally co-coherent and the finitely co-generated submodules form a set of cogenerators of co-coherent modules in $\sigma[M]$.

(2) If N is finitely copresented, then it is finitely cogenerated and by (see [19], pages 248-249) , there is an exact sequence

$$0 \longrightarrow N \longrightarrow \bigoplus_{i \leq k} U_i \longrightarrow K \longrightarrow 0$$

and also the central expression co-coherent and K is finitely cogenerated, then, by 1 1 N it is locally co-coherent and it is co-coherent in $\sigma[M]$. □

2.2 Finitely copresented cogenerators and co-coherent modules

Theorem 3. *Let M be an R-module, U a finitely copresented module in $\sigma[M]$ and $N \in \sigma[M]$. If every submodule of N is U-cogenerated, then the following assertions are equivalent:*

1. *N is locally co-coherent in $\sigma[M]$;*
2. *For every $f \in Hom(N, U^k), k \in \mathbb{N}$, the submodule $Kerf$ is finitely cogenerated (Imf is finitely copresented);*
3. *(a)For any $f \in Hom(N, U)$, the submodule $Kerf$ is finitely cogenerated and*
(b)The intersection of any two finitely cogenerated submodules of N is finitely cogenerated.

Proof: :

- (1) ⇔ (2) Under the given assumptions, for every finitely cogenerated submodule $K \subset U^k$, as a dually of 26.3 there is an epimorphism $f : N \longrightarrow K$, for some $k \in \mathbb{N}$.
- (1) ⇒ (3) follows from 1,(4) and 2.
- (3) ⇒ (2) We prove this by induction on $k \in \mathbb{N}$. The case $k = 1$ is given by (a).

Assume that, for $k \in \mathbb{N}$, all homomorphic images of N in U^k are finitely copresented, and consider $g \in Hom(N, U^k)$. In the exact sequence

$$0 \longrightarrow g(N) \longrightarrow g(N) \bigoplus g(L) \longrightarrow g(L) \cap g(N) \longrightarrow 0$$

(where L is a submodule of N) the central expression is finitely copresented by assumption and $g(N) \cap g(L)$ is finitely cogenerated because of (b) hence Imf is finitely copresented and $Kerf$ is finitely cogenerated.

□

2.3 *Characterizations of locally co-coherent modules in* $R - MOD$

Theorem 4. *For an* R*-module* N *the following assertions are equivalent:*

1. N *is locally co-coherent in R-MOD;*
2. *For every* $f \in Hom(N, R^k), k \in \mathbb{N}$*, the submodule* $Kerf$ *is finitely cogenerated* $(Imf$ *is finitely copresented);*
3. *(a)For any* $f \in Hom(N, U)$*, the submodule* $Kerf$ *is finitely cogenerated and*
(b)The intersection of any two finitely cogenerated submodules of N *is finitely cogenerated.*

Proof: : For locally co-coherence in $R - MOD$ we obtain from the proof of 3. □

Conclusion

This study sought to introduce and explore the concept of locally co-coherent modules, a novel dual notion. Initially, the definition and basic properties of locally co-coherent modules were established. Their relationship to locally coherent modules and other algebraic structures was then investigated. Specifically, connections were drawn between locally co-coherent modules and previously established concepts. A rigorous examination of the fundamental characteristics of locally co-coherent modules was undertaken. In sum, the foundational aspects and theoretical significance of locally co-coherent modules were delineated through a comprehensive theoretical analysis. While requiring further empirical validation, this work provides a conceptual framework for advancing the understanding and utility of this dual algebraic notion.

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Approaching Geometric Problems with the Particle Swarm Optimization Method

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Abstract: In our study, which is based on the lack of applications and problems in the literature, the aim is to develop different methods by approaching 7 different analytical geometry problems using the particle swarm optimization method. Besides its strong theoretical structure, a method has been designed that can be used in many real-life applications. It has been demonstrated that effective solutions can be generated in a short time in architectural, landscaping, cadastral and land-sharing, urban planning, water distribution, and other design-oriented areas through the software.The method used in our study can be transformed into a design that can be used in complex systems and problems containing many variables. Mathematical expressions are then converted into functions to which Particle Swarm Optimization is applied, allowing integration of any problem that can be written as a function of multiple variables. The theoretical solutions have been tested and proven accurate. At the same time, from the generated graphics, it has been demonstrated how important the number of iterations is to approach the correct solution.

Keywords: Area problems, Analytical geometry, Particle Swarm Optimization

1 Introduction

Particle Swarm Optimization (PSO) represents an optimization technique devised by Kennedy and Eberhart (1995), drawing inspiration from the collective movement of fish and insects in swarms. It serves as a fundamentally swarm intelligence-based algorithm, capitalizing on the observation that random movements exhibited by animals within swarms, particularly in contexts involving food and safety, enhance their ability to achieve objectives.

In the context of Particle Swarm Optimization, individual problem-solving entities are referred to as "particles," and collectively, they form the "population." To begin, the swarm members designated to search for the solution and the essential parameters are initially determined. A fitness function is utilized to evaluate the proximity of each particle to the sought-after solution. Subsequently, a change rate function guides each particle's movement towards a closer solution. The process iterates, repeatedly evaluating proximity to the solution with the fitness function, until the desired outcomes are attained. It is widely applied to target tracking, positioning and navigation, mode identification etc. by virtue of its advantages of simple concept, ease in actualization, fewer parameters, and effectiveness in solving complicated optimization and so on.[2] With an increasing number of iterations, the solution set's elements progressively approach the actual values of the solution. Given an infinite number of iterations, the optimization converges towards these ideal values.

2 Materials and Methods

The main methodology in this paper is forming proper equations for each problem by using basic analytic geometry knowledge, as it was used for creating the equation of lines, finding the intersection points of lines, calculating the area of polygons with known coordinates, creating parabolas, and using definite integral for parabolic area calculations. In the software part of our study, the particle swarm optimization method was implemented using Python. The "matplotlib" library was utilized to create necessary graphics, and GeoGebra was used for testing solutions and visualising problems.

2.1 Calculating the Area of a Triangle or Quadrilateral in Analytical Plane

In the analytical plane, the area of the triangle ABC, defined by the coordinates of its edges as $\mathrm{A}\,(x_a, y_a)$, $\mathrm{B}\,(x_b, y_b)$, and $\mathrm{C}\,(x_c, y_c)$, can be calculated as follows:

$$A(ABC) = \frac{1}{2}\left|(x_a y_b + x_b y_c + x_c y_a) - (x_b y_a + x_c y_b + x_a y_c)\right| \quad (1)$$

Similarly, by selecting the edges in a counterclockwise direction, the area of the quadrilateral $ABCD$, defined by the coordinates of its vertices as A (x_a, y_a) , B (x_b, y_b) , C (x_c, y_c), D (x_d, y_d) can be calculated as follows:

$$A(ABCD) = \frac{1}{2} \mid (x_a y_b + x_b y_c + x_c y_d + x_d y_a) - (x_b y_a + x_c y_b + x_d y_c + \ x_a y_d) \mid \quad (2)$$

3 Problems

3.1 Problem 1

For which values of m and n, the division of the ABC triangle into three equal-area parts by the $[BD$ and $[BE$ rays satisfies the equality $S_1 = S_2 = S_3$?

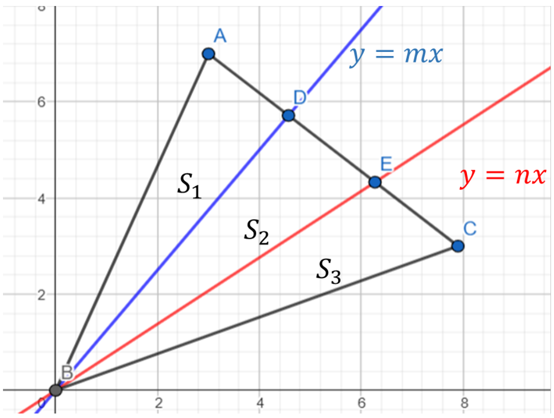

Fig. 1: Problem 1 Model

For the solution, we start by finding the equation of the BC line:

$$y - y_c = \frac{y_a - y_c}{x_a - x_c}(x - x_c) \quad (3)$$

$$y_d - y_c = \frac{y_a - y_c}{x_a - x_c}(x_d - x_c) \quad (4)$$

$$m x_d - y_c = \frac{y_a - y_c}{x_a - x_c}(x_d - x_c) \quad (5)$$

$$(x_a - x_c) m x_d - y_c(x_a - x_c) = (y_a - y_c)x_d - x_c y_a + x_c y_c \quad (6)$$

$$(m x_a - m x_c - y_a + y_c)x_d = y_c x_a - y_c x_c - x_c y_a + x_c y_c \quad (7)$$

$$x_d = \frac{y_c x_a - x_c y_a}{m x_a - m x_c - y_a + y_c} \quad (8) \quad y_e = m\left(\frac{y_c x_a - x_c y_a}{m x_a - m x_c - y_a + y_c}\right) \quad (9)$$

The point E (x_e, y_e) is also on the line BC and on the line $y = nx$, so we can express the coordinates of point E as the intersection of these two lines.

$$x_e = \frac{y_c x_a - x_c y_a}{x_a - n x_c - y_a + y_c} \quad (10) \quad y_e = n\left(\frac{y_c x_a - x_c y_a}{x_a - n x_c - y_a + y_c}\right) \quad (11)$$

We can calculate the area of triangle ABC using the equation numbered as S (1), as shown below.

$$\text{A(ABC)} = \text{S} = \frac{1}{2} |(x_a y_b + x_b y_c + x_c y_a) - (x_b y_a + x_c y_b + x_a y_c)| \quad (12)$$

$$\text{S} = \frac{1}{2} |(x_c y_a) - (x_a y_c)| \quad (13)$$

For the areas of the other three smaller triangles to be equal, satisfying the equation $S_1 = S_2 = S_3$, each of their areas must be $\frac{S}{3}$. We can also calculate the areas of triangles ABD, DBE, and EBC as shown below.

$$\text{A(ABD)} = S_1 = \frac{1}{2}\left|(x_a y_b + x_b y_d + x_d y_a) - (x_b y_a + x_d y_b + x_a y_d)\right| \quad (14)$$

$$\text{A(DBE)} = S_2 = \frac{1}{2}\left|(x_d y_b + x_b y_e + x_e y_d) - (x_b y_d + x_e y_b + x_d y_e)\right| \quad (15)$$

$$\text{A(EBC)} = S_3 = \frac{1}{2}\left|(x_e y_b + x_b y_c + x_c y_e) - (x_b y_e + x_c y_b + x_e y_c)\right| \quad (16)$$

$$S_1 = \frac{1}{2}\left|(x_d y_a) - (x_a y_d)\right| = \frac{S}{3} \quad (17) \qquad S_2 = \frac{1}{2}\left|(x_e y_d) - (x_d y_e)\right| = \frac{S}{3} \quad (18)$$

$$S_3 = \frac{1}{2}\left|(x_c y_e) - (x_e y_c)\right| = \frac{S}{3} \quad (19)$$

After obtaining these equations, we can define the function $F_1(m,n)$.

$$F_1(m,n) = \left|S_1 - \frac{S}{3}\right| + \left|S_2 - \frac{S}{3}\right| + \left|S_3 - \frac{S}{3}\right| \quad (20)$$

Using particle swarm optimization, we can solve for the values of m and n that make our function equal to 0. This way, we can find the values of m and n that satisfy previous equations.

$$F_1(m,n) \longrightarrow 0 \quad (21)$$

When solving this problem with the values $\text{A}(0,4)$ and $\text{C}(4.47,0)$ using our software, we obtained the following graph and solution set below.

For the key (m, n), we obtain the values $(1.789, 0.447)$. We also observe that as the iteration count increases, the solution gets closer.

(Used code for the problem 1 can be found in subsection 4.2)

3.2 Problem 2

For the ABC triangle with the side $|BC|$ lying on the x-axis, which values of m and x_e allow the division of the triangle into four equal-area parts by two perpendicular lines, one with the equation $y = mx$ and the other intersecting the x-axis at point x_e ?

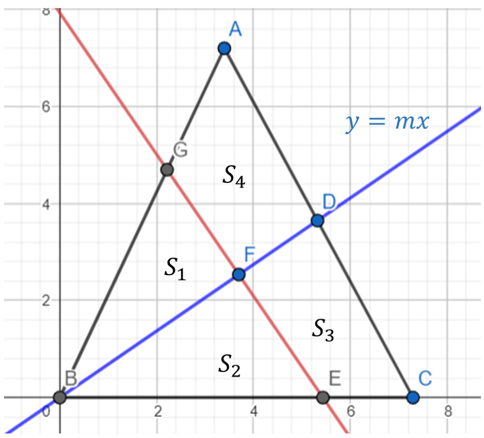

Fig. 2: Problem 2 Model

In this problem, firstly, we should find the slope of the line GE. Since it is perpendicular to the line DB, its slope can be calculated as $\frac{-1}{m}$. Knowing that it passes through point E, we can express the equation of the line as follows:

$$y - y_e = \frac{-(x - x_e)}{m} \quad (22) \quad \text{E}(x_e, 0), \quad y = \frac{x_e - x}{m} \quad (23)$$

To find the coordinates of point $\text{D}(x_d, y_d)$, we should examine the intersection of lines AC and BD. For this purpose, based on equation (4), we can express it as shown below:

$$y_d - y_c = \frac{y_a - y_c}{x_a - x_c}(x_d - x_c) \quad (24)$$

$$x_d = \frac{x_c y_a}{(m x_c + y_a - m x_{ai})} \quad (25) \quad y_d = m\left(\frac{x_c y_a}{(m x_c + y_a - m x_a)}\right) \quad (26)$$

To find the coordinates of point $\text{G}(x_g, y_g)$, we should examine the intersection of lines BA and GE. For this purpose, if we start from equations (4) and (23), we can express it as follows:

$$y_g - y_b = \frac{y_a - y_b}{x_a - x_b}(x_g - x_b) \quad (27) \quad y_g = \frac{x_e - x_g}{m} \quad (28)$$

$$\text{for } B(0,0), \quad y_g = \frac{y_a}{x_a} x_g \quad (29)$$

$$\frac{x_e - x_g}{m} = \frac{y_e}{x_a} x_g \quad (30)$$

$$x_g = \frac{x_k x_a}{(m y_a + x_a)} \quad (31) \quad y_g = \frac{x_e y_a}{(m y_a + x_a)} \quad (32)$$

To be able to calculate the areas that are formed, finally, we can find the coordinates of point F $\left(x_f, y_f\right)$ as the intersection of lines BD and GE.

$$y_f = \frac{x_e - x_f}{m} \quad (33) \quad y_f = m x_f \quad (34)$$

$$m x_f = \frac{x_e - x_f}{m} \quad (35)$$

$$m^2 x_f = x_e - x_f \quad (36)$$

$$x_f = \frac{x_e}{(m^2+1)} \quad (37) \quad y_f = \frac{m x_e}{(m^2+1)} (38)$$

We can calculate the area of triangle ABC as shown below.

$$\mathrm{A(ABC)} = S = \frac{1}{2}\left|(x_a y_b + x_b y_c + x_c y_a) - (x_b y_a + x_c y_b + x_a y_c)\right| \quad (39)$$

$$\mathrm{A(ABC)} = S = \frac{1}{2}\left|x_c y_a\right| \quad (40)$$

The areas of triangles GBF and FBE, with coordinates $\mathrm{B}(0,0)$, $\mathrm{E}(x_e, 0)$, $\mathrm{F}\left(x_f, y_f\right)$, and $\mathrm{G}(x_g, y_g)$, can be calculated as below.

$$\mathrm{A(GBF)} = S_1 = \frac{1}{2}\left|\left(x_g y_b + x_b y_f + x_f y_g\right) - \left(x_b y_g + x_f y_b + x_g y_f\right)\right| \quad (41)$$

$$S_1 = \frac{1}{2}\left|\left(x_f y_g\right) - \left(x_g y_f\right)\right| \quad (42)$$

$$S_1 = \frac{1}{2}\left|\left(\frac{x_e}{(m^2+1)} \frac{x_e y_a}{(m y_e + x_a)}\right) - \left(\frac{x_e x_a}{(m y_a + x_e)} \frac{m x_e}{(m^2+1)}\right)\right| \quad (43)$$

$$\mathrm{A(FBE)} = S_2 = \frac{1}{2}\left|\left(x_f y_b + x_b y_e + x_e y_f\right) - \left(x_b y_f + x_e y_b + x_f y_e\right)\right| \quad (44)$$

$$S_2 = \frac{1}{2}\left|\left(x_e y_f\right)\right| \quad (45)$$

$$S_2 = \frac{1}{2}\left|\left(x_e \frac{m x_e}{(m^2+1)}\right)\right| \quad (46)$$

The areas of quadrilaterals $AGFD$ and $DFEC$, with coordinates $\mathrm{A}(x_a, y_a)$, $\mathrm{C}(x_c, y_c)$, $\mathrm{D}(x_d, y_d)$, $\mathrm{E}(x_e, 0)$, $\mathrm{F}\left(x_f, y_f\right)$, and $\mathrm{G}(x_g, y_g)$, can be calculated using equation (2) as shown below.

$$A(AGFD) = S_3 = \frac{1}{2}\left|\left(x_a y_g + x_g y_f + x_f y_d + x_d y_a\right) - \left(x_g y_a + x_f y_g + x_d y_f + x_a y_d\right)\right| \quad (47)$$

$$A(DFEC) = S_4 = \frac{1}{2}\left|\left(x_d y_f + x_f y_e + x_e y_c + x_c y_d\right) - \left(x_f y_d + x_e y_f + x_c y_e + x_d y_c\right)\right| \quad (48)$$

$$S_3 = \frac{1}{2} \mid x_a \frac{x_e y_a}{m y_a + x_a} + \frac{x_a x_a}{m y_a + x_a} \frac{m x_c}{m^2+1} + \frac{x_e}{m^2+1} \frac{m x_c y_a}{m x_c + y_a - m x_a} + \frac{x_c y_a}{m x_c + y_a - m x_a} y_a) -$$
$$\frac{x_a x_a}{m y_a + x_a} y_a + \frac{x_e}{m^2+1} \frac{x_e y_a}{m y_a + x_a} + \frac{x_c y_a}{m x_c + y_a - m x_a} \frac{m x_c}{m^2+1} + x_a m x_c y_a (m x_c + y_a - m x_a) \mid \quad (49)$$

$$S_4 = \frac{1}{2} \left| \frac{x_c y_a}{m x_c + y_a - m x_a} \frac{m x_c}{m^2+1} + x_c m \frac{x_c y_a}{m x_c + y_a - m x_a}\right) - \frac{x_e}{m^2+1} m \frac{x_c y_a}{m x_c + y_a - m x_a} + x_e \frac{m x_c}{(m}) \right| \quad (50)$$

After obtaining these equations, we can define the function $F_2(m, x_e)$. Since it's desired that all areas are equal, each region's area should be $\frac{S}{4}$. We can define this function as shown below and solve the problem using the particle swarm optimization method to find values where the function approaches 0^+.

$$F_2(m, x_e) = \left|S_1 - \frac{s}{4}\right| + \left|S_2 - \frac{s}{4}\right| + \left|S_3 - \frac{s}{4}\right| + \left|S_4 - \frac{s}{4}\right| \quad (51)$$

$$F_2(m, x_e) \longrightarrow 0 \quad (52)$$

(Used code for the problem 2 can be found in subsection 4.3)

3.3 Problem 3

For the ABC triangle with point B at the origin, which values of m, x_h, and x_i allow the division of the triangle into four equal-area parts by two perpendicular lines: one with slope m and intersecting the x-axis at point x_h, and the other intersecting the x-axis at point x_i ? (The case $0 < x_h < x_c$ will be examined.)

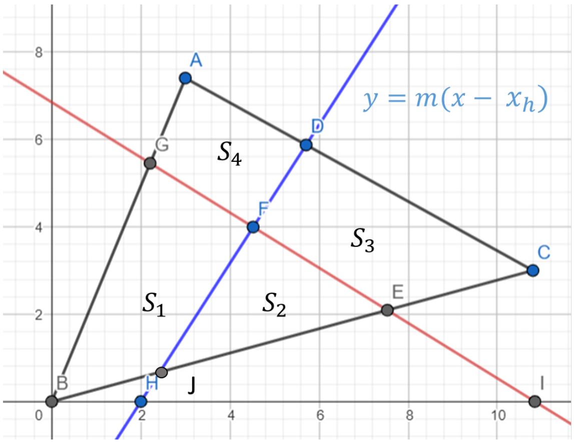

Fig. 3: Problem 3 Model

Similar to Problem 2, by considering that the slopes of the known perpendicular lines DH and GI must multiply to -1, we can deduce that the slope of line GI is $\frac{-1}{m}$. The equation of line GI can be expressed as follows:

$$y - y_i = \frac{-(x - x_i)}{m} \quad (53) \quad \text{for } I(x_i, 0) \quad y = \frac{x_i - x}{m} \quad (54)$$

To find the coordinates of point D (x_d, y_d), we need to examine the intersection of lines AC and HD. Deriving from equation (4), we can express it as follows:

$$y_d - y_c = \frac{y_a - y_c}{x_a - x_c}(x_d - x_c) \quad (4) \quad y_d = m(x_d - x_h) \quad (55)$$

$$x_d = \frac{m x_h (x_a - x_c) + x_a y_c - x_c y_a}{m (x_a - x_c) - y_a + y_c} \quad (56) \quad y_d = m(\frac{x_a y_c - x_c y_a + x_h y_e - x_b y_c}{m(x_a - x_c) - y_a + y_c}) \quad (57)$$

To find the coordinates of point G (x_g, y_g), we should examine the intersection of lines BA and GI. By starting from equations (27) and (54), we have the following expression:

$$y_g - y_b = \frac{y_a - y_b}{x_a - x_b}(x_g - x_b) \quad (27) \quad y_g = \frac{x_i - x_g}{m} \quad (54)$$

$$\text{for } B(0,0) \quad y_g = \frac{y_a}{x_a} x_g \quad (58)$$

$$\frac{x_i - x_g}{m} = \frac{y_a}{x_a} x_g \quad (59)$$

$$x_g = \frac{x_i x_a}{(m y_a + x_e)} \quad (60) \quad y_g = \frac{x_j y_a}{(m y_a + x_a)} \quad (61)$$

To find the coordinates of point J (x_j, y_j), we need to examine the intersection of lines BC and DH:

$$y_j - y_b = \frac{y_c - y_b}{x_c - x_b}(x_j - x_b) \quad (62) \quad y_j = m(x_j - x_k) \quad (63)$$

$$\text{for } B(0,0) \quad y_j = \frac{y_c}{x_c} x_j \quad (64)$$

$$m(x_j - x_h) = \frac{y_c}{x_c} x_j \quad (65)$$

$$m x_c x_j - m x_c x_h = y_c x_j \quad (66)$$

$$x_j = \frac{m x_h x_c}{m x_c - y_c} (67) \quad y_j = \frac{m x_h y_c}{m x_c - y_c} \quad (68)$$

To find the coordinates of point E (x_e, y_e), we need to examine the intersection of lines BC and GI. By using equations (62) and (54), we can express it as follows:

$$y_e - y_b = \frac{y_c - y_b}{x_c - x_b}(x_e - x_b) \quad (69) \quad y_e = \frac{x_i - x_e}{m} \quad (70)$$

$$\text{for } B(0,0) \quad y_e = \frac{y_c}{x_c} x_e \quad (71)$$

$$\frac{x_i - x_e}{m} = \frac{y_c}{x_c} x_e \quad (72)$$

$$x_e = \frac{x_i x_c}{m y_c + x_c} \quad (73) \quad y_e = \frac{x_i y_c}{m y_c + x_c} \quad (74)$$

To calculate areas, we can find the coordinates of point F $\left(x_f, y_f\right)$ as the intersection of lines DH and GI:

$$\text{Line } GI: \quad y_f = \frac{x_i - x_f}{m} \quad (75) \quad \text{Line } DH: \quad y_f = m\left(x_f - x_h\right) \quad (76)$$

$$\frac{x_i - x_f}{m} = m\left(x_f - x_h\right) \quad (77)$$

$$x_i - x_f = m^2 x_f - m^2 x_h \quad (78)$$

$$x_f = \frac{m^2 x_h + x_i}{(m^2+1)} \quad (79) \quad y_f = \frac{m x_i - m x_h}{(m^2+1)} \quad (80)$$

We can calculate the area of triangle ABC using equation (1) as shown below.

$$\text{A(ABC)} = S = \frac{1}{2}\left|(x_a y_b + x_b y_c + x_c y_a) - (x_b y_a + x_c y_b + x_a y_c)\right| \quad (1)$$

$$S = \frac{1}{2} \mid x_c y_a - x_a y_c \mid \quad (81)$$

The area of triangle JEF with coordinates J $\left(x_j, x_j\right)$, F $\left(x_f, y_f\right)$, and E (x_e, y_e) can be calculated as shown below.

$$\text{A(JEF)} = S_2 = \frac{1}{2}\left|\left(x_j y_e + x_e y_f + x_f y_j\right) - \left(x_e y_j + x_f y_e + x_j y_f\right)\right| \quad (82)$$

$$S_2 = \frac{1}{2} \mid \left(\frac{m x_{\text{h}} x_c}{m x_c - y_c}\frac{x_i y_c}{m y_c + x_c} + \frac{x_i x_c}{m y_c + x_c}\frac{m x_i - m x_{\text{h}}}{(m^2+1)} + \frac{m^2 x_{\text{h}} + x_i}{(m^2+1)}\frac{m x_{\text{h}} y_c}{m x_c - y_c}\right) -$$
$$\left(\frac{x_i x_c}{m y_c + x_c}\frac{m x_k y_c}{m x_c - y_c} + \frac{m^2 x_h + x_c}{(m^2+1)}\frac{x_1 y_c}{m y_c + x_c} + \frac{m x_b x_c}{m x_c - y_c}\frac{m x_1 - m x_h}{(m^2+1)}\right) \mid \quad (83)$$

The areas of quadrilaterals $AGFD$, $DFEC$, and $GBJF$ with coordinates A (x_a, y_a) , B$(0,0)$, C (x_c, y_c) , D (x_d, y_d) , E (x_e, y_e) , F $\left(x_f, y_f\right)$, and G (x_g, y_g) , J $\left(x_j, y_j\right)$ can be calculated using equation (2) as shown below:

$$A(GBJF) = S_1 = \frac{1}{2}\left|\left(x_g y_b + x_b y_j + x_j y_f + x_f y_g\right) - \left(x_b y_g + x_j y_b + x_f y_j + x_g y_f\right)\right| \quad (84)$$

$$S_1 = \frac{1}{2}\left|\left(\frac{m x_h x_c}{m x_c - y_c}\frac{m x_i - m x_h}{(m^2+1)} + \frac{m^2 x_h + x_i}{(m^2+1)}\frac{x_j y_a}{(m y_a + x_a)}\right) - \left(\frac{m^2 x_b + x_j}{(m^2+1)}\frac{m x_h y_c}{m x_c - y_c} + \frac{x x_a}{(m y_a + x_a)}\frac{m x_i - m x_k}{(m^2+1)}\right)\right| \quad (85)$$

$$A(DFEC) = S_3 = \frac{1}{2}\left|\left(x_d y_f + x_f y_e + x_e y_c + x_c y_d\right) - \left(x_f y_d + x_e y_f + x_c y_e + x_d y_c\right)\right| \quad (86)$$

$$S_3 = \frac{1}{2} \mid \left(\frac{m x_h (x_e - x_c) + x_e y_c - x_c y_e}{m(x_a - x_c) - y_a + y_c}\frac{m x_i - m x_k}{(m^2+1)} + \frac{m^2 x_h + x_j}{(m^2+1)}\frac{x_i y_c}{m y_c + x_c} + \frac{x_j x_c}{m y_c + x_c} y_c + x_c m\left(\frac{x_a y_c - x_c y_a + x_h y_a - x_k y_c}{m(x_a - x_c) - y_a + y_c}\right)\right) -$$
$$\left(\frac{m^2 x_k + x_i}{(m^2+1)} m\left(\frac{x_a y_c - x_c y_a + x_h y_e - x_b y_c}{m(x_a - x_c) - y_e + y_c}\right) + \frac{x_i x_c}{m y_c + x_c}\frac{m x_i - m x_h}{(m^2+1)} + x_c\frac{x_d y_c}{m y_c + x_c} + \frac{m x_h (x_e - x_c) + x_e y_c - x_c y_e}{m(x_a - x_c) - y_a + y_c} y_c\right) \mid \quad (87)$$

$$A(AGFD) = S_4 = \frac{1}{2}\left|\left(x_a y_g + x_g y_f + x_f y_a + x_d y_a\right) - \left(x_g y_a + x_f y_g + x_d y_f + x_a y_d\right)\right| \quad (88)$$

$$S_4 = \frac{1}{2} \mid \left(x_a\frac{x_a y_a}{(m y_a + x_a)} + \frac{x_i x_a}{(m y_a + x_a)}\frac{m x_i - m x_h}{(m^2+1)} + \frac{m^2 x_h + x_i}{(m^2+1)} m\left(\frac{x_a y_c - x_c y_a + x_h y_a - x_b y_c}{m(x_a - x_c) - y_a + y_c}\right) + \frac{m x_h (x_a - x_c) + x_a y_c - x_c y_a}{m(x_a - x_c) - y_a + y_c}\right) -$$
$$\left(\frac{x_i x_a}{(m y_a + x_a)} y_a + \frac{m^2 x_b + x_i}{(m^2+1)}\frac{x_j y_a}{(m y_a + x_a)} + \frac{m x_k (x_a - x_c) + x_a y_c - x_c y_a}{m(x_a - x_c) - y_a + y_c}\frac{m x_h}{(m^2+1)} + x_a m\left(\frac{x_a y_c - x_c y_a + x_b y_a - x_h y_c}{m(x_a - x_c) - y_a + y_c}\right)\right) \mid \quad (89)$$

After obtaining these equations, we can define the function $F_3(m, x_i, x_h)$. Since it's desired that all areas are equal, each region's area should be $\frac{S}{4}$. We can define this function as shown below and solve the problem using the particle swarm optimization method to find values where the function approaches 0.

$$F_3(m, x_i, x_h) = \left|S_1 - \frac{s}{4}\right| + \left|S_2 - \frac{s}{4}\right| + \left|S_3 - \frac{s}{4}\right| + \left|S_4 - \frac{s}{4}\right| \quad (90)$$
$$F_3(m, x_i, x_h) \longrightarrow 0 \quad (91)$$

(Used code for the problem 3 can be found in subsection 4.4)

3.4 Problem 4

For the trapezoid $ABCD$ with point B at the origin and side $|BC|$ lying on the x-axis, which values of x_h, x_f, and m allow the division of the trapezoid into four equal parts by a line with slope m intersecting the x-axis at point x_e and a line perpendicular to it intersecting the y-axis at point y_f?

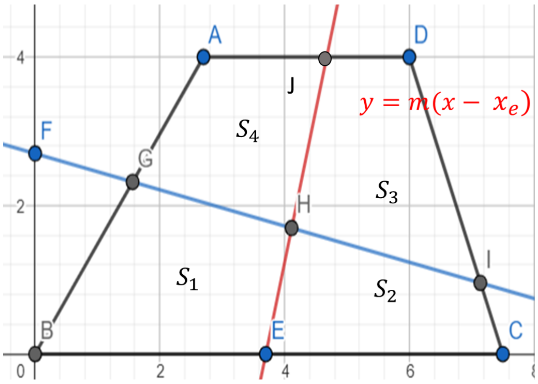

Fig. 4: Problem 4 Model

Similar to Problem 2, by considering that the slopes of the known perpendicular lines EJ and FI must multiply to -1, we can deduce that the slope of line FI is $\frac{-1}{m}$. The equation of line FI can be expressed as follows:

$$y - y_f = \frac{-(x - x_f)}{m} \quad (92)$$

$$\text{for F } (0, y_f) \quad y = \frac{-x}{m} + y_f \quad (93)$$

We can calculate the equations of the lines AD, BA, and DC as follows in the figure:

$$\text{line } BA \quad y - y_b = \frac{y_a - y_b}{x_a - x_b}(x - x_b) \quad \text{for } B(0,0) y = \frac{y_a}{x_a} x \quad (94)$$

$$\text{line } DC \quad y - y_c = \frac{y_d - y_c}{x_d - x_c}(x - x_c), \quad \text{for } C(x_c, 0) \quad y = \frac{y_d}{x_d - x_c}(x - x_c) \, (95) \quad \text{line } AD \quad y - y_a = \frac{y_a - y_d}{x_a - x_d}(x - x_a) \, (96)$$

The coordinates of point H (x_h, y_h) can be found as the intersection of lines EJ and FI.

$$\text{line } FI \quad y_h = \frac{-x_h}{m} + y_f \quad (97) \quad \text{line } EJ \quad y_h = m(x_h - x_e) \, (98)$$

$$\frac{-x_h}{m} + y_f = m(x_h - x_e) \quad (99)$$
$$-x_h + m y_f = m^2 x_h - m^2 x_e \quad (100)$$
$$m^2 x_e + m y_f = m^2 x_h + x_h \quad (101)$$

$$x_h = \frac{m^2 x_e + m y_f}{(m^2 + 1)} \quad (102) \quad y_h = \frac{m^2 y_f - m x_e}{(m^2 + 1)} \quad (103)$$

To find the coordinates of point J (x_d, y_d), we need to examine the intersection of lines AD and EJ. By deriving from equation (96), we can express it as follows:

$$y_j - y_a = \frac{y_a - y_d}{x_a - x_d}\left(x_j - x_a\right) \quad (104) \quad y_j = m(x_j - x_e) \quad (105)$$

$$\frac{y_a - y_d}{x_a - x_d}\left(x_j - x_a\right) + y_a = m\left(x_j - x_e\right) \quad (106)$$

$$x_j = \frac{y_a x_d - x_a x_{d-} m x_e\left(x_a - x_d\right)}{y_a - y_d - m x_a + m x_d} \quad (107) \quad y_j = m\left(\frac{y_e x_{dd} - x_a x_d - x_e y_a + x_e y_d}{y_a - y_d - m x_a + m x_d}\right) \quad (108)$$

To find the coordinates of point G (x_g, y_g), we need to examine the intersection of lines BA and FI. By deriving from equation (94), we can express it as follows:

$$y_g = \frac{y_a}{x_a} x_g \quad (109) \quad y_g = \frac{-x_g}{m} + y_f \quad (110)$$

$$\frac{y_a}{x_a} x_g = \frac{-x_g}{m} + y_f \quad (111)$$

$$m y_a x_g + x_a x_g = m x_a y_f \quad (112)$$

$$x_g = \frac{m x_a y_f}{m y_a + x_a} \quad (113) \quad y_g = \frac{m y_a y_f}{m y_a + x_a} \quad (114)$$

To find the coordinates of point I (x_i, y_i), we need to examine the intersection of lines DC and FI. By deriving from equation (95), we can express it as follows:

$$y_i = \frac{y_d}{x_d - x_c}\left(x_i - x_c\right) \quad (115) \quad y_i = \frac{-x_i}{m} + y_f \quad (116)$$

$$\frac{-x_i}{m} + y_f = \frac{y_d}{x_d - x_c}\left(x_i - x_c\right) \quad (117)$$

$$\left(x_d - x_c\right)\left(m y_f - x_i\right) = m y_d\left(x_i - x_c\right) \quad (118)$$

$$x_i\left(x_c - x_d\right) + x_d m y_f - m x_c y_f = m y_d x_i - m y_d x_c \quad (119)$$

$$x_i = \frac{m x_c y_f - x_d m y_f - m y_d x}{\left(x_c - x_d - m y_d\right)} \quad (120) \quad y_i = \frac{y_d x_c - m y_d y_f}{\left(x_c - x_d - m y_d\right)} \quad (121)$$

The area of the trapezoid $ABCD$ can be calculated using equation (2) as shown below.

$$\text{A(ABCD)} = S = \frac{1}{2}\left|\left(x_a y_b + x_b y_c + x_c y_d + x_d y_a\right) - \left(x_b y_a + x_c y_b + x_d y_c + x_a y_d\right)\right| \quad (2)$$

$$S = \frac{1}{2}\left|\left(x_c y_d + x_d y_a\right) - \left(x_a y_d\right)\right| \quad (122)$$

The areas of the quadrilaterals $GBEH$, $HECI$, $JHID$, and $AGHJ$ with coordinates A (x_a, y_a), B$(0,0)$, C $(x_c, 0)$, D (x_d, y_d), E $(x_e, 0)$, F $\left(0, y_f\right)$, G (x_g, y_g), H (x_h, y_h), I (x_i, y_i), and J $\left(x_j, y_j\right)$ can be calculated using equation (2) as shown below:

$$A(GBEH) = S_1 = \frac{1}{2}\left|\left(x_g y_b + x_b y_e + x_e y_h + x_h y_g\right) - \left(x_b y_g + x_e y_b + x_h y_e + x_g y_h\right)\right| \quad (123)$$

$$S_1 = \frac{1}{2}\left|\left(x_e \frac{m^2 y_f - m x_e}{(m^2+1)} + \frac{m^2 x_e + m y_f}{(m^2+1)} \frac{m y_a y_f}{m y_a + x_a}\right) - \left(\frac{m x_a y_f}{m y_a + x_a} \frac{m^2 y_f - m x_e}{(m^2+1)}\right)\right| \quad (124)$$

$$A(HECI) = S_2 = \frac{1}{2}\left|\left(x_h y_e + x_e y_c + x_c y_i + x_i y_h\right) - \left(x_e y_h + x_c y_e + x_i y_c + x_h y_i\right)\right| \quad (125)$$

$$S_2 = \frac{1}{2}\left|\left(x_c \frac{y_d x_c - m y_d y_f}{\left(x_c - x_d - m y_d\right)} + \frac{m x_c y_f - x_d m y_f - m y_d x_c}{\left(x_c - x_d - m y_d\right)} \frac{m^2 y_f - m x_e}{(m^2+1)}\right) - \left(x_e \frac{m^2 y_f - m x_e}{(m^2+1)} + \frac{m^2 x_e + m y_f}{(m^2+1)} \frac{y_d x_c - m y_d y_f}{\left(x_c - x_d - m y_d\right)}\right)\right| (126)$$

$$A(JHID) = S_3 = \frac{1}{2}\left|\left(x_j y_h + x_h y_i + x_i y_d + x_d y_j\right) - \left(x_h y_j + x_i y_h + x_d y_i + x_j y_d\right)\right| \quad (127)$$

$$S_3 = \frac{1}{2} \left| \left(\frac{y_a x_d - x_a x_{d-} - m x_e\left(x_a - x_d\right)}{y_a - y_d - m x_a + m x_d} \frac{m^2 y_f - m x_e}{(m^2+1)} + \frac{m^2 x_e + m y_f}{(m^2+1)} \frac{y_d x_c - m y_d y_f}{\left(x_c - x_d - m y_d\right)} + \frac{m x_c y_f - x_d m y_f - m y_d x_c}{\left(x_c - x_d - m y_d\right)} y_d + \right.\right.$$

$$\left.\left. x_d m\left(\frac{y_a x_d - x_a x_d - x_e y_a + x_e y_d}{y_a - y_d - m x_a + m x_d}\right)\right) - \left(\frac{m^2 x_e + m y_f}{(m^2+1)} m\left(\frac{y_a x_d - x_a x_d - x_e y_a + x_e y_d}{y_a - y_d - m x_a + m x_d}\right) + \frac{m x_c y_f - x_d m y_f - m y_d x_c}{\left(x_c - x_d - m y_f\right)} \frac{m x_e}{(m^2+1)} + \right.\right.$$

$$\left.\left. x_d \frac{y_d x_c - m y_d y_f}{\left(x_c - x_d - m y_d\right)} + \frac{y_a x_d - x_a x_d - m x_e\left(x_a - x_d\right)}{y_a - y_d - m x_a + m x_d} y_d\right) \right| \quad (128)$$

$$A(AGHJ) = S_4 = \frac{1}{2}\left|\left(x_a y_g + x_g y_h + x_h y_j + x_j y_a\right) - \left(x_g y_a + x_h y_g + x_j y_h + x_a y_j\right)\right| \quad (129)$$

$$S_4 = \frac{1}{2} \mid x_a \frac{m y_a y_f}{m y_a + x_a} + \frac{m x_a y_f}{m y_a + x_a}\frac{m^2 y_f - m x_e}{m^2+1} + \frac{m^2 x_e + m y_f}{m^2+1} m\left(\frac{y_a x_d - x_a x_d - x_e y_a + x_e y_d}{y_a - y_d - m x_a + m x_d}\right) +$$
$$\frac{y_a x_d - x_a x_{d-} m x_e (x_a - x_d)}{y_a - y_d - m x_a + m x_d} y_a - \frac{m x_a y_f}{m y_a + x_a} y_a - \frac{m^2 x_e + m y_f}{m^2+1}\frac{m y_a y_f}{m y_a + x_a} - \frac{y_a x_d - x_a x_d - m x_e\left(x_a - x_d\right)}{y_a - y_d - m x_a + m x_d}\frac{m_f - m x_e}{m^2+1} -$$
$$x_a m\left(\frac{y_a x_d - x_a x_d - x_e y_a + x_e y_d}{y_a - y_d - m x_a + m x_d}\right) \mid \quad (130)$$

After obtaining these equations, we can define the function $F_4\left(m, x_e, y_f\right)$. Since it's desired that all areas are equal, each region's area should be $\frac{S}{4}$. We can define this function as shown below and solve the problem using the particle swarm optimization method to find values where the function equals 0.

$$F_4\left(m, x_e, y_f\right) = \left|S_1 - \frac{S}{4}\right| + \left|S_2 - \frac{S}{4}\right| + \left|S_3 - \frac{s}{4}\right| + \left|S_4 - \frac{s}{4}\right| \quad (131)$$
$$F_4\left(m, x_e, y_f\right) \longrightarrow 0 \quad (132)$$

(Used code for the problem 4 can be found in subsection 4.5)

If we examine the case where $ABCD$ is a right trapezoid and calculate the values of points H, J, I, and G when we have $\mathrm{A}(0,2), \mathrm{B}(0,0), \mathrm{C}(6,0), \mathrm{D}(4,2)$.

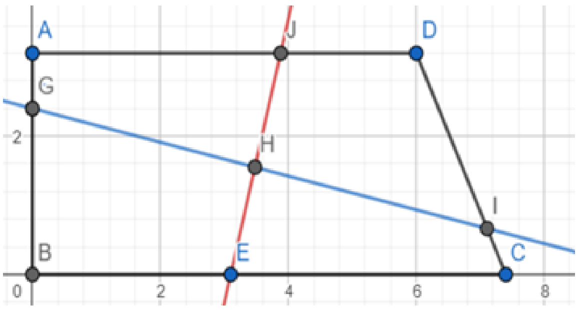

Fig. 5: ABCD right trapezoid

$$x_g = \frac{m x_a y_f}{m y_a + x_a} = 0 \quad x_j = \frac{y_a x_d - x_a x_{d-} m x_e (x_a - x_d))}{y_a - y_d - m x_a + m x_d} = \frac{2 + m x_e}{m} \quad x_i = \frac{2 m y_f - 12m}{(2-2m)} \quad x_h = \frac{m^2 x_e + m y_f}{(m^2+1)}$$
$$y_g = \frac{m y_a y_f}{m y_a + x_a} = y_f \quad y_j = m\left(\frac{y_a x_d - x_a x_d - x_e y_a + x_e y_d}{y_a - y_d - m x_a + m x_d}\right) = 2m \quad y_i = \frac{12 - 2 m y_f}{(2-2m)} \quad y_h = \frac{m^2 y_f - m x_e}{(m^2+1)}$$

The total area of the trapezoid ABCD can be calculated as shown below.

$$\mathrm{A(ABCD)} = \mathrm{S} = \frac{1}{2}\left|+x_c y_d + x_d y_a\right| = \frac{1}{2}|12+8| = 10$$

The areas of the four regions can be calculated as shown below.

$$A(GBEH) = S_1 = \frac{1}{2}\left|\left(x_e \frac{m^2 y_f - m x_e}{(m^2+1)} + \frac{m^2 x_e + m y_f}{(m^2+1)} y_f\right)\right|$$
$$A(HECI) = S_2 = \frac{1}{2}\left|\left(6\frac{12 - 2 m y_f}{(2-2m)} + \frac{2 m y_f - 12m}{(2-2m)}\frac{m^2 y_f - m x_e}{(m^2+1)}\right) - \left(x_e \frac{m^2 y_f - m x_e}{(m^2+1)} + \frac{m^2 x_e + m y_f}{(m^2+1)}\frac{12 - 2 m y_f}{(2-2m)}\right)\right|$$
$$A(JHID) = S_3 = \frac{1}{2} \mid \left(\frac{2 + m x_e}{m}\frac{m^2 y_f - m x_e}{(m^2+1)} + \frac{m^2 x_e + m y_f}{(m^2+1)}\frac{12 - 2 m y_f}{(2-2m)} + 2\frac{2 m y_f - 12m}{(2-2m)} + 8m\right)$$
$$- \left(\frac{m^2 x_e + m y_f}{(m^2+1)} 2m + \frac{2 m y_f - 12m}{(2-2m)}\frac{m^2 y_f - m x_e}{(m^2+1)} + 4\frac{12 - 2 m y_f}{(2-2m)} + 2\frac{2 + m x_e}{m}\right) \mid$$
$$A(AGHJ) = S_4 = \frac{1}{2}\left|\left(\frac{m^2 x_e + m y_f}{(m^2+1)} 2m + 2\frac{2 + m x_e}{m}\right) - \left(\frac{m^2 x_e + m y_f}{(m^2+1)} y_f + \frac{2 + m x_e}{m}\frac{m^2 y_f - m x_e}{(m^2+1)}\right)\right|$$

3.5 Problem 5

For the pentagon $ABCDJ$ with point B at the origin, side $|AB|$ lying on the y-axis, and side $|BC|$ lying on the x-axis, which values of x_e, x_f, and m allow the division of the pentagon into four equal parts by a line with slope m intersecting the $|JD|$ side and the x-axis at point x_e, and a line perpendicular to it intersecting the $|AJ|$ side and the x-axis at point x_f?

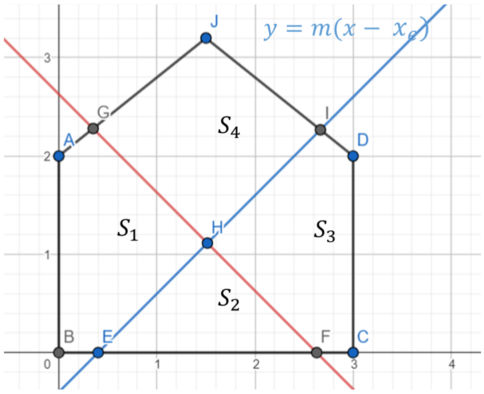

Fig. 6: Problem 5 Model

The slope of the line FG can be determined as $\frac{-1}{m}$ by considering that it intersects perpendicular to known lines EI and FG in a manner similar to Problem 2, where the product of their slopes should be -1. The equation of the line FG can be expressed as follows:

$$y - y_f = \frac{-\left(x - x_f\right)}{m} \quad (133)$$

$$\text{As F}\left(x_f, 0\right) \qquad y = \frac{x_f - x}{m} \quad (134)$$

We can calculate the equations of the lines AJ, JD, and DC as below:

$$\text{line} \quad DC \quad y - y_c = \frac{y_d - y_c}{x_d - x_c}\left(x - x_c\right) \quad for \quad C\left(x_c, 0\right) \quad y = \frac{y_d}{x_d - x_c}\left(x - x_c\right) \quad (135)$$

$$\text{line} \quad AJ \quad y - y_a = \frac{y_j - y_a}{x_j - x_a}\left(x - x_a\right) \quad for \quad A\left(0, y_a\right) \quad y = \frac{y_j - y_a}{x_i} x + y_a \quad (136)$$

$$\text{line} \quad JD \quad y - y_j = \frac{y_j - y_d}{x_i - x_d}\left(x - x_j\right) \quad (137)$$

We can find the coordinates of point H (x_h, y_h) as the intersection of the lines EI and FG.

$$\text{line } FG: \quad y_h = \frac{x_f - x_h}{m} \quad (138) \quad \text{line } EI: \quad y_h = m\left(x_h - x_e\right) (139)$$

$$\frac{-x_h}{m} + y_f = m\left(x_h - x_e\right) \quad (140)$$

$$-x_h + x_f = m^2 x_h - m^2 x_e \quad (141)$$

$$m^1 2x_e + x_f = m^2 x_h + x_h \quad (142)$$

$$x_h = \frac{m^2 x_e + x_f}{\left(m^2 + 1\right)} \quad (143) \quad y_h = \frac{m\left(x_f - x_e\right)}{\left(m^2 + 1\right)} \quad (144)$$

To find the coordinates of point G (x_g, y_g), we should examine the intersection of the lines AJ and FG. By using equation (135) as a starting point, we can express it as shown below:

$$y_g = \frac{y_j - y_a}{x_j} x_g + y_a \quad (145) \quad y_g = \frac{x_f - x_g}{m} \quad (146)$$

$$\frac{y_j - y_a}{x_j} x_g + y_a = \frac{x_f - x_g}{m} \quad (147)$$

$$m x_g\left(y_j - y_a\right) + m y_a x_j = x_f x_j - x_g x_j \quad (148)$$

$$x_g\left(m y_j - m y_a + x_j\right) = x_j\left(x_f - m y_a\right) \quad (149)$$

$$x_g = \frac{x_j\left(x_f - m y_a\right)}{\left(m y_j - m y_a + x_j\right)} \quad (150) \quad y_g = \frac{x_f y_j - x_f y_a - x_j y_a}{\left(m y_j - m y_a + x_j\right)} \quad (151)$$

To find the coordinates of point I (x_i, y_i), we need to consider the intersection of the lines DC and EI. By using equation (136) as a starting point, we can express it as shown below:

$$y_i - y_j = \frac{y_j - y_d}{x_j - x_d}\left(x_i - x_j\right) \quad (152) \quad y_i = m\left(x_i - x_e\right) \quad (153)$$

$$\frac{y_j - y_d}{x_j - x_d}\left(x_i - x_j\right) + y_j = m\left(x_i - x_e\right) \quad (154)$$

$$x_i\left(y_j - y_d - mx_j + mx_d\right) = mx_e x_d - mx_e x_j + y_j x_d - y_d x_j \quad (155)$$

$$x_i = \frac{mx_e x_d - mx_e x_j + y_j x_d - y_d x_j}{\left(y_j - y_d - mx_j + mx_d\right)} \quad (156) \quad y_i = m\left(\frac{x_e y_d - x_e y_j + y_j x_d - y_d x_j}{\left(y_j - y_d - mx_j + mx_d\right)}\right) \quad (157)$$

The area of the pentagon $ABCDJ$ can be calculated using equations (1) and (2), expressing it as the sum of the areas of triangle JAD and quadrilateral $ABCD$, as shown below:

$$A(ABCD) = S_a = \frac{1}{2}\left|(x_a y_b + x_b y_c + x_c y_d + x_d y_a) - (x_b y_a + x_c y_b + x_d y_c + x_a y_d)\right| \quad (158)$$

$$S_a = \frac{1}{2}\left|(x_c y_d + x_d y_a)\right| \quad (159)$$

$$\mathrm{A(JAD)} = S_b = \frac{1}{2}\left|\left(x_j y_a + x_a y_a + x_d y_j\right) - \left(x_a y_j + x_d y_a + x_j y_d\right)\right| \quad (160)$$

$$S_b = \frac{1}{2}\left|\left(x_j y_a + x_d y_j\right) - \left(x_d y_a + x_j y_d\right)\right| \quad (161)$$

$$A(ABCDJ) = S = S_a + S_b = \frac{1}{2}\left|(x_c y_d + x_d y_a)\right| + \frac{1}{2}\left|\left(x_j y_a + x_d y_j\right) - \left(x_d y_a + x_j y_d\right)\right| \quad (162)$$

The area of the pentagon $GABEH$ can be calculated using equations (1) and (2), expressing it as the sum of the areas of triangle GAH and quadrilateral $ABEH$, as shown below:

$$\mathrm{A(ABEH)} = S_{1a} = \frac{1}{2}\left|(x_a y_b + x_b y_e + x_e y_h + x_h y_a) - (x_b y_a + x_e y_b + x_h y_e + x_a y_h)\right| \quad (163)$$

$$S_{1a} = \frac{1}{2}\left|\left(x_e \frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m^2 x_e + x_f}{(m^2+1)} y_a\right)\right| \quad (164)$$

$$\mathrm{A(GAH)} = S_{1b} = \frac{1}{2}\left|(x_g y_a + x_a y_h + x_h y_g) - (x_a y_g + x_h y_a + x_g y_h)\right| \quad (165)$$

$$S_{1b} = \frac{1}{2} \mid \left(\frac{x_j\left(x_f - my_a\right)}{\left(my_j - my_a + x_j\right)} y_a + \frac{m^2 x_e + x_f}{(m^2+1)} \frac{x_f y_j - x_f y_a - x_j y_a}{\left(my_j - my_a + x_j\right)}\right) - \left(\frac{m^2 x_e + x_f}{(m^2+1)} y_a + \frac{x_j\left(x_f - my_a\right)}{\left(my_j - my_a + x_j\right)} \frac{m\left(x_f - x_e\right)}{(m^2+1)}\right) \quad (166)$$

$$A(\ GABEH\ ) = S_1 = S_{1a} + S_{1b} = \frac{1}{2}\left|\left(x_e \frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m^2 x_e + x_f}{(m^2+1)} y_a\right)\right| + \frac{1}{2} \mid \left(\frac{x_j\left(x_f - my_a\right)}{\left(my_j - my_a + x_j\right)} y_a + \right.$$

$$\left.\frac{m^2 x_e + x_f}{(m^2+1)} \frac{x_f y_j - x_f y_a - x_j y_a}{\left(my_j - my_a + x_j\right)}\right) - \left(\frac{m^2 x_e + x_f}{(m^2+1)} y_a + \frac{x_j\left(x_f - my_a\right)}{\left(my_j - my_a + x_j\right)} \frac{m\left(x_f - x_e\right)}{(m^2+1)}\right) \mid \quad (167)$$

The area of triangle HEF can be calculated as shown below:

$$\mathrm{A(HEF)} = S_2 = \frac{1}{2}\left|\left(x_h y_e + x_e y_f + x_f y_h\right) - \left(x_e y_h + x_f y_e + x_h y_f\right)\right| \quad (168)$$

$$S_2 = \frac{1}{2}\left|\left(x_f y_h\right) - \left(x_e y_h\right)\right| = \frac{1}{2}\left|\frac{m\left(x_f - x_e\right)}{(m^2+1)}\left(x_f - x_e\right)\right| \quad (169)$$

The area of the pentagon $IHFCD$ can be calculated using equations (1) and (2), expressing it as the sum of the areas of triangle IHD and quadrilateral $HFCD$, as shown below:

$$\mathrm{A(HFCD)} = S_{3a} = \frac{1}{2}\left|\left(x_h y_f + x_f y_c + x_c y_d + x_d y_h\right) - \left(x_f y_h + x_c y_f + x_d y_c + x_h y_d\right)\right| \quad (170)$$

$$S_{3a} = \frac{1}{2}\left|\left(x_c y_d + x_d \frac{m\left(x_f - x_e\right)}{(m^2+1)}\right) - \left(x_f \frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m^2 x_e + x_f}{(m^2+1)} y_d\right)\right| \quad (171)$$

$$\text{A(IHD)} = S_{3b} = \frac{1}{2}\left|(x_i y_h + x_h y_d + x_d y_i) - (x_h y_i + x_d y_h + x_i y_d)\right| \quad (172)$$

$$S_{3b} = \frac{1}{2}\,\Bigg|\left(\frac{m x_e x_d - m x_e x_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m^2 x_e + x_f}{(m^2+1)} y_d + x_d m\left(\frac{x_e y_d - x_e y_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\right)\right) - \left(\frac{m^2 x_e + x_f}{(m^2+1)} m\left(\frac{x_e y_d - x_e y_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\right) + x_d\frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m x_e x_d - m x_e x_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)} y_d\right)\Bigg| \quad (173)$$

$$A(\text{IHFCD}) = S_3 = S_{3a} + S_{3b} = \frac{1}{2}\left|\left(x_c y_d + x_d\frac{m\left(x_f - x_e\right)}{(m^2+1)}\right) - \left(x_f\frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m^2 x_e + x_f}{(m^2+1)} y_d\right)\right| +$$
$$\frac{1}{2}\,\Bigg|\left(\frac{m x_e x_d - m x_e x_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m^2 x_e + x_f}{(m^2+1)} y_d + x_d m\left(\frac{me y_d - x_e y_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\right)\right) - \left(\frac{m^2 x_e + x_f}{(m^2+1)} m\left(\frac{x_e y_d - x_e y_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\right) + x_d\frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m x_e x_d - m x_e x_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)} y_d\right)\Bigg| \quad (174)$$

The area of the quadrilateral $JGHI$ can be calculated using equation (2), as shown below:

$$\text{A(JGH)} = S_4 = \frac{1}{2}\left|\left(x_j y_g + x_g y_h + x_h y_i + x_i y_j\right) - \left(x_g y_j + x_h y_g + x_i y_h + x_j y_i\right)\right| \quad (175)$$

$$S_4 = \frac{1}{2}\,\Bigg|\left(x_j\frac{x_f y_j - x_f y_a - x_j y_a}{(m y_j - m y_a + x_j)} + \frac{x_j\left(x_f - m y_a\right)}{(m y_j - m y_a + x_j)}\frac{m\left(x_f - x_e\right)}{(m^2+1)} + \frac{m^2 x_e + x_f}{(m^2+1)} m\left(\frac{x_e y_d - x_e y_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\right) + \frac{m x_e x_d - m x_e x_j + y_j x_d - y_d x_j}{(y_j - y_d - mL x_j + m x_d)} y_j\right) - \left(\frac{x_j\left(x_f - m y_a\right)}{(m y_j - m y_a + x_j)} y_j + \frac{m^2 x_e + x_f}{(m^2+1)}\frac{x_f y_j - x_f y_a - x_j y_a}{(m y_j - m y_a + x_j)} + \frac{m x_e x_{dd} - m x_e x_j + y_j x_{dd} - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\frac{m\left(x_f - x_e\right)}{(m^2+1)} + x_j m\left(\frac{x_e y_d - x_e y_j + y_j x_d - y_d x_j}{(y_j - y_d - m x_j + m x_d)}\right)\right)\Bigg| \quad (176)$$

After obtaining these equations, we can define the function $F_S\left(m, x_e, x_f\right)$. Since it is desired that all areas are equal, each region's area should be $\frac{S}{4}$. We can define this function as follows and then solve the problem using the particle swarm optimization method to find the values where the function equals 0:

$$F_5\left(m, x_e, x_f\right) = \left|S_1 - \frac{s}{4}\right| + \left|S_2 - \frac{s}{4}\right| + \left|S_3 - \frac{s}{4}\right| + \left|S_4 - \frac{s}{4}\right| \quad (177)$$
$$F_5\left(m, x_e, x_f\right) \longrightarrow 0 \quad (178)$$

(Used code for the problem 5 can be found in subsection 4.6)

3.6 Problem 6

Which values of m, x_f, and x_g allow the division of the area between the y-axis and the parabolic curve $y = a^2 - x^2$ into four parts proportional to the numbers 1, 2, 3, and 12 by two perpendicular lines: one with positive slope m intersecting the x-axis at point x_f, and the other intersecting the x-axis at point x_g?

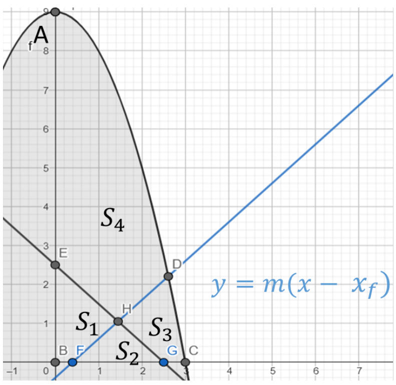

Fig. 7: Problem 6 Model

By considering that the slopes of the intersecting lines DF and GE should have a product of -1, we can deduce that the slope of the line GE is $\frac{-1}{m}$. The equation of the line GE can be represented as follows:

$$y - y_g = \frac{-(x - x_g)}{m} \quad (179)$$

$$\text{As G}\,(x_g, 0) \qquad y = \frac{x_g - x}{m} \quad (180)$$

To find the coordinates of point D (x_d, y_d), we need to look at the intersection of the parabola $y = a^2 - x^2$ and the line DF.

$$y_d = m\left(x_d - x_f\right) \quad (181) \quad y_d = a^2 - x_d^2 \quad (182)$$

$$mx_d - mx_f = a^2 - x_d^2 \quad (183) \qquad x_d^2 + mx_d - mx_f - a^2 = 0 \quad (184)$$

$$x_d = \frac{-m \pm \sqrt{\left(m^2 + 4\left(x_f + a^2\right)\right.}}{2} \quad (185)$$

Taking into account that the value of m is positive and considering that the point D is in the first quadrant, we can express x_d as follows:

$$x_d = \frac{-m + \sqrt{\left(m^2 + 4\left(mx_f + a^2\right)\right.}}{2} \quad (186)$$

$$y_d = m\left(\frac{-m + \sqrt{\left(m^2 + 4\left(mx_f + a^2\right)\right.}}{2} - x_f\right) \quad (187)$$

The point E (x_e, y_e) can be calculated as the intersection of the ordinate and the line GE, as shown below:

$$y_e = \frac{-(x_e - x_g)}{m} \quad (188) \quad E\,(0, y_e) \quad y_e = \frac{x_g}{m} \quad (189)$$

We can find the coordinates of point H (x_h, y_h) as the intersection of the DF and GE lines.

$$\text{line } GE: \quad y_h = \frac{x_g - x_h}{m} \quad (190) \quad \text{line } DH \quad y_h = m\left(x_h - x_f\right) \quad (191)$$

$$\frac{x_g - x_h}{m} = m\left(x_h - x_f\right) \quad (192)$$

$$x_g - x_h = m^2 x_h - m^2 x_f \quad (193)$$

$$x_h = \frac{m^2 x_f + x_g}{(m^2 + 1)} \quad (194) \quad y_h = \frac{m\left(x_g - x_f\right)}{(m^2 + 1)} \quad (195)$$

We can calculate the total area surrounded by the parabola (S) as shown below:

$$S = \int_{x_b}^{x_c} \left(a^2 - x^2\right) dx \quad (196)$$

$$S = \int_0^a \left(a^2 - x^2\right) dx \quad (197)$$

$$S = \left(a^2 x - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{2a^3}{3} \quad (198)$$

We can calculate the areas S_1 and S_2 as shown below:

$$A(EBFH) = S_1 = \frac{1}{2}\left|\left(x_e y_b + x_b y_e + x_f y_h + x_h y_e\right) - \left(x_b y_e + x_f y_b + x_h y_f + x_e y_h\right)\right| \quad (199)$$

$$S_1 = \frac{1}{2}\left|\left(x_f \frac{m\left(x_g - x_f\right)}{(m^2 + 1)} + \frac{m^2 x_f + x_g}{(m^2 + 1)} \frac{x_g}{m}\right)\right| \quad (200)$$

$$A(HFG) = S_2 = \frac{1}{2}\left|\left(x_h y_f + x_f y_g + x_g y_h\right) - \left(x_f y_h + x_g y_f + x_h y_g\right)\right| \quad (201)$$

$$S_2 = \frac{1}{2}\left|\left(x_g \frac{m\left(x_g - x_f\right)}{(m^2 + 1)}\right) - \left(x_f \frac{m\left(x_g - x_f\right)}{(m^2 + 1)}\right)\right| = \frac{1}{2}\left|\frac{m\left(x_g - x_f\right)^2}{(m^2 + 1)}\right| \quad (202)$$

To calculate the area S_3, we can compute it in two parts: by calculating the area of the quadrilateral $DHGC$ formed by connecting points D and C (S_{3a}), and through an integral calculation (S_{3b}), as shown below:

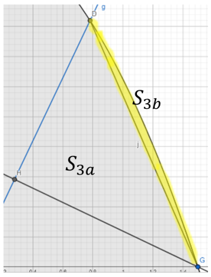

Fig. 8: S_{3a} and S_{3b}

$$S_3 = S_{3a} + S_{3b} \quad (203)$$

$$S_{3a} = A(DHGC) = \frac{1}{2} \mid (x_d y_h + x_h y_g + x_g y_c + x_c y_d) - (x_h y_d + x_g y_h + x_c y_g + x_d y_c) \mid \quad (204)$$

$$S_{3a} = \frac{1}{2} \mid \left(\frac{-m + \sqrt{(m^2 + 4\left(m x_f + a^2\right)}}{2} \frac{m\left(x_g - x_f\right)}{(m^2+1)} + ma\left(\frac{-m + \sqrt{(m^2 + 4\left(m x_f + a^2\right)}}{2} - x_f \right) \right) -$$

$$\left(\frac{m^2 x_f + x_g}{(m^2+1)} m \left(\frac{-m + \sqrt{(m^2 + 4\left(m x_f + a^2\right)}}{2} - x_f \right) + x_g \frac{m\left(x_g - x_f\right)}{(m^2+1)} \right) \mid \quad (205)$$

$$S_{3b} = \int_{x_d}^{x_c} \left(a^2 - x^2\right) dx - \frac{(x_c - x_d)\, y_d}{2} = \left(a^2 x - \frac{x^3}{3}\right)\Big|_{x_d}^{a} - \frac{(a - x_d)\, y_d}{2} \quad (206)$$

$$S_{3b} = \frac{2a^3}{3} - a^2 \frac{-m + \sqrt{(m^2 + 4\left(m x_f + a^2\right)}}{2} + \frac{\left(\frac{-m+\sqrt{(m^2+4(m x_f+a^2)}}{2}\right)^3}{3} - ma\left(\frac{-m + \sqrt{(m^2 + 4\left(m x_f + a^2\right)}}{2} - x_f \right) +$$

$$\frac{m\left(\frac{-m+\sqrt{(m^2+4(m x_f+a^2)}}{2} - x_f\right)\left(\frac{-m+\sqrt{(m^2+4(m x_f+a^2)}}{2}\right)}{2} \quad (207)$$

To calculate the area S_4, we can compute it in two parts: by calculating the area of the quadrilateral $AEHD$ formed by connecting points A and D (S_{4a}), and through an integral calculation (S_{4b}), as shown below:

$$S_4 = S_{4a} + S_{4b} \quad (208)$$

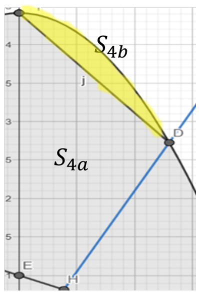

Fig. 9: S_{4a} and S_{4b}

$$S_{4a} = A(AEHD) = \frac{1}{2}\left|(x_a y_e + x_e y_h + x_h y_d + x_d y_a) - (x_e y_a + x_h y_e + x_d y_h + x_a y_d)\right| \quad (209)$$

$$S_{4a} = \frac{1}{2}\left| \left(\frac{m^2 x_f + x_g}{(m^2+1)} m \left(\frac{-m + \sqrt{(m^2 + 4(m x_f + a^2)}}{2} - x_f\right) + \frac{-m + \sqrt{(m^2 + 4(x_f + a^2)}}{2} a^2\right) - \right.$$
$$\left. \left(\frac{m^2 x_f + x_g}{(m^2+1)} \frac{x_g}{m} + \frac{-m + \sqrt{(m^2 + 4(x_f + a^2)}}{2} \frac{m(x_g - x_f)}{(m^2+1)}\right) \right| \quad (210)$$

$$S_{4b} = \int_{x_a}^{x_d} \left(a^2 - x^2\right) dx - \frac{(y_a + y_d)\, x_d}{2} = \left(a^2 x - \frac{x^3}{3}\right)\Bigg|_0^{x_d} - \frac{\left(a^2 + y_d\right) x_d}{2} = \left(a^2 x_d - \frac{{x_d}^3}{3}\right) - \frac{\left(a^2 + y_d\right) x_d}{2} \quad (211)$$

$$S_{4b} = \left(a^2 \frac{-m + \sqrt{(m^2 + 4(m x_f + a^2)}}{2} - \frac{\left(\frac{-m+\sqrt{(m^2+4(m x_f + a^2)}}{2}\right)}{3}\right) -$$
$$\frac{\left(a^2 + m\left(\frac{-m+\sqrt{(m^2+4(m x_f+a^2)}}{2} - x_f\right)\right) \frac{-m+\sqrt{(m^2+4(m x_f+a^2)}}{2}}{2} \quad (212)$$

After obtaining these equations, we can define the function $F_6\left(m, x_f, x_g\right)$. In order for the areas to be proportional to 1, 2, 3, and 12, we can define this function as shown below, and then solve the problem using the particle swarm optimization method to find the values where the function equals 0:

$$F_6\left(m, x_f, x_{fg}\right) = \left|S_1 - \frac{S}{18}\right| + \left|S_2 - \frac{S}{9}\right| + \left|S_3 - \frac{S}{6}\right| + \left|S_4 - \frac{2S}{3}\right| \quad (213)$$
$$F_6\left(m, x_f, x_{fg}\right) \longrightarrow 0 \quad (214)$$

(Used code for the problem 6 can be found in subsection 4.7)

3.7 Problem 7

What values of m, x_f, and x_g ensure the division of the area between the circular function $y = \sqrt{r^2 - x^2}$ and the x-axis into four equal parts using two perpendicular lines that intersect each other; one line being negatively sloped with an intersection at the point x_c on this function, and the other intersecting the same function at point x_d?

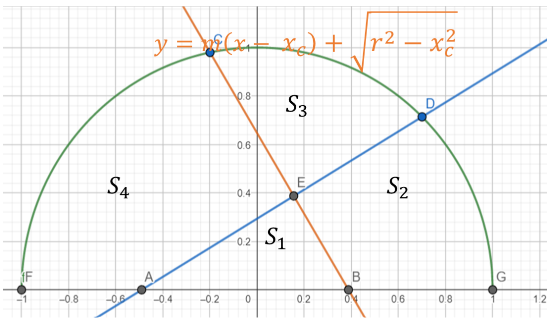

Fig. 10: Problem 7 Model

By considering that the slopes of the intersecting lines AD and BC should have a product of -1, we can deduce that the slope of the line AD is $\frac{-1}{m}$. The equation for the line AD is as follows:

$$y - y_d = \frac{-(x - x_d)}{m} \quad (215)$$

$$\text{As D}\left(x_d, \sqrt{r^2 - x_d^2}\right) \quad y = \frac{x_d - x}{m} + \sqrt{r^2 - x_d^2} \quad (216)$$

To find the coordinates of point E (x_e, y_e), we need to look at the intersection of lines AD and BC.

$$y_e = \frac{x_d - x_e}{m} + \sqrt{r^2 - x_d^2} \quad (217) \quad y_e = m\left(x_e - x_c\right) + \sqrt{r^2 - x_c^2} \quad (218)$$

$$\frac{x_d - x_e}{m} + \sqrt{r^2 - x_d^2} = m\left(x_e - x_c\right) + \sqrt{r^2 - x_c^2} \quad (219)$$

$$x_e\left(\frac{m^2+1}{m}\right) = mx_c + \frac{x_d}{m} + \sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2} \quad (220)$$

$$x_e = \frac{m^2 x_c + x_d + m\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}}{m^2+1} \quad (221) \quad y_e = m\frac{x_d - x_c + m\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}}{m^2+1} + \sqrt{r^2 - x_c^2} \quad (222)$$

The point A $(x_a, 0)$ can be calculated as the intersection of the x-axis and the line AD, as shown below:

$$y_a = \frac{x_d - x_a}{m} + \sqrt{r^2 - x_d^2} \quad (223) \quad \mathrm{A}\left(x_a, 0\right) \quad 0 = \frac{x_d - x_a}{m} + \sqrt{r^2 - x_d^2} \quad (224)$$

$$x_a = m\sqrt{r^2 - x_d^2} + x_d \quad (225)$$

The point B $(x_b, 0)$ can be calculated as the intersection of the x-axis and the line BC, as shown below:

$$y_b = m\left(x_b - x_c\right) + \sqrt{r^2 - x_c^2} \quad (226) \quad \mathrm{B}\left(x_b, 0\right) \quad 0 = m\left(x_b - x_c\right) + \sqrt{r^2 - x_c^2} \quad (227)$$

$$x_b = \frac{-\sqrt{r^2 - x_c^2}}{m} + x_c \quad (228)$$

The total area of the circular region (S) can be calculated as shown below:

$$S = \frac{\pi r^2}{2} \quad (229)$$

$$A(AEB) = S_1 = \frac{1}{2}|x_a y_e + x_e y_b + x_b y_a - x_e y_a + x_b y_e + x_a y_b| \quad (230)$$

The area of triangle $A(AEB) = S_1$ can be calculated using equation (1) as shown below:

$$S_1 = \frac{1}{2}\left|(x_a y_e) - (x_b y_e)\right| \quad (231)$$

To calculate the area S_2, we can compute it in two parts: by calculating the area of the quadrilateral $DEBG$ formed by connecting points D and C (S_{2a}), and through an integral calculation (S_{2b}), as shown below:

$$S_2 = S_{2a} + S_{2b} \quad (232)$$

$$S_{2a} = A(DEBG) = \frac{1}{2} \mid (x_d y_e + x_e y_b + x_b y_g + x_g y_d) - (x_e y_d + x_b y_e + x_g y_b + x_d y_g) \mid \quad (233)$$

$$S_{2a} = \frac{1}{2} \mid \left(x_d m\left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1}\right) + \sqrt{r^2 - x_c^2} + r\sqrt{r^2 - x_d^2}\right) - \left(\frac{m^2 x_c + x_d + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1}\right.$$

$$\sqrt{r^2 - x_d^2} + \left(\frac{-\sqrt{r^2 - x_c^2}}{m} + x_c\right) m\left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1}\right) + \sqrt{r^2 - x_c^2}) \mid \quad (234)$$

$$S_{2b} = I - \frac{\left(x_g - x_d\right) y_d}{2} = \int_{x_d}^{x_g} \sqrt{(r^2 - x^2)}dx - \frac{\left(x_g - x_d\right) y_d}{2} \quad (235)$$

By substituting x with $x = r\cos(t)$ in the integral equal to I, our expression becomes as shown below:

$$x = r\cos t \quad dx = -r\sin t.dt \quad (236)$$

$$I = \int_{\arccos\left(\frac{x}{r}\right)}^{0} \sqrt{r^2 - r^2\cos^2 t}(-r\sin t)dt \quad (237)$$

$$I = r^2\int_{0}^{\arccos\left(\frac{x}{r}\right)} \sqrt{1 - \cos^2 t}\sin t dt \quad (238)$$

$$I = r^2\int_{0}^{\arccos\left(\frac{x_d}{r}\right)} \sin^2 t dt \quad (232) \quad \sin^2 t = \frac{1 - \cos^2 t}{2} \quad (239)$$

$$I = r^2\int_{0}^{\arccos\left(\frac{x_d}{r}\right)} \frac{1 - \cos^2 t}{2}dt = r^2\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Big|_{0}^{\arccos\left(\frac{x_d}{r}\right)} \quad (240)$$

$$S_2 = \frac{1}{2} \mid \left(x_d m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right) + \sqrt{r^2 - x_c^2} + r\sqrt{r^2 - x_d^2} \right) - (\frac{m^2 x_c + x_d + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1}$$

$$\sqrt{r^2 - x_d^2} + \left(\frac{-\sqrt{r^2 - x_c^2}}{m} + x_c \right) m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right) + \sqrt{r^2 - x_c^2}) \mid + r^2 \left(\frac{\arccos\left(\frac{x_d}{r}\right)}{2} - \frac{\sin 2\arccos\left(\frac{x_d}{r}\right)}{4} \right)$$

$$- \frac{(x_g - x_d)\, y_d}{2} \quad (241)$$

To calculate the area S_3, we can divide it into two parts: calculating the area of the triangle DEC formed by connecting points D and C (S_{3a}), and through an integral calculation (S_{3b}), as shown below:

$$S_3 = S_{3a} + S_{3b} \quad (242)$$

$$S_{3a} = A(DEC) = \frac{1}{2} \mid (x_d y_e + x_e y_c + x_c y_d) - (x_e y_d + x_c y_e + x_d y_c) \mid \quad (243)$$

$$S_{3a} = A(DEC) = \frac{1}{2} \mid x_d m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right) + \sqrt{r^2 - x_c^2} + \frac{m^2 x_c + x_d + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1}$$

$$\sqrt{r^2 - x_c^2} + x_c\sqrt{r^2 - x_d^2} - \frac{m^2 x_c + x_d + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \sqrt{r^2 - x_d^2} - x_c m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right)$$

$$- \sqrt{r^2 - x_c^2} - x_d\sqrt{r^2 - x_c^2} \mid \quad (244)$$

By examining the steps in equation (236) and simplifying the integral, we obtain the expression below. This time, by subtracting the area of the trapezoid below the CD line segment from the integral over the interval, we define the region:

$$S_{3b} = I' - \frac{(y_c + y_d)(x_b - x_a)}{2} = \int_{x_c}^{x_d} \sqrt{(r^2 - x^2)} dx - \frac{(y_c + y_d)(x_b - x_a)}{2} \quad (245)$$

$$S_{3b} = r^2 \left(\frac{t}{2} - \frac{\sin 2t}{4} \right) \Bigg|_{\arccos\left(\frac{x_c}{r}\right)}^{\arccos\left(\frac{x_d}{r}\right)} - \frac{(y_c + y_d)(x_b - x_a)}{2} \quad (246)$$

$$S_3 = \frac{1}{2} \mid x_d m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right) + \sqrt{r^2 - x_c^2} + \frac{m^2 x_c + x_d + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \sqrt{r^2 - x_c^2} +$$

$$x_c\sqrt{r^2 - x_d^2} \quad - \frac{m^2 x_c + x_d + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \sqrt{r^2 - x_d^2} - x_c m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right) - \sqrt{r^2 - x_c^2}$$

$$- x_d\sqrt{r^2 - x_c^2} \mid + r^2 \left(\frac{\arccos\left(\frac{x_d}{r}\right) - \arccos\left(\frac{x_c}{r}\right)}{2} - \frac{\sin 2\arccos\left(\frac{x_d}{r}\right) - \arccos\left(\frac{x_c}{r}\right)}{4} \right) - \frac{\left(\sqrt{r^2 - x_c^2} + \sqrt{r^2 - x_d^2}\right)(x_b - x_a)}{2} \quad (247)$$

To calculate the area S_4, we can compute it in two parts: by calculating the area of the quadrilateral $CFAE$ formed by connecting points F and C (S_{4a}), and through an integral calculation (S_{4b}), as shown below:

$$S_4 = S_{4a} + S_{4b} \quad (248)$$

$$S_{4a} = A(CFAE) = \frac{1}{2} \mid \left(x_c y_f + x_f y_a + x_a y_e + x_e y_c\right) - \left(x_f y_c + x_a y_f + x_e y_a + x_c y_e\right) \mid \quad (249)$$

$$S_{4a} = \frac{1}{2} \mid m\sqrt{r^2 - x_d^2} + x_d m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right) + \sqrt{r^2 - x_c^2} + \frac{m^2 x_c + x_d + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1}$$

$$\sqrt{r^2 - x_c^2} - \left(r\sqrt{r^2 - x_c^2} + x_c m \left(\frac{x_d - x_c + m\left(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2}\right)}{m^2+1} \right) + \sqrt{r^2 - x_c^2} \right) \mid \quad (250)$$

$$S_{4b} = I'' - \frac{y_c\,(x_c - -r)}{2} = \int_{x_f}^{x_c} \sqrt{(r^2 - x^2)}dx - \frac{\sqrt{r^2 - x_c^2}\,(x_c + r)}{2} \quad (251)$$

$$S_{4b} = r^2 \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Bigg|_{\pi}^{\arccos\left(\frac{x_c)}{r}\right)} - \frac{\sqrt{r^2 - x_c^2}\,(x_c + r)}{2} \quad (252)$$

$$\begin{aligned} S_4 =& \frac{1}{2} \mid m\sqrt{r^2 - x_d^2} + x_d m(\frac{x_d - x_c + m\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2})}{m^2 + 1}) + \sqrt{r^2 - x_c^2} + \frac{m^2 x_c + x_d + m(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2})}{m^2 + 1} \\ & \sqrt{r^2 - x_c^2} \quad) - (r\sqrt{r^2 - x_c^2} + x_c m \frac{x_d - x_c + m(\sqrt{r^2 - x_d^2} - \sqrt{r^2 - x_c^2})}{m^2 + 1}) + \sqrt{r^2 - x_c^2}) \mid \\ & + r^2 (\frac{\arccos(\frac{x_c}{r}) - \pi}{2} - \frac{\sin 2\arccos(\frac{x_c}{r})}{4}) - \frac{\sqrt{r^2 - x_c^2}(x_c + r)}{2} \quad (253) \end{aligned}$$

After deriving these equations, we can define the function $F_7\,(m, x_c, x_d)$. In order for the areas to be equal, we can define this function as shown below, and then solve the problem using the particle swarm optimization method to find the values where the function equals 0:

$$F_7\,(m, x_c, x_d) = \left|S_1 - \frac{S}{4}\right| + \left|S_2 - \frac{S}{4}\right| + \left|S_3 - \frac{S}{4}\right| + \left|S_4 - \frac{S}{4}\right| \quad (254)$$

$$F_7\,(m, x_c, x_d) \longrightarrow 0 \quad (255)$$

(Used code for the problem 7 can be found in subsection 4.8)

4 Solution Codes

4.1 Code for PSO

```
import random
import time
import matplotlib.pyplot as plt
# ------------------------------------------------------------------------------
a: float = 6
b: float = 7
c: float = 10
def problem2(X):
   global a, b, c
   u = X[0]
   m = X[1]
   xd = (c * b) / (m * c + b - m * a)
   yd = m * xd
   xg = u * a / (m * b + a)
   yg = u * b / (m * b + a)
   xf = u / (m**2 + 1)
  yf = m * xf
   s = abs(c * b) / 2
   s1 = abs(xf * yg - xg * yf) / 2
   s2 = abs(u * yf) / 2
   s3 = abs((a * yg + xg * yf + xf * yd + xd * b) - (xg * b + xf * yg + xd * yf + a * yd)) / 2
   s4 = abs((xd * yf + c * yd) - (xf * yd + u * yf)) / 2
   q = s / 4
   return abs(s1 - q) + abs(s2 - q) + abs(s3 - q) + abs(s4 - q)

bounds = [(-20,20), (-20,20)] # upper and lower bounds of variables
nv = 2 # number of variables
mm = -1 # if minimization problem, mm = -1; if maximization problem, mm = 1


  # PARAMETERS OF PSO
particle_size = 120 # number of particles
iterations = 200 # max number of iterations
w = 0.8 # inertia constant
c1 = 1 # cognative constant
c2 = 2 # social constant

  # Visualization
fig = plt.figure()
ax = fig.add_subplot()
fig.show()
```

```
plt.title('Fonksiyonumuzun değerinin iterasyonla \verbdeğişimi')|
plt.xlabel("İterasyon")
plt.ylabel("Fonksiyonumuz")
# ------------------------------------------------------------
class Particle:
   def __init__(self, bounds):
   self.particle_position = [] # particle position
   self.particle_velocity = [] # particle velocity
   self.local_best_particle_position = [] # best position of the particle
   self.fitness_local_best_particle_position = inital_fitness
   self.fitness_particle_position = inital_fitness
   for i in range(nv):
   self.particle_position.append(random.uniform(bounds[1][0], bounds[i][1]))
      self.particle_velocity.append(random.uniform(-1,1))
  def evaluate(self, objective_function):
    self.fitness_particle_position = objective_function(self.particle_position)
     if mm == -1:
      if self.fitness_particle_position < self.fitness_local_best_particle_position:
        self.local_best_particle_position = self.particle_position # update the local best
        self.fitness_local_best_particle_position = self.fitness_particle_position
        # update the fitness of the local best
    if mm == 1:
      if self.fitness_particle_position > self.fitness_local_best_particle_position:
        self.local_best_particle_position = self.particle_position # update the local best
        self.fitness_local_best_particle_position = self.fitness_particle_position
        # update the fitness of the local best
  def update_velocity(self, global_best_particle_position):
     for i in range(nv):
       r1 = random.random()
       r2 = random.random()
       cognitive_velocity = c1 * r1 * (self.local_best_particle_position[i] - self.particle_position[i]
       social_velocity = c2 * r2 * (global_best_particle_position[i] - self.particle_position[i])
      self.particle_velocity[i] = w * self.particle_velocity[i] + cognitive_velocity + social_velocity
  def update_position(self, bounds):
    for i in range(nv):
       self.particle_position[i] = self.particle_position[i] +
       self.particle_velocity[i]
       # check and repair to satisfy the upper bounds
      if self.particle_position[i] > bounds[i][1]:
        self.particle_position[i] = bounds[i][1]
       # check and repair to satisfy the lower bounds
       if self.particle_position[i] < bounds[i][0]:
         self.particle_position[i] = bounds[i][0]
class PSO:
   def __init__(self, objective_function, bounds, particle_size, iterations):
     fitness_global_best_particle_position = inital_fitness
     global_best_particle_position = []
     swarm_particle = []
     for i in range(particle_size):
       swarm_particle.append(Particle(bounds))
     A = []
     for i in range (iterations):
       for j in range(particle_size):
         swarm_particle[j].evaluate(objective_function)
         if mm == -1:
           if swarm_particle[j].fitness_particle_position <  fitness_global_best_particle_position:
             global_best_particle_position = list(swarm_particle[j].particle_position)
             fitness_global_best_particle_position = float(swarm_particle[j].fitness_particle_position)
         if mm == 1:
           if swarm_particle[j].fitness_particle_position > fitness_global_best_particle_position:
             global_best_particle_position = list(swarm_particle[j].particle_position)
             fitness_global_best_particle_position = float(swarm_particle[j].fitness_particle_position)
       for j in range(particle_size):
        swarm_particle[j].update_velocity(global_best_particle_position)
        swarm_particle[j].update_position(bounds)
       A.append(fitness_global_best_particle_position)
       ax.plot(A, color="r")
       fig.canvas.draw()
       ax.set_xlim(left=max(0, i-iterations), right=i+3)
       time.sleep(0.01)
    print("Result:")
    print("Optimal Solution", global_best_particle_position)
```

```
    print("Objective function value:", fitness_global_best_particle_position)
    if mm == -1:
       inital_fitness = float("inf")
    if mm == 1:
      inital_fitness = -float("inf")
PSO(problem2,bounds,particle_size,iterations)
plt.show()
```

4.2 Problem 1

```
import ParticleSwarm as ps
# corner coordinates
a: float = 0
b: float = 4
c: float = 4.4723317897697
d: float = 0
def problem1(X):
  global a, b, c, d
  x1 = (-b * c + c * d + a * d + b * d) / (X[0] * a - X[0] * c - b + d)
  y1 = X[0] * x1
  x2 = (c * (d - b) + d * (a - c)) / (X[1] * a - X[1] * c + d - b)
  y2 = X[1] * x2
  s = abs(a * d - b * c) / 2
  s1 = abs(a * y1 - b * x1) / 2
  s2 = abs(x1 * y2 - x2 * y1) / 2
  s3 = abs(d * x2 - c * y2) / 2
  return abs(s1 - s / 3) + abs(s2 - s / 3) + abs(s3 - s / 3)
dimensions=2
dimension_bounds=[-6,6]
bounds=[0]*dimensions #creating 5 dimensional bounds
for i in range(dimensions):
  bounds[i]=dimension_bounds

#creates bounds [[x1,x2],[x3,x4],[x5,x6]....]

p=60 #shouldn't really change
vmax=(dimension_bounds[1]-dimension_bounds[0])*0.75
c1=2.8 #shouldn't really change
c2=1.3 #shouldn't really change
tol=0.00000000000001
ps.particleswarm(problem1, bounds,p,c1,c2,vmax,tol)
```

4.3 Problem 2

```
import ParticleSwarm as ps
a: float = 6
b: float = 7
c: float = 10
def problem2(X):
   global a, b, c
   m = X[0]
   u = X[1]
   xd = (c * b) / (m * c + b - m * a)
   yd = m * xd
   xg = u * a / (m * b + a)
   yg = u * b / (m * b + a)
   xf = u / (m**2 + 1)
   yf = m * xf
   s = abs(c * b) / 2
   s1 = abs(xf * yg - xg * yf) / 2
   s2 = abs(u * yf) / 2
   s3 = abs((a * yg + xg * yf + xf * yd + xd * b) - (xg * b + xf * yg + xd * yf + a * yd)) / 2
   s4 = abs((xd * yf + c * yd) - (xf * yd + u * yf)) / 2
   q = s / 4
   return abs(s1 - q) + abs(s2 - q) + abs(s3 - q) + abs(s4 - q)
```

4.4 Problem 3

```
import ParticleSwarm as ps
a: float = 5
```

```
b: float = 7
c: float = 8
d: float = 3
def problem3(X):
  global a, b, c,d
  u = X[0]
  v = X[1]
  m = X[2]
  x1 = (m**2 * u + v) / (m**2 + 1)
  y1 = m * (x1 - u)
  x2 = v * a / (b * m + a)
  y2 = x2 * (b / a**2)
  x3 = ((a - c) * (m * u + d) + c * (d - b)) / (m + d - b)
  y3 = m * (x3 - u)
  x4 = -m * u / (d / c - m)
  y4 = -d * m * u / (d - c * m)
  x5 = v * c / (d + c * m)
  y5 = d / c * x5
  s = (b * c - a * d) / 2
  s1 = abs(x4 * y1 + x1 * y2 - x2 * y1 - x1 * y4) / 2
  s2 = ((x1 * y4 + x4 * y5 + x5 * y1) - (x1 * y5 + x5 * y4 + x4 * y1)) / 2
  s3 = abs(x3 * y1 + x1 * y5 + x5 * d + c * y3 - x3 * d - c * y5 - x5 * y1 - x1 * y3) / 2
  s4 = abs(a * y2 + x2 * y1 + x1 * y3 + x3 * b - a * y3 - x3 * y1 - x1 * y2 - x2 * b) / 2
  q = s / 4
  return abs(s1 - q) + abs(s2 - q) + abs(s3 - q) + abs(s4 - q)
dimensions=3
dimension_bounds=[-6,6]
bounds=[0]*dimensions #creating 5 dimensional bounds
for i in range(dimensions):
   bounds[i]=dimension_bounds
#creates bounds [[x1,x2],[x3,x4],[x5,x6]....]
p=60 #shouldn't really change
vmax=(dimension_bounds[1]-dimension_bounds[0])*0.75
c1=2.8 #shouldn't really change
c2=1.3 #shouldn't really change
tol=0.00000000000001
ps.particleswarm(problem3, bounds,p,c1,c2,vmax,tol)
```

4.5 Problem 4

```
import ParticleSwarm as ps
a: float = 3
b: float = 5
c: float = 7
d: float = 4
e: float = 10
def problem4(X):
  u = X[0]
  v = X[1]
  m = X[2]
  global a, b, c, d, e
  x1 = (m**2 * u + m * v) / (m**2 + 1)
  y1 = m * (x1 - u)
  x2 = m * a * v / (b * m + a)
  y2 = (b / a) * x2
  x3 = ((c - a) * (-m * v - b) + a*d - a*b) / (d - b - m*c + m*a)
  y3 = m * (x3 - u)
  x4 = m * (v * c - e * v + e * d) / (d * m + c - e)
  y4 = d / (c-e) * (x4 - e)
  s = abs(e * d + b * c - a * d) / 2
  s1 = abs(a * y2 + x2 * y1 + x1 * y3 + x3 * b - a * y3 - x3 * y1 - x1 * y2 - x2 * b) / 2
  s2 = abs(u * y1 + x1 * y2 - x2 * y1) / 2
  s3 = abs(c * y4 + x4 * y1 - x1 * y4 - u * y1) / 2
  s4 = abs(x3 * y1 + x1 * y4 + x4 * d + c * y3 - x3 * d - c * y4 - x4 * y1 - x1 * y3) / 2
  q = s / 4
  return abs(s1 - q) + abs(s2 - q) + abs(s3 - q) + abs(s4 - q)
```

4.6 Problem 5

```
import ParticleSwarm as ps
xa: float = 0
ya: float = 2
xb: float = 0
yb: float = 0
xc: float = 3
yc: float = 0
xd: float = 2
yd: float = 2
yj: float = 3.2
xj: float = 1.5
xh: float = 1.2
def problem5(X):
 xe=X[0]
 xf=X[1]
 m=X[2]
 global xa, ya, xb,yb, xc,yc, xd, yd, yj, xj
 yh=(m*(xf-xe)) / (m**2 +1)
 yi=m*((xe*yd-xe*yj+yj*xd-yd*xj) / (yj-yd-m*xj+m*xd))
 xi=(m*xe*xd-m*xe*xj+yj*xd-yd*xj)/(yj-yd-m*xj+m*xd)
 xg=(xj*(xf-m*ya) / (m*yj-m*ya+xj))
 yg=((xf*yj-xf*ya-xj*ya) / (m*yj-m*ya+xj))
 sa=abs((xa*yb+xb*yc+xc*yd+xd*ya) -(xb*ya+xc*yb+xd*yc+xa*yd))/2
 sb=abs((xj*ya+xd*yj) - (xd*ya+xj*yd)) / 2
 s=sa+sb
 s1a=abs(xe*(m*(xf-xe)/(m**2+1))+ (m**2 * xe + xf) / (m**2+1)* ya) / 2
 s1b=(abs((xj*(xf-m*ya))/(m*yj-m*ya+xj)*ya + ((m**2 * xe + xf)/(m**2 +1)) * (xf*yj-xf*ya-xj*ya)
     /(m*yj-m*ya+xj)-(m**2 * xe + xf)/(m**2 + 1) * ya +(xj*(xf-m*ya))/(m*yj-m*ya+xj) * (m*(xf-xe))/
     (m**2 + 1)) / 2)
 s1 = s1a+s1b
 s2=abs((m*(xf-xe))/(m**2 + 1) * (xf-xe)) / 2
 s3a=(abs(xc*yd + (xd*(m*(xf-xe) / (m**2 + 1))) - (xf* (m*(xf-xe) / (m**2 + 1))) + (m**2 * xf - xe) /
     (m**2 + 1) * yd) / 2)
 s3b=abs((xi*yh + xh*yd + xd*yi) - (xh*yi + xd*yh + xi*yd))/2
 s3=s3a+s3b
 s4=abs(xj*yg + xg*yh + xh*yi + xi*yj) - (xg*yj+xh*yg+xi*yh+xj*yi) / 2
 return abs(s1-(s/4)) + abs(s2-(s/4)) + abs(s3-(s/4)) + abs(s4-(s/4))
```

4.7 Problem 6

```
import ParticleSwarm as ps
a: float = 3
b: float = 0
c: float = 3
def problem6(X):
 m = X[0]
 xf = X[1]
 xg = X[2]
 global a, b, c
 xd= -m+sqrt((m**2+4(m*xf+a**2)))/2
 yd= m*xd/2
 ye= xg/m
 xh=((m**2)*xf+xg)/((m**2)+1)
 yh=(m(xg-xf))/(m**2+1)
 s = 2* (a**3) / 3
 s1=abs((xf*(m*(xg-xf)/(m**2 + 1))) + (m**2 * xf + xg)/(m**2 + 1)*(xg/m))/2
 s2=abs((xg* (m(xg-xf))/(m**2+1)) - (xf * (m(xg-xf)/(m**2+1))))/2
 s3a=(abs((((-m+sqrt(m**2+(4*(m*xf+a**2))))/2)* (m*(xg-xf)) / (m**2+1) + (m*a*(-m+sqrt(m**2 + 4 *
     (m*xf + a**2)))/2 - xf)) - (((m**2 * xf + xg) / (m**2+1)) * m * (-m * sqrt((m**2+ 4 *
     (m * xf + a**2)))/ 2 - xf)) + xg * (m*(xg-xf)/(m**2+1))))
 s3b=(2*(a**3)/3 - (a**2 * (-m + sqrt(m**2 + 4 * (m*xf + a**2))) / 2 + ((-m + sqrt(m**2 + 4*(m*xf+ a**2
     / 3) - m * a((-m + sqrt(m**2+4*(m*xf+a**2))) / 2 -xf) + (m * ((-m+sqrt(m**2 + 4*(m*xf+ a**2))) / 2
     ((-m+sqrt(m**2+4*(m*xf+a**2)))/2))/2)
 s3=s3a+s3b
 s4a=(abs((m**2 * xf + xg)/ (m**2+1) * m *( (-m + sqrt(m**2 + 4 * (m * xf + a ** 2))) / 2 - xf) +
     (-m + sqrt(m**2 + 4 * (xf+ a**2)))/ 2* a**2 - ((m**2 * xf + xg)/ (m**2+1) * (xg/m) +
     ((-m+sqrt(m**2+4*(xf+a**2)))/2 * (m * (xg-xf)) / (m**2+1)))))
 s4b=(a**2 * (-m+sqrt(m**2+ 4 * (m*xf+a**2)))/2 - (((-m+(sqrt(m**2+ 4 * (m*xf+a**2)))) / 2) / 3)  ) -
```

```
    (((a**2+m * (-m+sqrt(m**2+ 4 *(m * xf + a**2))) / 2 - xf)) * ((-m+sqrt(m**2+4*(m*xf+a**2))) / 2)/2
 s4=s4a+s4b
 return abs(s1-(s/18)) + abs(s2-(s/9)) + abs(s3-(s/6)) + abs(s4-(2*s/3))
```

4.8 Problem 7

```
import ParticleSwarm as ps
import numpy
import math
r = 5
def problem7(X):
 m = X[0]
 xc = X[1]
 xd = X[2]
 yc = -(math.sqrt(r**2 - xc**2))
 yd = math.sqrt(r**2 - xd**2)
 xe = (m**2 * xc + xd + m * (math.sqrt(r**2 - xd**2) - math.sqrt(r**2 - xc**2))) / (m**2 + 1)
 ye = (m * ((xd - xc + m * (math.sqrt(r**2 - xd**2) - math.sqrt(r**2 - xc**2))) / (m**2 + 1)) +
      math.sqrt(r**2 - xc**2))
 xa = m * math.sqrt(r**2 - xd**2) + xd
 xb = -(math.sqrt(r**2 - xc**2)) / m + xc
 s1 = abs(xa * ye - xb * ye) / 2
 s2a = abs((xd * ye + r * yd) - (xe * yd + xb * ye)) / 2
 s2b = r**2 * (numpy.arccos(xd / r) / 2 - math.sin(2) * numpy.arccos(xd / r) / 4) - yd * (r - xd) / 2
 s2 = s2a + s2b
 s3a = abs(xc * ye + xe * yd + xd * yc - (xe * yc + xd * ye + xc * yd)) / 2
 s3b = (r**2 * ((numpy.arccos(xb / r) - numpy.arccos(xa/ r)) / 2 - math.sin(2) * (numpy.arccos(xb / r)
       math.sin(2) * numpy.arccos(xa / r)) / 4) - (math.sqrt(r**2 - xc**2) + math.sqrt(r**2 - xd**2)) /
       2 * (xb - xa))
 s3 = s3a + s3b
 s4a = abs(((math.sqrt(r**2 - xd**2) + xd) * ye + xe * yc) * (r * yc + xc * ye)) / 2
 s4b = r**2 * ((numpy.arccos(xc / r) - math.pi) / 2 - math.sin(2) * numpy.arccos(xc / r) / 4) -
       math.sqrt(r**2 -xc**2) * (xc + r) / 2
 s4 = s4a + s4b
 q = math.pi * r**2 / 8
 return abs(s1 - q) + abs(s2 - q) + abs(s3 - q) + abs(s4 + q)
 bounds = [(-10,0), (-r + 0.01, -0.01), (0.01, r - 0.01)]
 # upper and lower bounds of variables
nv = 3
```

5 Conclusion

The problems we have tackled in our study can be developed further and adapted to address any kind of problem that requires optimization of different variables. The initial values in the 3rd, 4th, and 5th problems can be refined for different sets of values. The parabolic function in the 6th problem can be solved for different equations or adapted to other problems requiring integral calculations. Investigating the initial values in these problems across different ranges will alter both the resulting geometric pattern and the equations required for the solution. Similarly, in physics or engineering problems, after defining variables and equations, solutions can be found using appropriate function definitions.

As seen in sections 7 and 6, since the application of shapes requiring integration is quite practical, it can be attempted for various irregular shapes with known equations.

Additionally, it has been demonstrated that the number of iterations plays a crucial role in approaching an accurate solution The method employed in our study has been transformed into a design applicable not only in these fields but also in complex systems and problems involving multiple variables.In essence, the approach used in our study can be extended and applied to a wide range of scenarios that involve optimization, equation solving, and pattern generation across different fields.

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Pre & Post Pandemic Formative Assessments Methodologies to Improve Mathematics Learning for University Students

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Abstract: Formative assessments help teachers identify concepts that students are struggling to understand, skills they are having difficulty acquiring, or learning standards they have not yet achieved so that adjustments can be made to lessons, instructional techniques, and academic support. This paper focuses on a study at the American University of Sharjah, Mathematics Department of integrating the use of technology with formative assessments in order to identify students who are struggling and focus on specific points prior to exams rather that doing a general review. More specifically, the study was conducted over two different semesters. The first semester (before the pandemic), classroom response systems with Poll Everywhere was used while in the second semester (after the pandemic), ILearn Formative Feedback was used. The uniqueness of this research study is the integration of formative assessments and feedback with technology in the delivery of Mathematics Subjects in Higher Education. Both quantitative and qualitative results were collected and there was evident and significant improvements in students' performance when the proposed formative feedback prior to exams was used. It was also apparent that after the pandemic, the use of Ilearn formative feedback was very beneficial for the students and helped improve their performances significantly in Mathematics.

Keywords: Formative Assessments, Formative Feedback, Ilearn, Learning Objectives, Lecture Capture, LMS, Mathematics Teaching, Poll Everywhere, Technology in Education

1 INTRODUCTION

During the last decade, researchers and academics have shown a great prerequisite to reengineer mathematics education to move away from the lecture-homework format to a more technology centric innovative approach focused on student needs. When developing technological services to support students in higher education, it is crucial to account for flexibility, diversity, and time-saving in options. Lecture Capture (LC) encompasses these criteria, and many institutions around the globe are currently using it [4]; a lot of research is ongoing about its present-day use and growth [3]. Such research led towards the latest, innovative advancement in LC which mainly includes interaction. Although there are many recognized pros for LC [1–4], it does not replace physically attending a lecture; it could only be a supplementary learning aid to students. To overcome the limitation of the requirement of students to attend class to participate in activities that can only be administered there, the new, interactive advancement in LC allows student participation via captured lectures similarly to classroom interaction. A major disadvantage in using video lectures is the lack of customized feedback or focus review sessions. There is an evident need in order to combine video lectures with customized focused material based on student feedback.

LC is a hardware/software process that involves recording classroom sessions and storing the recordings digitally to make them available electronically for students to watch the entire lecture, and LC is becoming increasingly popular in universities around the world. However, merely recording an entire lecture and uploading it is not a precisely effective learning method [4]. The current era is more student oriented and demands a greater focus on students by having them constantly engaged during class. To have LC in classrooms comply accordingly, students are able to see lecturer notes and explanations on a captured video, to answer lecturers' quizzes and questions relevant to a video, to search for keywords that will refer to a part of the given recorded lecture, and to access the published lectures anywhere from any device. ***This paper explores the impact and effects on students' academic performance of innovative teaching techniques that combines online review sample exams prior to the course exams with customized focused in-class recorded review sessions based on students results in the online sample exams.***

Our proposed teaching methodology bridges the gap between live lectures and current, non-interactive video lectures. Because current LC is widely supported and proven extremely effective, customized review video sessions based on student results has magnified the positive effects in a directly proportional manner. These positive effects are showcased in this paper. More specifically, this paper reports on the results of a one semester study in a second year Mathematics subject (Quantitative Methods) in which a traditional lecture course (**TC**) was used in one section of the course and a traditional lecture combined with customized pre-exam review video lectures based student results from and online sample exam (**CC**).

The research objectives of the study were the following:

1. To analyze students acceptance of the proposed teaching methodologies

2. To evaluate how CC affects student understanding of course material and their respective performance compared to traditional TC systems.

2 RELATED WORK

LC technology is becoming a more integral part of the digital classroom. To improve this aspect of digitalized education, a certain set of criteria has to be met. Essentially, in basic terms, LC is a technological setup where lecturers record a lecture and upload it online for students to access anytime, anywhere for review. This means that LC is for both students who have attended the corresponding in-class lecture and students who have not attended that class; it serves as a supplement to attending class while providing the flexibility of missing a class sometimes by blending with the lecturers' workflows combining learning and LC. A pronounced LC system should have a feedback method with analytical capabilities for lecturers to examine how students use the uploaded content to determine the motive behind students repeatedly watching a particular video: interesting video or struggle to comprehend a concept. This method is similar to how a lecturer can assess student reactions through body language, hands raised, etc. and by conducting surveys [2]. Moreover, when integrating an LC system in an institution, it should be of maximum compatibility with the existing technology in that institution, by making use of the apparatus and software already available instead of investing in additional equipment, to lessen capital expenditures. Compatibility is also important when regarding the platform through which a student would view the video; an LC recording should be available on all operating systems and computers/tablets/mobiles [3]. The innovative LC discussed in this paper fulfills all the aforementioned criteria of a distinct LC system.

A web seminar at the Seattle Pacific University discusses innovative ways of using LC relative to the LC system discussed in this paper. One lecturer highlighted several aspects that Seattle Pacific focused on to capture lecture content: an LC software, namely Camtasia, is used to record the lectures; the faculty personnel administering LC are familiar with it; LC recordings resolve the issue of absences by students and lecturers: students who could not attend a class are able to watch LC videos and lecturer absences can be substituted for by giving students access to previously recorded videos of the same lecture; LC recordings facilitate lecturer and student time management. Another lecturer has stated the importance of LC in providing a one-to-one experience in addition to uploading 20 minute videos. A third lecturer pinpointed further uses of an LC platform: quizzes can be taken online; communication takes place in a medium students of the technology age are quite familiar with. The LC used in this study uses Screencast instead of Camtasia, so it requires minimal, readily accessible resources making the technology with marginal technical support facets; it is also used by personnel familiar with the technology, resolves any absences, and enhances time management. Instead of 20 minute segments, LC uploads have an auxiliary add-on to the video that allow students to traverse to any point in the lecture-length videos using a tag or a keyword. Students can also take quizzes administered online [4].

Other currently available LC systems allow the option of lecturers customizing the library of LC recordings available to students in the same way the LC system used in this study has a library of LC recordings in addition to a search option. Besides the search option, the LC system has a tagging aspect to facilitate locating any second through a video lecture as opposed to the use of short videos to explain a concept used in other LC systems, none of which have the aforementioned tagging option. Also, similarly to how Eastern New Mexico University arranged for foreign students LC videos, to learn English, that they can access and study from before attending university, the LC system used in this study can be accessed anytime from any platform using any device and operating system enabling students to study the LC material if they are abroad [1].

A research at Queen's University Belfast [5] presented a number of figures to prove that LC is operative in supplementing the learning experience in a university. Contrary to the popular misconception that providing students with LC videos would decrease lecture attendance, only 27% of the surveyed agreed. Also, 98% of the students said that viewing those videos is integral prior to assessments. Another report [6] stated that the purpose of LC from a student point of view is to make up for missed lectures and to review lectures as a preparation for assessments along with the possibility of aid for students with learning disabilities; students also preferred a mixture of LC, live lectures, course materials, and additional classes. Yet, a third research [7] ascertained that LC generally accompanies better test scores, it is effective in fact-focused courses that do not involve discussions, and it provides an overall heightened learning comprehension of the course material.

In recorded lectures viewing, students usually follow one of four types of viewing strategies [8]: Linear Watch where students would watch everything in one uninterrupted pass; Elaboration Watch which would come after an initial linear pass; Maintenance rehearsal where selected sections are watched repeatedly; and Zapping where student would skip through the lecture and watch short sections only. Students usually use a combination of those viewing behaviors in order to grasp the concepts. This is problematic in many cases as the student might get bored or demotivated after several passes looking for particular concepts he had difficulty understanding. Some researchers have tried to use a hybrid approach and introduced the idea of a Flipped (or inverted) Classroom [9] which is the practice of recording lectures and distributing them electronically to students to watch at their convenience before contact time. More recently, the authors in [10] also investigated the effects of in-video quizzes on a flipped classroom environment. There are evident benefits of this approach is that contact time can be used for something more interactive than content delivery however such approaches are not applicable to all subject deliveries which require in-class physical interaction with the lecturer with hands on exercises. The inability for students to receive feedback while viewing the recorded lectures or search for a particular concept and get instant results (without zapping multiple times over a set of video lectures) remains a key limitation of using video delivery.

In order to enrich the learning experience provided by video lectures [11], the authors decided to trial the use of the in-video quiz approach. The approach involved presenting automatically assessed quiz questions within electronically recorded lectures and programming demonstrations. An intended benefit of making videos interactive in this way is that knowledge acquisition is no longer passive, but an active process, with an opportunity for students to test their understanding and get feedback periodically during consumption of the content. They distinguish in-video quiz questions from a post video quiz in a number of ways. In-video quiz questions: are designed to appear, and be answered, during normal video playback, with the video automatically pausing for the student to answer the question. It is clear that the technique of using in-video quizzes is not novel since they are used by some of the larger MOOC platforms such as Coursera [12] and have been trialled as part of other flipped classroom style investigations [13]. To the best of our knowledge, there is no academic work focusing on the combination of online sample exams prior to the real exams and providing customized in-class recorded review sessions that are based on the results of the online sample. The ability to focus only on the concepts that were not fully understandable by the students as evident from the online sample exam provides a number of benefits. One of the most important of these is the ability to quickly deliver feedback to students. This allows students to take some form of corrective action if necessary to support the learning process. The data available after students have engaged with the lecture material are also used to improve support for individuals in face-to face sessions, or indeed, to identify common issues that can be addressed in later video sessions. Adding to that, students will have the motivation to watch the customized recorded video sessions as they focus only on their mistakes and they do not have to waste any time on concepts they already knew.

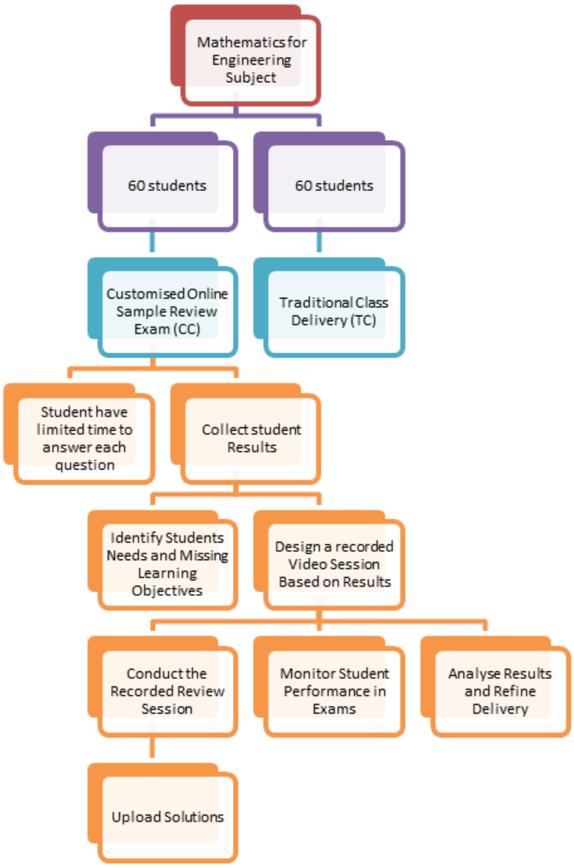


Fig. 1: Flowchart depicting our research methodology

3 RESEARCH METHODOLOGY

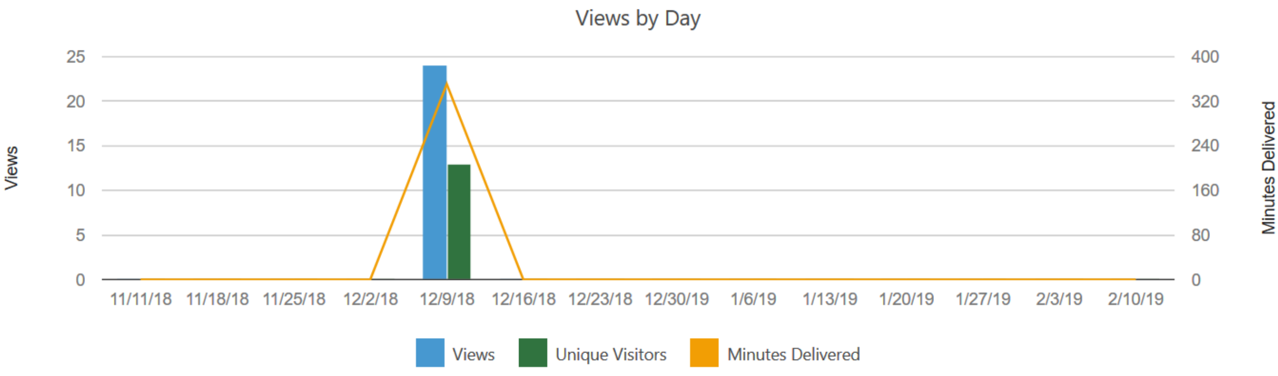
To prove the positive impact of our proposed teaching and learning methodology, a class of 120 students was sampled in 2018/2019 in the Mathematics Department at the American University of Sharjah (AUS), UAE and another set of 2 classes of 80 students were sampled in spring 2023 in the Mathematics Department at the American University of Sharjah (AUS), UAE. The subject under study was Math001. This course emphasizes the basic algebraic skills and techniques. Topics included were real and complex numbers, basic arithmetic, equations and inequalities, study of functions, polynomial and rational functions, exponential and logarithmic functions, trigonometric functions, and introduction to limits.

More specifically, upon completion of the course, students will be able to:

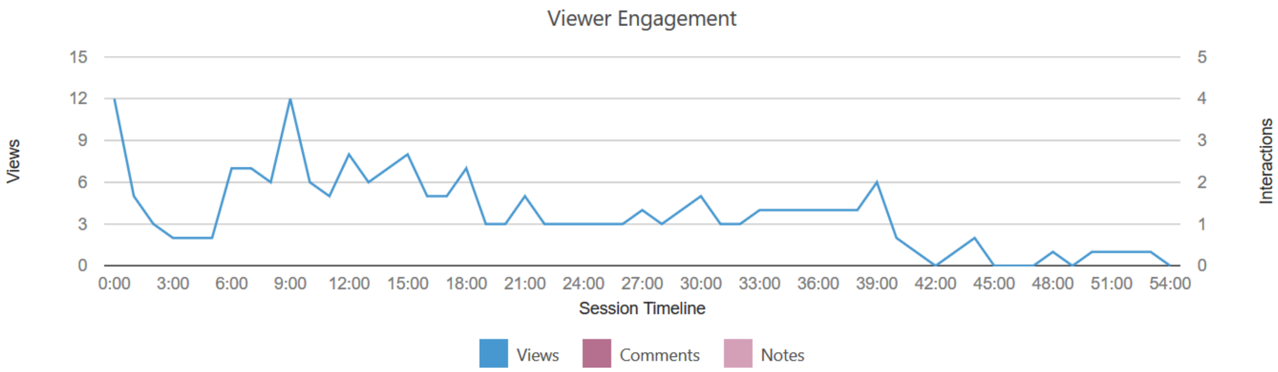
1. Develop the basic properties of real and complex numbers.
2. Solve, rational, radical equations and polynomial inequalities.
3. Define the basic concepts of functions, the concepts of domain and range, and composition of functions and sketch functions by transformation.
4. Find the inverse of a function, if exists, and use it to define and sketch the graph of logarithmic and exponential functions and solve equations with exponential and logarithmic expressions.
5. Sketch trigonometric functions and identify domain, periods, amplitudes, and define some basic trigonometric identities

The learning objectives above are a major part of the whole design of course delivery based on the proposed techniques. The class was equipped with lecture capture technology that allows the lecturer to record lectures when needed combining the lecture slides with audio/video recordings. Throughout one semester, student behavior and performance as well as technology stability were monitored, and data was collected. In addition, a survey of the lecture content and lecturer delivery was conducted to receive 60 student reviews. The collected data was cleared of blank records and exported to SPSS to perform statistical analysis to determine the statistical significance of the data by finding associations among different items of the survey and the criteria that pertains to overall satisfaction with the technology when they are tabulated against each other. There were a total of four sections (of 30 students each) and half of the students were taught using tradition course deliveries (PowerPoint, in-class exercises, uploaded notes) while the other half started up till Exam1 using traditional course delivery and shifted to customized course delivery after Exam 1 using our proposed delivery model that encompasses the use of online sample exams prior to a focused recorded review lecture on students weakness. Our research framework is presented in Fig.1. To conduct the online sample exam, *Poll Everywhere* software was used. It is an online service for classroom response and audience response systems and has many interactive features that were used by the instructor to analyze student feedback.

Also, to prove the positive impact of Sample Exams introduction, 2 classes of 80 students were sampled in spring 2023 in the Mathematics Department at the American University of Sharjah (AUS), UAE. The subject under study was also Math001. A sample exam was posted on ilearn about the material for exam 2. The sample exam had 17 questions with a limited time period of 75 minutes same as the time of the real exam 2. A recorded review class was done focusing on students' mistakes. 40 students out of 80 students did the sample exam and the results



(a)



(b)

Fig. 2: (a) Views per day and (b) Viewer Engagement

were collected. A recorded review class was delivered in class for the whole 80 students. The review session focused on the mistakes done in the sample exam.

4 RESULTS AND STATISTICS

This new, innovative method of CC has been largely perceived as a useful method to aid students' understanding of lectures for a given course and its assessments prior to exams as noted by the students. A main feature is that students can revise the exam content at their own pace focusing only on the mistakes they had in the sample review exam which helps enhance revision skills. The following highlight some of the results of our proposed methodology in terms of usability, student acceptance, and finally effects on student performance compared to a traditional course delivery.

4.1 Usability & Viewer Engagement

To evaluate the usability of the online sample exam and corresponding video review lectures uploaded online, students were tracked online to identify how many viewed the material. At the end of the semester, the results showed that although students generally missed a few face-to-face lectures, the review video lectures online were accessed a total number of more than 100 sessions throughout the course of the semester. It can be concluded that students used the online review material to further understand concepts they were lectured about and more specifically material they had difficulty in as the review sessions were very focused on student mistakes and were not general. Also, 45/60 (75%) of the students conducted the sample Exam 2 online using poll-everywhere while 38/60 (64%) conducted the online sample final exam review.

Regarding the recorder review sessions and viewer engagement, it can be noted from Fig. 2. that there was a peak viewership just before the exams on 12/9/2018 and also a very interesting observation can be noted in Fig. 2 (b) which indicates the number of views per minute of video decreased significantly indicating that students only watch parts of the video they find interesting and also shorter (2-3 minutes) videos per learning objective is more efficient compared to a one hour recorded video.

4.2 Student Acceptance

A survey was conducted to evaluate student perception of the proposed methodology and it was evident that most students who were involved in the customized sessions and online sample exam were satisfied. More specifically, as shown in Fig 3, most students strongly agree that the proposed methodologies had too much effect on their understanding of the subject in both Exam 2, and Final Exam. More than 80% of the students believed that the customized delivery of the sample review exam combined with the in class review had either too much or medium effect on their understanding and performance in the subject under study.

Also, when asked which mode of delivery they preferred, 92% of the students indicated that they preferred the customized review sessions combined with online sample exam compared to the traditional mode of delivery.

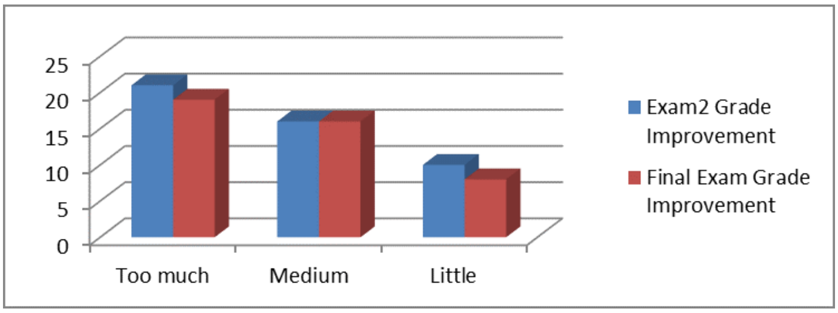


Fig. 3: Students' perception of the proposed methodologies

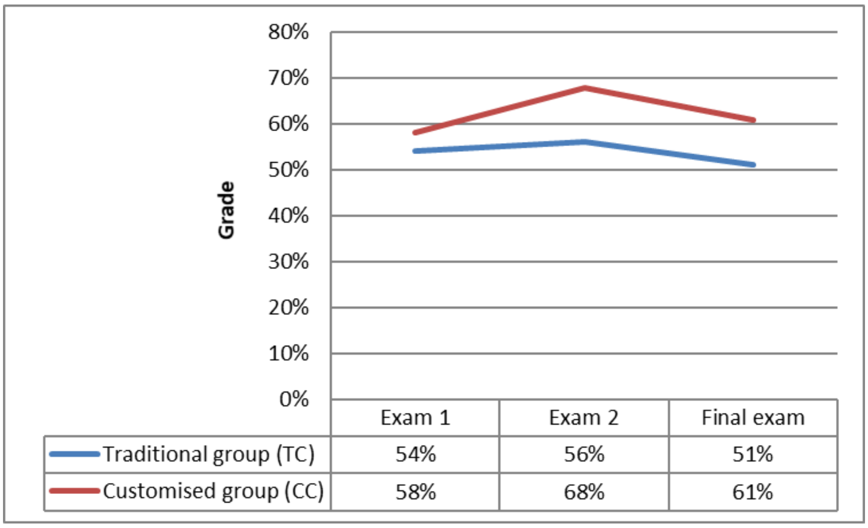


Fig. 4: Comparison between TC and CC in terms of student exam performance

4.3 Effect on Student Performance (1)

Student grades were monitored in all of Exam 1, Exam 2 and Final Exam for both sections (one with a traditional way of teaching (TC), and one with the proposed delivery (CC)) and the average student grades are shown in Fig 4.

As can be noted from Fig. 4, the average exam grades from Exam1, Exam 2 and Final Exam using our proposed methodologies outperformed the traditional mode of delivery performance although based on prior student GPA before they enrolled in the subject both groups showed similar averages and standard deviations. Also, a significant observation is the performance of the same group of students in Exam 1 compared to Exam 2 where in Exam 1, traditional techniques were used while in Exam 2, the customized methodologies were used. It is evident that students performed better in Exam 2 considering the added value the proposed methodologies had to offer allowing both students and teachers to focus only on their mistakes in the online sample exam.

Student qualitative feedback regarding the effects the proposed techniques had on their improvements was also collected in a survey at the end of the semester and some notable comments included:

- "It was good and gave me more experience and confidence to solve the exam."
- "It is a great way to test your readiness before entering the exam hall and learning from your own mistakes."
- "It helped me practice an exam style questions focusing only on what I don't know. This improved my overall grade."
- "Practice made it perfect. It helped me understand more."
- "The one in class helped a lot, unlike the one at home."
- "It helped me in solving difficult equations."

4.4 Effect on Student Performance (2) – Sample Exam

The introduction of sample exams with follow up sessions focusing on students mistakes was investigated during Spring 2023 on a sample of 80 students with 40 students taking part in the sample exam and the results are discussed below.

A sample of three questions in both the sample exam and similar questions in the real exam are shown below followed by the correct answer and the percentage of Students with Correct Answers.

4.4.1 Sample Exam Performance:

1. Find the inverse of $f(x) = \frac{x-9}{x-6}$?

   Correct answer: $f^{-1}(x) = \frac{9-6x}{1-x}$

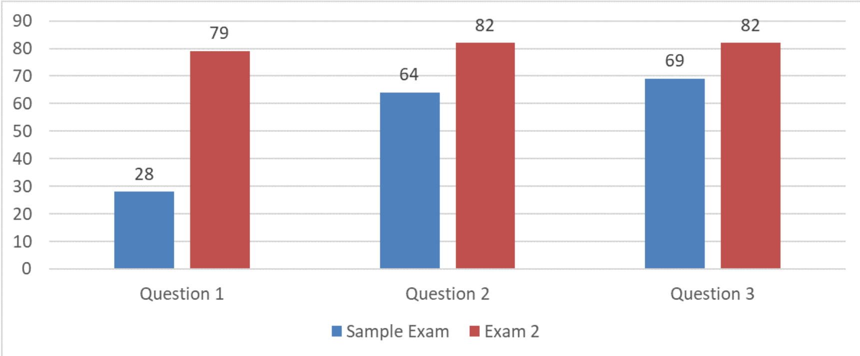


Fig. 5: Effects on Student Performance after the introduction of Sample Exam Formative Feedback.

% of Students with correct answers: 28%

2. $f(x) = \frac{3}{x^2}$, $(x) = \sqrt{x - 19}$, find the domain of $fog(x)$?

   Correct answer: $x > 19$

   % of Students with correct answers: 64%

3. $f(x) = log(x - 2) + log_3(11 - x)$, find the domain of f(x)?

   Correct answer: $(2, 11)$

   % of Students with correct answers: 69%

4.4.2 Real Exam 2 Performance:

1. Find the inverse of $f(x) = e^{x-2} + 1$?

   Correct answer: $f^{-1}(x) = ln(x - 1) + 2$

   % of Students with correct answers: 79%

2. $f(x) = \sqrt{x^2 - 5}$, $(x) = \sqrt{x}$, find the domain of $fog(x)$?

   Correct answer: $[0, 5]$

   % of Students with correct answers: 82%

3. $f(x) = log_2(x - 3)$, $g(x) = \sqrt{x}$, find the domain of $f(x) + g(x)$?

   Correct answer: $(3, \infty)$

   % of Students with correct answers: 82%

It is evident that student's performance was improved significantly on the three question types after going through the formative assessment stage as summarized in Fig. 5.

Comparing the performance of students, who did the sample exam with those who did not, also validates the effectiveness of the introduction of formative assessments in the form of sample exams on improving students' performance as shown in Fig. 6. The Exam 1, Exam 2 and Final average for students who did the sample exam was 62%, 70%, and 60% while the Exam 1, Exam 2 and Final average for all students was 53%, 62%, and 53% which was lower in all three exams. This highlights the fact that the students who did the sample exam performed significantly better than those who did not.

5 Conclusion

In conclusion, several observations are made from this study. Technology has invaded our education space in different ways and educators need to utilize technology to suit their own teaching styles. Also, students tend to accept technology if it is simple to use, more direct and gives them an incentive. This paper discusses the positive effects the use of technology had in a Mathematics Subject for Engineers. More specifically, the use of online polling systems as a sample home exam with time constraints allowed the instructor to gather insights regarding concepts students had difficulty with prior to the exam itself. Those insights were used to conduct an in-class recorded review session that focused on student mistakes. Both the video and the solutions were uploaded for the students. This methodology showed significant improvements in student performance in corresponding exams and show great promise for a wider deployment across all subjects. Also, a significant observation is the

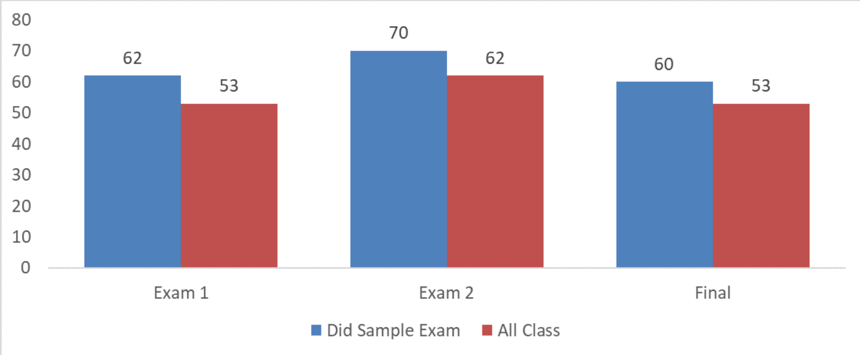


Fig. 6: The exam average of students who did the Sample Exam compare to the whole class in Exam 1, Exam 2, and Final.

length of the recorded videos as many students did not make it towards the end of the video and skipped though different sessions. Future work need to explore the use of shorter 1-2 minutes videos focusing only on specific learning objectives.

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Embedding the stationary spacetimes into Brans-Dicke cosmology via conformal transformations

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Abstract: A conformal transformation of a static or stationary spacetime by a time dependent conformal scale factor $S(\tau)^2$ is one of the methods of producing a cosmological spacetime. Using this knowledge and Brans-Dicke (BD) field equations, we investigate two time dependent metrics, including Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime and conformally transformed Kerr-Newman black hole, and we obtain solutions that allow different expansion rates for each geometry. These expansion rates depend on the matter content of the conformally transformed geometry. We state that the BD scalar field yields accelerated expansion of the conformal spacetime if the original metric has vacuum geometry, and no acceleration if the original spacetime has some curvature or matter content in it.
Keywords: Cosmology, FLRW spacetime, Kerr-Newman black hole, Scalar tensor theories.

1 Introduction

The expansion of universe was observed by Hubble in 1920s and also recently discovered that the universe is not only expanding but also accelerating [1, 2]. Observational evidence, coming from the type Ia Supernova explosion, implies that the correct spacetime geometry must be nonstatic both in astrophysics and cosmology. Since gravitational interactions can be well described in the General Theory of Relativity, to understand the structure and behavior of the universe, we use the General Relativistic formulations and pseudo-Riemannian geometry as a mathematical tool. On sub-galactic regions like the Solar system, the effects of gravity are not strong enough and spacetime can be characterized as nearly flat. In these scales, the General Theory of Relativity (GR) has been accurately tested and verified [3–8]. However, when we use GR in cosmology especially at the large scale structures or for the evolving universe in time, we need some new phenomena which we have not understood and explained theoretically and observationally yet, for example, dark matter and dark energy [9, 10]. Also, black holes provide strong gravitational fields and there are large deviations from GR at high field strength [11–14]. This means that, for a more general theory of gravitation, we need to understand the strong gravitational regimes and the large scale structure of the universe [15–17]. Brans-Dicke (BD) scalar tensor theory is a well known scenario of gravitational field [18–21], and in general, it is considered as an alternative theory to GR. In our sense, this is not an alternative to GR but a more general theory of gravity and it can be related with the $f(R)$ theory [22–24], string theory [25–27] and Kaluza Klein theory [28, 29] in the appropriate limits. Also, BD theory involves Mach's principle which says that all of the matter in the universe affects each other, hence a universe, filled with a scalar field, might be a reasonable candidate for this interaction of the masses. Therefore, motivated to find the solutions for an accurate cosmological model and also a theory for highly gravitating cosmological environment, it might be convenient to study BD theory of gravity.

In general, the cosmological expansion in time has been defined by a time dependent scale factor in front of the spatial part of the metric components as in the FLRW metric. For more general cases, we do not restrict ourselves with the FLRW case, we can also include Kerr-Newman black hole. Using the method of conformal transformation by rescaling static or stationary spacetime, we try to produce a cosmological model for asymptotically non-flat cosmological black holes. In this context, conformal transformation of Kerr-Newman black hole also provide to obtain inhomogeneities in the FLRW backgrounds. BD scalar tensor theory adds the system a scalar degree of freedom which is represented by a scalar field ϕ. Using this property, we find a relation between the BD scalar field ϕ and the conformal scale factor $S(\tau)$ in which the transformed spacetime has the stress-energy tensor for a perfect fluid. This means that the stress-energy tensor arises from the curvature and matter content of spacetime, and also that the expansion parameter is closely related to the scalar field ϕ. Also, in reference [30], the author works on the scalar field and perfect fluid and concludes that the scalar field and a perfect fluid are not equivalent but a convenient correspondence for the formal purposes.

A stationary/static submanifold can be embedded in a cosmological background by a scale factor of $S(\tau)$ as,

$$\tilde{g}_{ab} = S(\tau)^2 g_{ab}. \tag{1}$$

This expression is a conformal transformation of a metric tensor with a conformal factor $S(\tau)^2$. In the rest of the paper, the metric g_{ab} will be called as original metric or submanifold $\mathcal{M}$ and chosen as static or stationary spacetime, and the transformed metric $\tilde{g}_{ab}$ will be named as conformally rescaled frame or cosmological background $\tilde{\mathcal{M}}$, which describes a spacetime evolving in time. Depending on the properties of the chosen original frame, the $\tilde{g}_{ab}$ may characterize a cosmological spacetime or a dynamical object. In the rest of the paper, all of the geometric quantities in this conformally rescaled cosmological frame will be denoted by a "tilde". In this work, we will first start with the vacuum case of original frame in which $G_{ab}= 0$. Next, we will consider other contents of matter sources such as $G_{ab}=T_{ab}^{EM}$ and for the more general case $G_{ab} = T_{ab}$, where T_{ab}^{EM} is Maxwell stress-energy tensor and T_{ab} represents the energy-momentum tensor of a perfect fluid.

The conformal transformation rules (1) in GR frame require Einstein tensor to be [31–34],

$$G_{ab} = \tilde{G}_{ab} + 3\tilde{g}_{ab}\frac{\tilde{\nabla}_c S \tilde{\nabla}^c S}{S^2} + \frac{2}{S}(\tilde{\nabla}_a \tilde{\nabla}_b S - \tilde{g}_{ab}\tilde{\Box} S), \tag{2}$$

where, $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ is the Einstein tensor for the static/stationary submanifold and $\tilde{G}_{ab} = \tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R}$ is the Einstein tensor for the conformally rescaled space-time, the covariant dervative $\tilde{\nabla}_a$ and d'Alembertian $\tilde{\Box}$ are taken with respect to the metric $\tilde{g}_{ab}$. This equation shows that even if there is no matter in the untilted manifold ($G_{ab} = 0$), under the conformal transformation (1), the transformed spacetime geometry may contain any form of matter which may be responsible for the expansion of the universe in time. In this work, we suppose that, the spacetime geometry $\tilde{g}_{ab}$, obtained by the conformal transformation (1) of a static or stationary spacetime, would be a cosmological solution of Brans-Dicke theory in the Jordan frame and we can find a relation between the conformal factor $S(\tau)$ and the BD scalar scalar field ϕ. In other word, the scalar field would be responsible for the expansion of the spacetime. Hence, we will use BD scalar tensor theory, and its action in the conformally transformed frame as,

$$S_{BD} = \int d^4x\sqrt{-\tilde{g}}\left\{\tilde{\phi}\tilde{R} - \frac{\omega}{\tilde{\phi}}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \tilde{\mathcal{L}}_m\right\}, \tag{3}$$

which also includes matter Lagrangian $\tilde{\mathcal{L}}_m$. Here, $\tilde{\phi}$ is called as BD scalar field might be a function of both time and spatial coordinates, and ω is a free dimensionless BD parameter. The variation of the BD action (3) with respect to $\tilde{g}^{ab}$ gives the field equations as,

$$\begin{aligned} &\tilde{G}_{ab} - \frac{\omega}{\tilde{\phi}^2}\left(\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{2}\,\tilde{g}_{ab}\tilde{g}^{cd}\tilde{\nabla}_c\tilde{\phi}\nabla_d\tilde{\phi}\right) - \frac{1}{\tilde{\phi}}\left(\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} - \tilde{g}_{ab}\tilde{\Box}\tilde{\phi}\right) \\ &-\frac{8\pi}{\tilde{\phi}}\,\tilde{T}_{ab} = 0\,, \end{aligned} \tag{4}$$

and variation with respect to $\tilde{\phi}$ gives the scalar field equation as,

$$\tilde{\Box}\tilde{\phi} = \frac{8\pi}{2\omega+3}\tilde{T}. \tag{5}$$

Using equations (2) and (4), we can write the stress-energy tensor $\tilde{T}_{ab}$ in terms of the Einstein tensor of submanifold $\mathcal{M}$, scale factor $S(\tau)$ and scalar field $\tilde{\phi}$ as,

$$\begin{aligned} \frac{8\pi}{\tilde{\phi}}\,\tilde{T}_{ab} &= G_{ab} - 3\tilde{g}_{ab}\frac{\tilde{\nabla}_c S\tilde{\nabla}^c S}{S^2} - \frac{2}{S}(\tilde{\nabla}_a\tilde{\nabla}_b S - \tilde{g}_{ab}\tilde{\Box} S) \\ &\quad -\frac{\omega}{\tilde{\phi}^2}\left(\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{2}\,\tilde{g}_{ab}\tilde{\nabla}_c\tilde{\phi}\nabla\tilde{\phi}\right) - \frac{1}{\tilde{\phi}}\left(\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} - \tilde{g}_{ab}\tilde{\Box}\tilde{\phi}\right), \end{aligned} \tag{6}$$

here, even if we suppose $G_{ab}= 0$, there would be a nonzero stress energy tensor in the conformal frame. It means that the conformal transformation creates an extra term composed of the conformal factor, and this term can be related to the BD scalar field.

The energy-momentum tensor of the matter in conformal frame is,

$$\tilde{T}_m^{ab} = \frac{2}{\sqrt{-\tilde{g}}}\frac{\delta}{\delta\tilde{g}_{ab}}\left(\sqrt{-\tilde{g}}\tilde{L}_m\right), \tag{7}$$

and in the form of perfect fluid of matter it is defined as,

$$\tilde{T}_{ab} = \tilde{T}_{ab}^{(pf)} = (\tilde{P} + \tilde{\rho})\tilde{u}_a\tilde{u}_b + \tilde{g}_{ab}\tilde{P}, \tag{8}$$

the four velocity $\tilde{u}_a$ and $\tilde{u}_a\tilde{u}^a = -1$, $\tilde{\rho}$ and $\tilde{P}$ are energy density and pressure respectively.

In BD theory, the matter part of Lagrangian $\tilde{\mathcal{L}}_m$ is not coupled with the scalar field $\tilde{\phi}$, this is the main difference between the BD and Jordan models. But as we see in (6), the stress-energy tensor of the matter part seems to couple with the scalar $\tilde{\phi}$, however, it is not a coupling, but an interaction between the scalar field and metric tensor field. Therefore, the weak equivalence principle is respected [35]. To see this interaction explicitly, we begin by assuming an ansatz given by [36],

$$\tilde{\phi} = \phi_0 S(\tau)^{\alpha}, \tag{9}$$

with $\tilde{\phi}_0$ and α constants and $\tilde{\phi}_0 \geq 0$. This equation yields BD scalar field depends only on time [37]. In this work we suppose that the BD scalar field naturally arises in the universe and stating the ansatz (9), this scalar field may provide different expansion rates depending on the matter in the submanifold of the metric in (1).

Field equations (6) indicate that if the relation between scalar field and conformal factor becomes as in (9), each term with S and $\tilde{\phi}$ on the right will be related with eachother, hence the field equations become easy to solve and the stress-energy tensor for this system satisfies the perfect fluid description of matter [38].

Now, we apply these transformation rules to some known geometries and obtain the expansion rates for each spacetime generated by this method. Therefore, we shall see that the vacuum energy provides the expansion of the universe to accelerate, and the matter content causes it to decelerate in time.

Our paper is organised as follows. In section 2, we study on the Friedmann-Lemaitre-Robertson-Walker geometry and write the line element in the conformal form. First we apply time dependent conformal factor acting on Minkowski geometry and we obtain the expansion rates as a function of scalar field. Next, we consider that the conformal factor is acting on a curved static spacetime, and obtain that expansion rate decreases due to curvature of the submanifold. In Section 3, we introduce a conformally rescaled, charged, rotating Kerr Black hole with time dependent conformal factor and we obtain a relation between scale factor and BD scalar field. Finally, we end the paper with a brief summary and concluding remarks.

2 Friedmann-Lemaitre-Robertson-Walker cosmological solution

FLRW line element is a well known cosmological metric and given by,

$$d\tilde{s}^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right), \tag{10}$$

here, t is a cosmological time and $a(t)$ is the scale factor, k is the spatial curvature parameter and $d\Omega^2$ is two sphere metric. If we rescale the cosmological time as $dt = S(\tau)d\tau$ and reorganize the metric suitably, we can write FLRW line element in the conformal form,

$$\begin{aligned} d\tilde{s}^2 &= S(\tau)^2 ds^2, \\ &= S(\tau)^2 \left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right], \end{aligned} \tag{11}$$

by denoting τ as a conformal time and $S(\tau)^2$ as the conformal factor where $a\,(t)^2 = S[\tau\,(t)]^2$. Therefore, the static part in square brackets, which we call as the submanifold, is transformed to the cosmological spacetime by a time dependent conformal factor $S(\tau)^2$. The Ricci scalar of the cosmological space-time is

$$\tilde{R} = \frac{6}{S^2} \left[k + \frac{\ddot{S}}{S} \right], \tag{12}$$

where the overdot represents derivative with respect to conformal time τ. The Ricci scalar of submanifold reads $R = 6\,k$, which will be vanished for $k = 0$, hence, we obtain a Minkowski line element that implies the flat submanifold. The BD solution for FLRW metric has been explicitly given by [39–41]. Also, the work [42] reviews all possible solutions in this subject. Strictly speaking that, the main purpose of our work is not to obtain all the solutions and repeat the literature but try to understand the effect of the scalar field for some time dependent spacetimes with static or stationary submanifold and compare their expansion rates in the subject of stress-energy tensor for the perfect fluid.

Using the and equation (8) we solve the BD field equation (6) through the computer algebra and obtain that,

$$\frac{8\pi S(\tau)^2}{\tilde{\phi}\,(\tau)} \tilde{\rho}\,(\tau) = 3k + \frac{3\dot{S}(\tau)^2}{S(\tau)^2} + \frac{3\dot{S}(\tau)\dot{\phi}(\tau)}{S(\tau)\phi(\tau)} - \frac{\omega}{2} \frac{\dot{\phi}\,(\tau)^2}{\phi\,(\tau)^2} \tag{13}$$

$$\frac{8\pi S(\tau)^2}{\tilde{\phi}\,(\tau)} \tilde{P}\,(\tau) = -k + \frac{\dot{S}\,(\tau)^2}{S\,(\tau)^2} - \frac{2\ddot{S}\,(\tau)}{S\,(\tau)} - \frac{\dot{S}\,(\tau)\,\dot{\phi}\,(\tau)}{S\,(\tau)\,\phi\,(\tau)} - \frac{\omega}{2} \frac{\dot{\phi}\,(\tau)^2}{\phi\,(\tau)^2} - \frac{\ddot{\phi}\,(\tau)}{\phi\,(\tau)}, \tag{14}$$

and using the ansatz (9), we obtain the energy density $\tilde{\rho}$ and pressure $\tilde{P}$ as,

$$\tilde{\rho} = \frac{S^{\alpha-2}}{16\pi} \left(6k + (6 + 6\alpha - \omega\alpha^2) \frac{\dot{S}^2}{S^2} \right), \tag{15}$$

$$\tilde{P} = -\frac{S^{\alpha-2}}{16\pi} \left[2k + ((2+\omega)\alpha^2 - 2) \frac{\dot{S}^2}{S^2} + 2(2+\alpha) \frac{\ddot{S}}{S} \right], \tag{16}$$

where the four velocity vector has only time component as $\tilde{u}_a = \{-S(\tau), 0, 0, 0\}$. Substituting $\tilde{\phi} = \phi_0\, S(\tau)^{\alpha}$, the scalar field equation (5) satisfies,

$$6k - \omega\alpha(2+\alpha) \frac{\dot{S}^2}{S^2} + 2(3 - \omega\alpha) \frac{\ddot{S}}{S} = 0. \tag{17}$$

Here, the energy conservation equation $\tilde{\nabla}^a \tilde{T}_{ab}^{(pf)} = 0$ becomes identical with the (17) and satisfies the same equation. The (17) is a key equation and responds to the question of how the flat subspace yields expansion of the spacetime to be accelerated or how any type of the content of this subspace affects the expansion rate of the universe.

If $\mathbf{k = 0}$; The untransformed submanifold has no curvature and becomes Minkowski line element, thus it contains no matter, therefore, solving (17), this flat submanifold yields conformal factor to be a power law,

$$S(\tau) = S_0\, \tau^{\frac{2\omega\alpha-6}{\omega\alpha(\alpha+4)-6}}\,, \tag{18}$$

where, S_0 is an integration constant Therefore, the empty submanifold is transformed to the cosmological vacuum era by a conformal factor. If we rescale the time parameter as $S(\tau)d\tau = dt$ and rewrite the scale factor and scalar field in terms of the cosmological time, we get

$$a(t) = a_0\, t^{\frac{2\omega\alpha-6}{\omega\alpha(\alpha+6)-12}} \quad \text{and} \quad \tilde{\phi}(t) = \phi_0\, a(t)^{\alpha}, \tag{19}$$

where the constant, $a_0 = S_0^{\frac{\omega\alpha(\alpha+4)-6}{\omega\alpha(\alpha+6)-12}} \left(\frac{\omega\alpha(\alpha+6)-12}{\omega\alpha(\alpha+4)-6}\right)^{\frac{2\omega\alpha-6}{\omega\alpha(\alpha+6)-12}}$. These solutions are consistent with previous results [42]. Here, note that if the power of τ, in (18), is equal to -1, the scale factor in (19) will be an exponential function, hence we obtain de-Sitter spacetime. Nevertheless, this result is a very special subcase of the solution presented in our work, and we prefer to stay in the power-law type solution. Another motivation to insist on this solution group is to keep the whole paper in the same context. That means we aim to compare the expansion rates of three differently curved spacetimes and the common properties of these geometries are all admit the power-law expansion parameter simultaneously.

Based on this setup, the deceleration parameter that gives how the universe accelerates, becomes

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{\omega\alpha^2 + 4\alpha - 6}{2\omega\alpha - 6}, \tag{20}$$

which strongly depends on the relation between the expansion parameter and the scalar field. In this result, α remains as a free parameter and we can determine the value of α from cosmological observations. Depending on the value of α, we may have acceleration or deceleration of the spacetime.

If $\mathbf{k \neq 0}$: This choice describes a submanifold with a constant curvature and causes a nonzero stress-energy tensor. Hence, the first term G_{ab} on the right side of (6) has some contribution to the system. Physically, that means we are studying a homogeneously curved submanifold, and hence this submanifold has some massive content and affects the expansion rate.

The solution of nonlinear differential equation (17) has the form of an exponential equation,

$$S(\tau) = S_0\, e^{\pm\left(\frac{6k}{\omega\alpha^2+4\omega\alpha-6}\right)^{1/2}\tau}, \tag{21}$$

and the expansion parameter with respect to cosmological time becomes,

$$a(t) = \pm\left(\frac{6k}{\omega\alpha^2 + 4\omega\alpha - 6}\right)^{1/2} t \quad \text{and} \quad \tilde{\phi}(t) = \phi_0\, a(t)^{\alpha}, \tag{22}$$

which satisfies a linearly expanding spacetime and fits the result obtained in [42]. Here, note that we choose the plus sign for the consistency. This result could be interpreted as follows, the matter content in the conformally transformed submanifold prevents the acceleration of the spacetime, or we may say that the scalar field in the curved region could not accelerate the expansion of the spacetime. Neverheless, the vacuum submanifold that fills with a scalar field could speed up the expansion of spacetime. In summary, by taking into account $k = 0$ and $k \neq 0$ cases, we can propose the following statements: while a scalar field yields an accelerated expansion for the flat submanifold, it cannot accelerate the curved manifold filled with matter. This result might be applied to cosmology and interpreted as: a galactic system does not expand locally however, the vacuum parts of the universe are expanding much more and spreading apart the galaxies from each other.

3 The Charged and Rotating Time Dependent Black Hole Solution

In this part, using the same ansatz in previous part, we search for an allowable cosmological black hole solution. A cosmological black hole might be possible by means of embedding a static or stationary black hole in a cosmological background. There are some similar cosmological black hole geometries considered in the literature [43–48]. More realistic black holes are axially symmetric ones that have a mass and angular momentum. Although it is not necessary to have electrical charge for the physically reasonable black holes, to obtain a more general result, we include the electrical charge in this work. Therefore, we consider a Kerr-Newman (KN) metric that will be transformed into a spacetime varying in time with a time-dependent conformal factor. Since observations show that the realistic black holes curve the spacetime around themself, and also they are dynamical objects and interact with their environment, it will be convenient to work with this geometry that changes in time. From the previous part, we expect that, these massive objects, in which we may call the curved stationary submanifolds, might decelerate the expansion rate around itself or cause spacetime to be expanded linearly or there might be no expansion at all.

The simplest way to embed a black hole in a time dependent framework is to multiply all KN metric by a time dependent scale factor $S(\tau)^2$ [44]. Then the metric becomes,

$$\begin{aligned} \tilde{ds}^2 &= S(\tau)^2\, ds^2_{KN}\,, \\ &= S(\tau)^2\Big[-\Big(1 - \frac{2Mr - Q^2}{\Sigma}\Big)d\tau^2 - 2a\sin^2\theta\,\frac{(2Mr - Q^2)}{\Sigma}d\tau d\varphi \\ &\quad + \Sigma\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \Big(a^2 + r^2 + \frac{(2Mr - Q^2)a^2\sin^2\theta}{\Sigma}\Big)\sin^2\theta d\varphi^2\Big]\,, \end{aligned} \tag{23}$$

which can also be written as,

$$\tilde{ds}^2 = S(\tau)^2\Big[-d\tau^2+\Sigma\Big(\frac{dr^2}{\Delta}+d\theta^2\Big)+\frac{2Mr-Q^2}{\Sigma}\Big(-d\tau+a\sin^2\theta d\varphi\Big)^2 + (a^2+r^2)\sin^2\theta d\varphi^2\Big], \quad (24)$$

where,

$$\Sigma = r^2+a^2\cos^2\theta \quad (25)$$

$$\Delta = r^2+a^2+Q^2-2Mr, \quad (26)$$

where, M is the mass, Q is the electric charge and a is the the rotation parameter of the body. Here, for large radial distances, this geometry reduces to the spatially flat FLRW geometry. For the systems with electromagnetic field, the Brans-Dicke action is given by

$$\tilde{S}_{BD}=\int d^4x\sqrt{-\tilde{g}}\left\{\tilde{\phi}\tilde{R}-\omega\frac{\tilde{g}^{ab}}{\tilde{\phi}}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi}-\tilde{F}^{ab}\tilde{F}_{ab}+\tilde{\mathcal{L}}_m\right], \quad (27)$$

where, $\tilde{R}$ is the Ricci scalar of overall cosmological metric, $\tilde{\phi}$ is the BD scalar field, ω is the BD parameter, $\tilde{F}^{ab}$ is the Maxwell electromagnetic tensor and $\tilde{\mathcal{L}}_m$ is the Lagrangian density for the matter part. The total stress-energy tensor for this cosmological background contains electromagnetic and perfect fluid contributions as

$$\tilde{T}_{ab} = \tilde{T}_{ab}^{(EM)}+\tilde{T}_{ab}^{(m)}, = \frac{T_{ab}^{(EM)}}{S(\tau)^2}+\tilde{T}_{ab}^{(pf)}. \quad (28)$$

The Ricci scalar for this geometry becomes

$$\tilde{R}=\frac{6[\Delta\Sigma-(r^2+a^2)(Q^2-2Mr)]}{\Delta\Sigma}\frac{\ddot{S}}{S^3}=\frac{6\ddot{S}}{S}|g^{00}|. \quad (29)$$

Here, the matter part might be chosen as the form given in (8) and the velocity four vector has the following components:

$$\tilde{u}_a=\{-S(\tau)\sqrt{\frac{\Delta\Sigma}{\Delta\Sigma-(r^2+a^2)(Q^2-2Mr)}},0,0,0\}=\{-\frac{1}{\sqrt{|\tilde{g}^{00}|}},0,0,0\}, \quad (30)$$

and $\tilde{u}_a\tilde{u}^a=-1$.

The nonzero components of energy-momentum tensor for the matter part are

$$\tilde{T}_\tau^{\tau\,(pf)} = -\tilde{\rho}(\tau,r,\theta),$$

$$\tilde{T}_r^{r\,(pf)} = \tilde{T}_\theta^{\theta\,(pf)}=\tilde{T}_\varphi^{\varphi\,(pf)}=\tilde{P}(\tau,r,\theta),$$

$$\tilde{T}_\tau^{\varphi\,(pf)} = \frac{2a(Q^2-2Mr)(\tilde{\rho}+\tilde{P})}{\Delta\Sigma-(r^2+a^2)(Q^2-2Mr)}. \quad (31)$$

The electromagnetic stress-energy momentum tensor is given by

$$\tilde{T}_{ab}^{(EM)}=2(\tilde{F}_{ac}\tilde{F}_{bd}g^{cd}-\frac{1}{4}\tilde{F}_{cd}\tilde{F}^{cd}g_{ab}). \quad (32)$$

Here, the electromagnetic potential one form is,

$$A_a=\Big(-\frac{Qr}{\Sigma},0,0,\frac{Qra\sin^2\theta}{\Sigma}), \quad (33)$$

and the electromagnetic field tensor is given by $\tilde{F} = \tilde{d}A$ or in component form, it is defined as $\tilde{F}_{ab} = \tilde{\nabla}_a A_b - \tilde{\nabla}_b A_a$. The nonzero components of electromagnetic energy-momentum tensor are

$$\begin{aligned}
\tilde{T}_{\tau}^{\tau\,(EM)} &= -\tilde{T}_{\varphi}^{\varphi\,(EM)} = \frac{Q^2[\Sigma - 2(r^2+a^2)]}{\Sigma^3 S^4}, \\
\tilde{T}_{\varphi}^{\tau\,(EM)} &= \frac{2aQ^2(r^2+a^2)\sin^2\theta}{\Sigma^3 S^4}, \\
\tilde{T}_{r}^{r\,(EM)} &= T_{\theta}^{\theta\,(EM)} = \frac{Q^2}{2\Sigma^2 S^4}, \\
\tilde{T}_{\tau}^{\varphi\,(EM)} &= \frac{2aQ^2}{\Sigma^3 S^4}\,.
\end{aligned} \tag{34}$$

Note that, the conservation of energy-momentum tensor for the electromagnetic part, $\tilde{\nabla}^a \tilde{T}_{ab}^{(EM)} = 0$ is already satisfied.

Substituting the stress-energy tensor (28) in the Brans-Dicke field equations (6) and using computer algebra, from the (τ, r) term, we obtain the following differential equation,

$$2\tilde{\phi}(\tau)\dot{S(\tau)} + \dot{\tilde{\phi}}(\tau)S(\tau) = 0, \tag{35}$$

and the solution for the scalar field is given by,

$$\tilde{\phi}(\tau) = \phi_0\, S(\tau)^{-2}. \tag{36}$$

The components (τ, τ) and (r, r) of the field equations (6) are satisfied for $\phi_0 = 1$ then the energy density and pressure become,

$$\tilde{\rho}(\tau, r, \theta) = \tilde{P}(\tau, r, \theta) = -\frac{(2\omega+3)\Big(\Delta\Sigma - (Q^2 - 2Mr)(r^2+a^2)\Big)}{8\pi\Delta\Sigma}\frac{\dot{S}^2}{S^6}\,. \tag{37}$$

To satisfy positive energy density, this result requires to be $\omega < -\frac{3}{2}$ or $\Delta\Sigma < (Q^2 - 2Mr)(r^2 + a^2)$. In the works [44, 45], energy density becomes negative for the cosmological black hole geometries generated in this way. Therefore, using a straightforward conformal transformation, in BD theory, we have embedded a Kerr-Newman black hole in an expanding universe filled with matter. The equation of state, $\tilde{\rho} = \tilde{P}$ is known as Zeldovich's stiff fluid model and is used in general relativity to obtain the stellar and cosmological models for ultrahigh dense matter [49].

Now, we must satisfy the scalar field equation (5),

$$\frac{2(2\omega+3)\Big(\Delta\Sigma - (Q^2 - 2Mr)(r^2+a^2)\Big)}{\Delta\Sigma}\frac{\ddot{S}}{S^5} = 0\,. \tag{38}$$

This equation restricts the scale factor $S(\tau)$ to be linearly depending on time as,

$$S(\tau) = S_0\, \tau. \tag{39}$$

This value of scale factor also satisfies the conservation of energy-momentum tensor for the matter part and given by,

$$\tilde{\nabla}^a \tilde{T}_{ab}^{(pf)} = 0. \tag{40}$$

Therefore, all of the field equations are satisfied and the line element takes the folowing form,

$$\begin{aligned}
ds^2 &= {S_0}^2\tau^2\Big[-\Big(1 - \frac{2Mr - Q^2}{\Sigma}\Big)d\tau^2 - 2a\sin^2\theta\frac{(2Mr - Q^2)}{\Sigma}d\tau d\varphi \\
&\quad + \Sigma\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \Big(a^2 + r^2 + \frac{(2Mr - Q^2)a^2\sin^2\theta}{\Sigma}\Big)\sin^2\theta d\varphi^2\Big].
\end{aligned} \tag{41}$$

To analyse the singularity structure of this spacetime, since Ricci scalar (29) is zero for the result (39), we can look for the square of Riemann tensor

$$\tilde{R}_{abcd}\tilde{R}^{abcd} = \frac{f(r,\theta)}{\tau^8\Delta^2\Sigma^4}, \tag{42}$$

where the function, $f(r, \theta)$ in the numerator is an r and θ dependent complicated function and its explicit expression is not needed to determine the singularity structure of the geometry. This geometry posesses three singular points, namely, the initial big bang type singularity at $\tau = 0$, the ring singularity of Kerr-Newman solution at $\Sigma = 0$, and also the horizon singularity at $\Delta = 0$. These singularities also cause a singular fluid in which the energy density and pressure (37) diverge as well. If we rescale the conformal time, the line element (41) can be expressed as,

$$ds^2 = -\Big(1 - \frac{2Mr - Q^2}{\Sigma}\Big)dt^2 + t\Big[-2a\sin^2\theta \frac{(2Mr - Q^2)}{\sqrt{t}\Sigma} dt d\varphi + \Sigma\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \Big(a^2 + r^2 + \frac{(2Mr - Q^2)a^2\sin^2\theta}{\Sigma}\Big)\sin^2\theta d\varphi^2\Big]. \tag{43}$$

Here, the scale factor has the form $a(t) = \sqrt{t}$ hence, the deceleration parameter becomes $q = 1$ (where the integration constants are chosen to be unity as a convenience). This result shows that the spacetime around a charged rotating object with mass M and angular momentum a is not accelerating but decelerating as we expect. The scalar field in this curved submanifold does not yield the expansion to accelerate.

4 Conclusion

In this work, we have tried to explain how the vacuum energy provides the expansion of the universe to be accelerated in time and how the matter content of spacetime causes the universe to be decelerated. Using the rules for conformal transformation of a metric and the BD theory, we get some different cosmological spacetimes from the several static or stationary submanifolds. One of these submanifolds has chosen as Minkowskian spacetime, one has constant curvature, and the other has a massive content. The conformal factor has been set as a time dependent function, and the Brans-Dicke scalar field is directly related to this conformal factor as $\tilde{\phi} = \phi_0\, S(\tau)^\alpha$. Depending on the matter content in the submanifold, we obtain different expansion rates resulting in various scalar fields for each scenario. We conclude that the BD scalar field yields an accelerated expansion for the empty submanifold, but it is difficult to expand a spacetime filled with some pressure and energy. Therefore, the scalar field becomes responsible for the expansion in the vacuum. On the other hand, the gravitational sector prevents the expansion of spacetime even if there is a scalar field existing around the massive content. Cosmologically, the effect of scalar field can be explained as follows: If a spacetime has some massive content in it, this spacetime is not expanded so fast, nevertheless, an empty spacetime can have accelerated expansion due to the BD scalar field that might be interpreted as the effect of dark energy in the conventional cosmology.

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Gaussian Generalized John Numbers

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Abstract: In this work, we introduce Gaussian generalized John numbers and as particular cases, we examine Gaussian John and Gaussian John-Lucas numbers with their several properties. We exhibit the Binet's formulas, some identities, generating functions, sum formulas and matrix formulations of this new type Gaussian sequence and it's two special cases.
Keywords: Gaussian generalized John numbers, Gaussian John numbers, Gaussian John-Lucas numbers, John numbers.

1 Introduction and Preliminaries

Horadam [1], in 1963, carried out the concept of Fibonacci numbers to the complex sense and defined the complex Fibonacci numbers called with Gaussian Fibonacci numbers. Later, Jordan [2], in 1965, considered the complex Fibonacci numbers $\{\mathcal{GF}_n\}$ and the complex Fibonacci-Lucas numbers $\{\mathcal{GL}_n\}$ called as Gaussian Fibonacci-Lucas numbers by writing $G\mathcal{F}_n = \mathcal{F}_n + i\mathcal{F}_{n-1}$ where $\{\mathcal{F}_n\}$ is the Fibonacci sequence and $G\mathcal{L}_n = \mathcal{L}_n + i\mathcal{L}_{n-1}$ where $\{\mathcal{L}_n\}$ is the Fibonacci-Lucas sequence, respectively. Here, $\{\mathcal{F}_n\}$ and $\{\mathcal{L}_n\}$ are given by the relations $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ and $\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2}$ with $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$ and $\mathcal{L}_0 = 2$, $\mathcal{L}_1 = 1$ initial values, respectively. Later on, Berzsenyi [3], in 1977, also defined complex Fibonacci numbers with a different approach which is considering them as a set of complex numbers whose imaginary and real part are Gaussian integers with satisfying the Fibonacci second order recurrence relation at any triple of adjacent points. Then, Harman [4] in 1981 and Pethe and Horadam [5] in 1986 developed the idea and exhibited several properties of Gaussian Fibonacci numbers. Gaussian versions of other sequences of numbers other than Fibonacci were studied later. For instance, Aşçı and Gürel [6], in 2013, worked on Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers. Later on, Halıcı and Öz [7], in 2016, considered Gaussian Pell and Gaussian Pell-Lucas numbers and exhibited several properties of these special sequences. When we pass to the complex sequences given with the third order recurrence relation we come across with the work about Gaussian generalized Tribonacci and Tribonacci-Lucas numbers written by Soykan et. al. [8] in 2018. They defined the Gaussian generalized Tribonacci numbers $\{\mathcal{GV}_n\}$ by

$$\mathcal{GV}_n = \mathcal{GV}_{n-1} + \mathcal{GV}_{n-2} + \mathcal{GV}_{n-3} \tag{1.1}$$

with the initial conditions

$$\mathcal{GV}_0 = \mathcal{V}_0 + i(\mathcal{V}_2 - \mathcal{V}_1 - \mathcal{V}_0), \mathcal{GV}_1 = \mathcal{V}_1 + i\mathcal{V}_0, \mathcal{GV}_2 = \mathcal{V}_2 + i\mathcal{V}_1$$

not all being zero where $\{\mathcal{V}_n\}$ is a generalized Tribonacci sequence given with the relation from [9]

$$\mathcal{V}_n = \mathcal{V}_{n-1} + \mathcal{V}_{n-2} + \mathcal{V}_{n-3} \tag{1.2}$$

with initial conditions $\mathcal{V}_0, \mathcal{V}_1$ and $\mathcal{V}_2$ arbitrary real numbers. (1.1) and (1.2) are third order linear recurrence relations. Equivalently, this Gaussian sequence can be also defined by

$$\mathcal{GV}_n = \mathcal{V}_n + i\mathcal{V}_{n-1}. \tag{1.3}$$

Generalized Tribonacci numbers or $(r, s, t)-$numbers have been worked by many authors, see for example [9–19]. Other than these, there have been many studies about Gaussian sequences which are defined recursively, see [20–27], however, it never have been worked about generalized John numbers in Gaussian sense which we will see that it can be defined with third order linear recurrence relation. Therefore, we will recall generalized John numbers and Gaussian numbers in a quick background. Then we will pass to our problem which is to investigate several properties of a new complex sequence, named Gaussian generalized John sequence obtained with generalized John sequence, such as Binet's formula, summation formulas, generating function, identities and matrix formulation.

Soykan [28], in 2022, defined a new sequence named with generalized John sequence $\{\mathcal{W}_n\}_{n\geq 0} = \{\mathcal{W}_n\,(\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2)\}_{n\geq 0}$ having the third order recurrence relation, also related with Pell and Pell-Lucas sequences given by the second order recurrence relation, as follows:

$$\mathcal{W}_n = 3\mathcal{W}_{n-1} - \mathcal{W}_{n-2} - \mathcal{W}_{n-3}, \tag{1.4}$$

with the initial values $\mathcal{W}_0, \mathcal{W}_1$ and $\mathcal{W}_2$ not all being zero.

The sequence $\{\mathcal{W}_n\}_{n\geq 0}$ can be expanded to negative indices by describing

$$\mathcal{W}_{-n} = -\mathcal{W}_{-(n-1)} + 3\mathcal{W}_{-(n-2)} - \mathcal{W}_{-(n-3)}$$

for $n \in \mathbb{N}$. Hence, recurrence (1.4) satisfies for all integers n [28].

As $\{\mathcal{W}_n\}_{n\geq 0}$ is a sequence of third order recurrence, the characteristic equation associated to this recurrence is given with the equation

$$x^3 - 3x^2 + x + 1 = 0 \tag{1.5}$$

where roots are

$$\begin{aligned} \alpha &= 1+\sqrt{2}, \\ \beta &= 1-\sqrt{2}, \\ \gamma &= 1. \end{aligned} \tag{1.6}$$

It is given by Soykan [28] that generalized John numbers $\mathcal{W}_n(\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2)$ can be written, for each integers n, in the Binet's form

$$\mathcal{W}_n = \frac{P\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{Q\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{R\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.7}$$

where

$$\begin{aligned} P &= \mathcal{W}_2 - (\beta+\gamma)\mathcal{W}_1 + \beta\gamma\mathcal{W}_0 = \mathcal{W}_2 - (3-\alpha)\mathcal{W}_1 - \frac{1}{\alpha}\mathcal{W}_0 = \mathcal{W}_2 - \left(2-\sqrt{2}\right)\mathcal{W}_1 - \left(\sqrt{2}-1\right)\mathcal{W}_0, \\ Q &= \mathcal{W}_2 - (\alpha+\gamma)\mathcal{W}_1 + \alpha\gamma\mathcal{W}_0 = \mathcal{W}_2 - (3-\beta)\mathcal{W}_1 - \frac{1}{\beta}\mathcal{W}_0 = \mathcal{W}_2 - \left(2-\sqrt{2}\right)\mathcal{W}_1 - \left(-\sqrt{2}-1\right)\mathcal{W}_0, \\ R &= \mathcal{W}_2 - (\alpha+\beta)\mathcal{W}_1 + \alpha\beta\mathcal{W}_0 = \mathcal{W}_2 - (3-\gamma)\mathcal{W}_1 - \frac{1}{\gamma}\mathcal{W}_0 = \mathcal{W}_2 - 2\mathcal{W}_1 - \mathcal{W}_0. \end{aligned}$$

Then it can be repreheased the Binet's form as

$$\mathcal{W}_n = \frac{P\alpha^n}{4} + \frac{Q\beta^n}{4} - \frac{R\gamma^n}{2} = \frac{P\alpha^n + Q\beta^n - 2R\gamma^n}{4}.$$

We now consider two special cases of $\mathcal{W}_n$ in according to the initial values. The first one is John numbers which is determined with $\mathcal{W}_n(0,1,3) = \mathcal{J}_n$ and the second one is John-Lucas numbers which is determined with $\mathcal{W}_n(3,3,7) = \mathcal{H}_n$, so we can obtain the Binet's formula of John and John-Lucas sequence as follows:

$$\begin{aligned} \mathcal{J}_n &= \frac{\alpha^{n+1} + \beta^{n+1} - 2\gamma^{n+1}}{4}, \\ \mathcal{H}_n &= \alpha^n + \beta^n + \gamma^n. \end{aligned}$$

by [28].

Now let us recall the definiton of a Gaussian integer. A Gaussian integer z is a complex number where it's imaginary and real parts are both integers. Gauss searched these type of numbers in 1832 and $\mathbb{Z}[i]$ is the denotion of these numbers. $\mathbb{Z}[i]$ composes an integral domain with the usual addition and multiplication of complex numbers,. The norm of a Gaussian integer $a+ib$, $a, b \in \mathbb{Z}$ is it's Euclidean norm, i.e., $N(a+ib) = \sqrt{a^2+b^2} = \sqrt{(a+ib)(a-ib)}$. See [29] for more information and details about the Gaussian integers.

2 Gaussian Generalized John Numbers

Gaussian generalized John numbers $\{G\mathcal{W}_n\}_{n\geq 0} = \{G\mathcal{W}_n(G\mathcal{W}_0, G\mathcal{W}_1, G\mathcal{W}_2)\}_{n\geq 0}$ are defined by

$$G\mathcal{W}_n = 3G\mathcal{W}_{n-1} - G\mathcal{W}_{n-2} - G\mathcal{W}_{n-3}, \tag{2.1}$$

with the initial conditions

$$\begin{aligned} G\mathcal{W}_0 &= \mathcal{W}_0 + i(-\mathcal{W}_2 + 3\mathcal{W}_1 - \mathcal{W}_0), \\ G\mathcal{W}_1 &= \mathcal{W}_1 + i\mathcal{W}_0, \\ G\mathcal{W}_2 &= \mathcal{W}_2 + i\mathcal{W}_1, \end{aligned}$$

not all being zero. The sequences $\{G\mathcal{W}_n\}_{n\geq 0}$ can be expanded to negative indices by describing

$$G\mathcal{W}_{-n} = -G\mathcal{W}_{-(n-1)} + 3G\mathcal{W}_{-(n-2)} - G\mathcal{W}_{-(n-3)}$$

for $n \in \mathbb{N}$. Hence, recurrence (2.1) satisfies for each integers n. We can note that for each integer n, this sequence can be defined equivalently by

$$G\mathcal{W}_n = \mathcal{W}_n + i\mathcal{W}_{n-1}. \tag{2.2}$$

The first few Gaussian generalized John numbers with negative and positive indices are given in the next tables:

n	$G\mathcal{W}_n$
0	$\mathcal{W}_0+i(-\mathcal{W}_2+3\mathcal{W}_1-\mathcal{W}_0)$
1	$\mathcal{W}_1+i\mathcal{W}_0$
2	$\mathcal{W}_2+i\mathcal{W}_1$
3	$3\mathcal{W}_2-\mathcal{W}_1-\mathcal{W}_0+i\mathcal{W}_2$
4	$(8+3i)\,\mathcal{W}_2-(4+i)\,\mathcal{W}_1-(3+i)\,\mathcal{W}_0$
5	$(20+8i)\,\mathcal{W}_2-(11+4i)\,\mathcal{W}_1-(8+3i)\,\mathcal{W}_0$
6	$(49+20i)\,\mathcal{W}_2-(28+11i)\,\mathcal{W}_1-(20+8i)\,\mathcal{W}_0$
7	$(119+49i)\,\mathcal{W}_2-(69+28i)\,\mathcal{W}_1-(49+20i)\,\mathcal{W}_0$
8	$(288+119i)\,\mathcal{W}_2-(168+69i)\,\mathcal{W}_1-(119+49i)\,\mathcal{W}_0$
9	$(696+288i)\,\mathcal{W}_2-(407+168i)\,\mathcal{W}_1-(288+119i)\,\mathcal{W}_0$
10	$(1681+696i)\,\mathcal{W}_2-(984+407i)\,\mathcal{W}_1-(696+288i)\,\mathcal{W}_0$
11	$(4059+1681i)\,\mathcal{W}_2-(2377+984i)\,\mathcal{W}_1-(1681+696i)\,\mathcal{W}_0$
12	$(9800+4059i)\,\mathcal{W}_2-(5740+2377i)\,\mathcal{W}_1-(4059+1681i)\,\mathcal{W}_0$
	$\vdots$

Table 1 A few values of Gaussian generalized John numbers with positive subscripts.

n	$G\mathcal{W}_{-n}$
0	$(1-i)\,\mathcal{W}_0+3i\mathcal{W}_1-i\mathcal{W}_2$
1	$-(1-4i)\,\mathcal{W}_0+(3-4i)\,\mathcal{W}_1-(1-i)\,\mathcal{W}_2$
2	$(4-8i)\,\mathcal{W}_0-(4-13i)\,\mathcal{W}_1+(1-4i)\,\mathcal{W}_2$
3	$-(8-21i)\,\mathcal{W}_0+(13-28i)\,\mathcal{W}_1-(4-8i)\,\mathcal{W}_2$
4	$(21-49i)\,\mathcal{W}_0-(28-71i)\,\mathcal{W}_1+(8-21i)\,\mathcal{W}_2$
5	$-(49-120i)\,\mathcal{W}_0+(71-168i)\,\mathcal{W}_1-(21-49i)\,\mathcal{W}_2$
6	$(120-288i)\,\mathcal{W}_0-(168-409i)\,\mathcal{W}_1+(49-120i)\,\mathcal{W}_2$
7	$-(288-697i)\,\mathcal{W}_0+(409-984i)\,\mathcal{W}_1-(120-288i)\,\mathcal{W}_2$
8	$(697-1681i)\,\mathcal{W}_0-(984-2379i)\,\mathcal{W}_1+(288-697i)\,\mathcal{W}_2$
9	$-(1681-4060i)\,\mathcal{W}_0+(2379-5740i)\,\mathcal{W}_1-(697-1681i)\,\mathcal{W}_2$
10	$(4060-9800i)\,\mathcal{W}_0-(5740-13\,861i)\,\mathcal{W}_1+(1681-4060i)\,\mathcal{W}_2$
11	$-(9800-23\,661i)\,\mathcal{W}_0+(13\,861-33\,460i)\,\mathcal{W}_1-(4060-9800i)\,\mathcal{W}_2$
12	$(23\,661-57\,121i)\,\mathcal{W}_0-(33\,460-80\,783i)\,\mathcal{W}_1+(9800-23\,661i)\,\mathcal{W}_2$
	$\vdots$

Table 2 A few values of Gaussian generalized John numbers with negative subscripts.

We now consider two special cases of $G\mathcal{W}_n$ in according to initial values. The first one is Gaussian John numbers which is determined with $G\mathcal{W}_n(0,1,3+i)=G\mathcal{J}_n$ and the second one is Gaussian John-Lucas numbers which is determined with $G\mathcal{W}_n(3-i,3+3i,7+3i)=G\mathcal{H}_n$. If we would like to define this special sequences, we can give the next defitions:

Gaussian John numbers are defined by

$$G\mathcal{J}_n=3G\mathcal{J}_{n-1}-G\mathcal{J}_{n-2}-G\mathcal{J}_{n-3}$$

with the initial conditions

$$G\mathcal{J}_0=0,G\mathcal{J}_1=1,G\mathcal{J}_2=3+i$$

and Gaussian John-Lucas numbers are defined by

$$G\mathcal{H}_n=3G\mathcal{H}_{n-1}-G\mathcal{H}_{n-2}-G\mathcal{H}_{n-3}$$

with the initial conditions

$$G\mathcal{H}_0=3-i,G\mathcal{H}_1=3+3i,G\mathcal{H}_2=7+3i.$$

Note that for all integers n

$$G\mathcal{J}_n=\mathcal{J}_n+i\mathcal{J}_{n-1}$$

and

$$G\mathcal{H}_n=\mathcal{H}_n+i\mathcal{H}_{n-1}.$$

If we combine the first few values of the Gaussian John and Gaussian John-Lucas numbers in a one list, we can give Table 3.

We next exhibit the Binet's formula for the Gaussian generalized John numbers which helps to express the terms of Gaussian generalized John numbers in function of roots (1.6) of the characteristic equation (1.5).

n	$G\mathcal{J}_n$	$G\mathcal{J}_{-n}$	$G\mathcal{H}_n$	$G\mathcal{H}_{-n}$
0	0	0	$3-i$	$3-i$
1	1	$-i$	$3+3i$	$-1+7i$
2	$3+i$	$-1+i$	$7+3i$	$7-13i$
3	$8+3i$	$1-4i$	$15+7i$	$-13+35i$
4	$20+8i$	$-4+8i$	$35+15i$	$35-81i$
5	$49+20i$	$8-21i$	$83+35i$	$-81+199i$
6	$119+49i$	$-21+49i$	$199+83i$	$199-477i$
7	$288+119i$	$49-120i$	$479+199i$	$-477+1155i$
8	$696+288i$	$-120+288i$	$1155+479i$	$1155-2785i$
9	$1681+696i$	$288-697i$	$2787+1155i$	$-2785+6727i$
10	$4059+1681i$	$-697+1681i$	$6727+2787i$	$6727-16\,237i$
11	$9800+4059i$	$1681-4060i$	$16\,239+6727i$	$-16\,237+39\,203i$
12	$23\,660+9800i$	$-4060+9800i$	$39\,203+16\,239i$	$39\,203-94\,641i$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3 The first few values of the Gaussian John and Gaussian John-Lucas numbers.

Theorem 1. *The Binet's formula for the Gaussian generalized John numbers is*

$$\begin{aligned} G\mathcal{W}_n &= \left(\frac{P\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{Q\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{R\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\right) \\ &\quad +i\left(\frac{P\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{Q\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{R\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}\right) \end{aligned} \tag{2.3}$$

*where P,Q and R are as in (*1.7*).*

Proof: The proof follows from (1.7) and (2.2). □

Theorem 1 about the Binet's formula gives the next results as special examples:

Corollary 1. *The Binet's formula for the Gaussian John numbers and Gaussian John-Lucas numbers are*

$$\begin{aligned} G\mathcal{J}_n &= \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\ &\quad +i\left(\frac{\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)}\right) \end{aligned}$$

and

$$G\mathcal{H}_n=\left(\alpha^n+\beta^n+\gamma^n\right)+i\left(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}\right),$$

respectively.

The next theorem exhibits the generating function of Gaussian generalized John numbers.

Theorem 2. *The generating function of Gaussian generalized John numbers is given as*

$$\mathcal{F}_{G\mathcal{W}_n}(x)=\sum_{n=0}^{\infty}G\mathcal{W}_nx^n=\frac{G\mathcal{W}_0+(G\mathcal{W}_1-3G\mathcal{W}_0)x+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)x^2}{1-3x+x^2+x^3}. \tag{2.4}$$

Proof: Let

$$\mathcal{F}_{G\mathcal{W}_n}(x)=\sum_{n=0}^{\infty}G\mathcal{W}_nx^n$$

be generating function of Gaussian generalized John numbers. If we use the definition of $G\mathcal{W}_n$ and substract $3x\sum_{n=0}^{\infty}G\mathcal{W}_nx^n$, $-x^2\sum_{n=0}^{\infty}G\mathcal{W}_nx^n$ and $-x^3\sum_{n=0}^{\infty}G\mathcal{W}_nx^n$ from $\sum_{n=0}^{\infty}G\mathcal{W}_nx^n$ then we obtain that

$$\begin{aligned}
(1-3x+x^2+x^3)\mathcal{F}_{G\mathcal{W}_n}(x) &= \sum_{n=0}^{\infty}G\mathcal{W}_nx^n-3x\sum_{n=0}^{\infty}G\mathcal{W}_nx^n+x^2\sum_{n=0}^{\infty}G\mathcal{W}_nx^n+x^3\sum_{n=0}^{\infty}G\mathcal{W}_nx^n \\
&= \sum_{n=0}^{\infty}G\mathcal{W}_nx^n-3\sum_{n=0}^{\infty}G\mathcal{W}_nx^{n+1}+\sum_{n=0}^{\infty}G\mathcal{W}_nx^{n+2}+\sum_{n=0}^{\infty}G\mathcal{W}_nx^{n+3} \\
&= \sum_{n=0}^{\infty}G\mathcal{W}_nx^n-3\sum_{n=1}^{\infty}G\mathcal{W}_{n-1}x^n+\sum_{n=2}^{\infty}G\mathcal{W}_{n-2}x^n+\sum_{n=3}^{\infty}G\mathcal{W}_{n-3}x^n \\
&= (G\mathcal{W}_0+G\mathcal{W}_1x+G\mathcal{W}_2x^2)-(3G\mathcal{W}_0x+3G\mathcal{W}_1x^2)+G\mathcal{W}_0x^2 \\
&\quad +\sum_{n=3}^{\infty}(G\mathcal{W}_n-3G\mathcal{W}_{n-1}+G\mathcal{W}_{n-2}+G\mathcal{W}_{n-3})x^n \\
&= G\mathcal{W}_0+G\mathcal{W}_1x+G\mathcal{W}_2x^2-3G\mathcal{W}_0x-3G\mathcal{W}_1x^2+G\mathcal{W}_0x^2 \\
&= G\mathcal{W}_0+(G\mathcal{W}_1-3G\mathcal{W}_0)x+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)x^2.
\end{aligned}$$

Rearranging above equation, we get

$$\mathcal{F}_{G\mathcal{W}_n}(x)=\frac{G\mathcal{W}_0+(G\mathcal{W}_1-3G\mathcal{W}_0)x+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)x^2}{1-3x+x^2+x^3}.$$

□

The previous theorem concerning the generating function gives the following results for Gaussian John and Gaussian John-Lucas numbers as particular examples:

Corollary 2. *The generating function of Gaussian John numbers and Gaussian John-Lucas numbers are given as*

$$\mathcal{F}_{G\mathcal{J}_n}(x)=\frac{x+ix^2}{1-3x+x^2+x^3}$$

and

$$\mathcal{F}_{G\mathcal{H}_n}(x)=\frac{(1-7i)\,x^2-6\,(1-i)\,x+3-i}{1-3x+x^2+x^3},$$

respectively.

3 Binet's Formula Obtained From Generating Function

We next find Binet's formula of Gaussian generalized John numbers $\{G\mathcal{W}_n\}$ by the help of generating function for $G\mathcal{W}_n$.

Theorem 3. *(Binet's formula of Gaussian generalized John numbers)*

$$G\mathcal{W}_n=\frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{3.1}$$

where

$$\begin{aligned}
d_1 &= G\mathcal{W}_0\alpha^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\alpha+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,, \\
d_2 &= G\mathcal{W}_0\beta^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\beta+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,, \\
d_3 &= G\mathcal{W}_0\gamma^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\gamma+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,.
\end{aligned}$$

Proof: Consider the equation

$$k(x)=1-3x+x^2+x^3.$$

For some α,β and γ we can write

$$1-3x+x^2+x^3=(1-\alpha x)(1-\beta x)(1-\gamma x). \tag{3.2}$$

Therefore, $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of $k(x)$. As a result, we have that α and β as the roots of

$$k(\frac{1}{x})=1-\frac{3}{x}+\frac{1}{x^2}+\frac{1}{x^3}=0.$$

This implies that $x^3-3x^2+x+1=0$. Now, by (2.4) and (3.2), it follows that

$$\sum_{n=0}^{\infty} G\mathcal{W}_n x^n = \frac{G\mathcal{W}_0+(G\mathcal{W}_1-3G\mathcal{W}_0)\,x+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we can write

$$\frac{G\mathcal{W}_0+(G\mathcal{W}_1-3G\mathcal{W}_0)\,x+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{Z_1}{(1-\alpha x)}+\frac{Z_2}{(1-\beta x)}+\frac{Z_3}{(1-\gamma x)}. \tag{3.3}$$

So,

$$G\mathcal{W}_0+(G\mathcal{W}_1-3G\mathcal{W}_0)x+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,x^2 = Z_1(1-\beta x)(1-\gamma x)+Z_2(1-\alpha x)(1-\gamma x)+Z_3(1-\alpha x)(1-\beta x).$$

Considering $x=\frac{1}{\alpha}$, we get $G\mathcal{W}_0+(G\mathcal{W}_1-3G\mathcal{W}_0)\frac{1}{\alpha}+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,\frac{1}{\alpha^2}=Z_1(1-\dfrac{\beta}{\alpha})(1-\dfrac{\gamma}{\alpha})$. This gives

$$Z_1=\frac{G\mathcal{W}_0\alpha^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\,\alpha+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)}{(\alpha-\beta)(\alpha-\gamma)}=\frac{d_1}{(\alpha-\beta)(\alpha-\gamma)}.$$

Considering $x=\frac{1}{\beta}$, we obtain

$$Z_2=\frac{G\mathcal{W}_0\beta^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\,\beta+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)}{(\beta-\alpha)(\beta-\gamma)}=\frac{d_2}{(\beta-\alpha)(\beta-\gamma)}.$$

Similarly, taking $x=\frac{1}{\gamma}$, we have

$$Z_3=\frac{G\mathcal{W}_0\gamma^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\,\gamma+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)}{(\gamma-\alpha)(\gamma-\beta)}=\frac{d_3}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} G\mathcal{W}_n x^n = Z_1(1-\alpha x)^{-1}+Z_2(1-\beta x)^{-1}+Z_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} G\mathcal{W}_n x^n = Z_1\sum_{n=0}^{\infty}\alpha^n x^n+Z_2\sum_{n=0}^{\infty}\beta^n x^n+Z_3\sum_{n=0}^{\infty}\gamma^n x^n=\sum_{n=0}^{\infty}(Z_1\alpha^n+Z_2\beta^n+Z_3\gamma^n)x^n. \tag{3.4}$$

Thus, comparing the coefficients on both sides of (3.4), we obtain

$$G\mathcal{W}_n = Z_1\alpha^n+Z_2\beta^n+Z_3\gamma^n$$

and then we get (3.1). □

Hence, we have immediately next corollary comparing (3.1) with (2.4) pointing out the relation between the first three values of generalized John numbers and the Gaussian generalized John numbers by using the roots 1.6 of 1.5.

Corollary 3. *The following identities hold:*

$$\begin{aligned}
(\mathcal{W}_2-(\beta+\gamma)\mathcal{W}_1+\beta\gamma\mathcal{W}_0)\left(1+\frac{i}{\alpha}\right) &= G\mathcal{W}_0\alpha^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\alpha+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,,\\
(\mathcal{W}_2-(\alpha+\gamma)\mathcal{W}_1+\alpha\gamma\mathcal{W}_0)\left(1+\frac{i}{\beta}\right) &= G\mathcal{W}_0\beta^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\beta+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,,\\
(\mathcal{W}_2-(\alpha+\beta)\mathcal{W}_1+\alpha\beta\mathcal{W}_0)\left(1+\frac{i}{\gamma}\right) &= G\mathcal{W}_0\gamma^2+(G\mathcal{W}_1-3G\mathcal{W}_0)\gamma+(G\mathcal{W}_2-3G\mathcal{W}_1+G\mathcal{W}_0)\,.
\end{aligned}$$

4 Sum Formulas

In the next theorem we present the sum formulas of Gaussian generalized John numbers.

Theorem 4. *We have the following sum formulas:*

(a)$\sum_{k=0}^{n} G\mathcal{W}_k=\frac{1}{2}(-(n+2)G\mathcal{W}_{n+2}+(2n+5)G\mathcal{W}_{n+1}+(n+3)G\mathcal{W}_n+2G\mathcal{W}_2-5G\mathcal{W}_1-G\mathcal{W}_0)$.
(b)$\sum_{k=0}^{n} G\mathcal{W}_{2k}=\frac{1}{4}(-(2n+3)\,G\mathcal{W}_{2n+2}+4\,(n+2)\,G\mathcal{W}_{2n+1}+(2n+3)\,G\mathcal{W}_{2n}+3G\mathcal{W}_2-8G\mathcal{W}_1+G\mathcal{W}_0)$.

(c)$\sum_{k=0}^{n} G\mathcal{W}_{2k+1} = \frac{1}{4}(-(2n+1)\,G\mathcal{W}_{2n+2} + 2\,(2n+3)\,G\mathcal{W}_{2n+1} + (2n+3)\,G\mathcal{W}_{2n} + G\mathcal{W}_2 - 2G\mathcal{W}_1 - 3G\mathcal{W}_0)$.
(d)$\sum_{k=0}^{n} G\mathcal{W}_{-k} = \frac{1}{2}(-(n+1)G\mathcal{W}_{-n+2} + (2n+1)G\mathcal{W}_{-n+1} + (n+2)G\mathcal{W}_{-n} + G\mathcal{W}_2 - G\mathcal{W}_1)$.
(e)$\sum_{k=0}^{n} G\mathcal{W}_{-2k} = -\frac{1}{4}(-(2n+5)G\mathcal{W}_{-2n-2} - 4(n+2)G\mathcal{W}_{-2n-1} + (2n+5)G\mathcal{W}_{-2n} - 3G\mathcal{W}_2 + 4G\mathcal{W}_1 + 3G\mathcal{W}_0)$.
(f)$\sum_{k=0}^{n} G\mathcal{W}_{-2k+1} = -\frac{1}{4}(-(2n+5)G\mathcal{W}_{-2n-2} - 2(2n+5)G\mathcal{W}_{-2n-1} + (2n+7)G\mathcal{W}_{-2n} - 5G\mathcal{W}_2 + 6G\mathcal{W}_1 + 3G\mathcal{W}_0)$.

Proof: When we take $r=3, s=-1, t=-1$ in [Theorem 62, [30]] we obtain the sum formulas of generalized John numbers. Then, if we modify the sum formulas to the Gaussian version, we get the sum formulas above of Gaussian generalized John numbers. □

As a special case, we can present the sum formulas of Gaussian John numbers in the next corollary.

Corollary 4. *We have the following sum formulas:*

(a)$\sum_{k=0}^{n} G\mathcal{J}_k = \frac{1}{2}(-(n+2)G\mathcal{J}_{n+2} + (2n+5)G\mathcal{J}_{n+1} + (n+3)G\mathcal{J}_n + 1 + 2i)$.
(b)$\sum_{k=0}^{n} G\mathcal{J}_{2k} = \frac{1}{4}(-(2n+3)\,G\mathcal{J}_{2n+2} + 4\,(n+2)\,G\mathcal{J}_{2n+1} + (2n+3)\,G\mathcal{J}_{2n} + 1 + 3i)$.
(c)$\sum_{k=0}^{n} G\mathcal{J}_{2k+1} = \frac{1}{4}(-(2n+1)\,G\mathcal{J}_{2n+2} + 2\,(2n+3)\,G\mathcal{J}_{2n+1} + (2n+3)\,G\mathcal{J}_{2n} + 1 + i)$.
(d)$\sum_{k=0}^{n} G\mathcal{J}_{-k} = \frac{1}{2}(-(n+1)G\mathcal{J}_{-n+2} + (2n+1)G\mathcal{J}_{-n+1} + (n+2)G\mathcal{J}_{-n} + 2 + i)$.
(e)$\sum_{k=0}^{n} G\mathcal{J}_{-2k} = -\frac{1}{4}(-(2n+5)G\mathcal{J}_{-2n-2} - 4(n+2)G\mathcal{J}_{-2n-1} + (2n+5)G\mathcal{J}_{-2n} - 5 - 3i)$.
(f)$\sum_{k=0}^{n} G\mathcal{J}_{-2k+1} = -\frac{1}{4}(-(2n+5)G\mathcal{J}_{-2n-2} - 2(2n+5)G\mathcal{J}_{-2n-1} + (2n+7)G\mathcal{J}_{-2n} - 9 - 5i)$.

Next, the sum formulas for Gaussian John-Lucas numbers are given.

Corollary 5. *We have the following sum formulas:*

(a)$\sum_{k=0}^{n} G\mathcal{H}_k = \frac{1}{2}(-(n+2)G\mathcal{H}_{n+2} + (2n+5)G\mathcal{H}_{n+1} + (n+3)G\mathcal{H}_n - 4 - 8i)$.
(b)$\sum_{k=0}^{n} G\mathcal{H}_{2k} = \frac{1}{4}(-(2n+3)\,G\mathcal{H}_{2n+2} + 4\,(n+2)\,G\mathcal{H}_{2n+1} + (2n+3)\,G\mathcal{H}_{2n} - 16i)$.
(c)$\sum_{k=0}^{n} G\mathcal{H}_{2k+1} = \frac{1}{4}(-(2n+1)\,G\mathcal{H}_{2n+2} + 2\,(2n+3)\,G\mathcal{H}_{2n+1} + (2n+3)\,G\mathcal{H}_{2n} - 8)$.
(d)$\sum_{k=0}^{n} G\mathcal{H}_{-k} = \frac{1}{2}(-(n+1)G\mathcal{H}_{-n+2} + (2n+1)G\mathcal{H}_{-n+1} + (n+2)G\mathcal{H}_{-n} + 4)$.
(e)$\sum_{k=0}^{n} G\mathcal{H}_{-2k} = -\frac{1}{4}(-(2n+5)G\mathcal{H}_{-2n-2} - 4(n+2)G\mathcal{H}_{-2n-1} + (2n+5)G\mathcal{H}_{-2n})$.
(f)$\sum_{k=0}^{n} G\mathcal{H}_{-2k+1} = -\frac{1}{4}(-(2n+5)G\mathcal{H}_{-2n-2} - 2(2n+5)G\mathcal{H}_{-2n-1} + (2n+7)G\mathcal{H}_{-2n} - 8)$.

5 Some Identities

In the present section, some identities of Gaussian John numbers and Gaussian John-Lucas numbers will be obtained. The next one represent the relation between Gaussian John and Gaussian John-Lucas numbers. Specifically, we write the terms of Gaussian John sequence in terms of Gaussian John-Lucas sequence, or vice versa.

Lemma 5. *For each integer, we have the next identities:*

(a) $G\mathcal{J}_n = \frac{3}{8}G\mathcal{H}_{n+2} - \frac{1}{2}G\mathcal{H}_{n+1} - \frac{3}{8}G\mathcal{H}_n$.
(b) $G\mathcal{J}_n = \frac{9}{8}G\mathcal{H}_n - G\mathcal{H}_{n-1} - \frac{5}{8}G\mathcal{H}_{n-2}$.
(c) $G\mathcal{H}_n = -G\mathcal{J}_{n+2} + 6G\mathcal{J}_{n+1} - 7G\mathcal{J}_n$.
(d) $G\mathcal{H}_n = 3G\mathcal{J}_n - 2G\mathcal{J}_{n-1} - 3G\mathcal{J}_{n-2}$.

Proof: We can oly proof of (a). The other identities in (b), (c) and (d) can be proven similarly. Writing

$$G\mathcal{J}_n = aG\mathcal{H}_{n+2} + bG\mathcal{H}_{n+1} + cG\mathcal{H}_n$$

and solving the system of equations

$$\begin{aligned} G\mathcal{J}_0 &= aG\mathcal{H}_2 + bG\mathcal{H}_1 + cG\mathcal{H}_0 \\ G\mathcal{J}_1 &= aG\mathcal{H}_3 + bG\mathcal{H}_2 + cG\mathcal{H}_1 \\ G\mathcal{J}_2 &= aG\mathcal{H}_4 + bG\mathcal{H}_3 + cG\mathcal{H}_2 \end{aligned}$$

we find that $a = \dfrac{3}{8}, b = -\frac{1}{2}, c = \dfrac{-3}{8}$. □

Now, we give the Simson's formula of Gaussian generalized John numbers.

Theorem 6. *(Simson's Formula) For every integer n, we have the next formula:*

$$\begin{vmatrix} G\mathcal{W}_{n+2} & G\mathcal{W}_{n+1} & G\mathcal{W}_n \\ G\mathcal{W}_{n+1} & G\mathcal{W}_n & G\mathcal{W}_{n-1} \\ G\mathcal{W}_n & G\mathcal{W}_{n-1} & G\mathcal{W}_{n-2} \end{vmatrix} = 4(-1)^{n+1}(\mathcal{W}_0 + 2\mathcal{W}_1 - \mathcal{W}_2)\left(\mathcal{W}_0^2 - 2\mathcal{W}_1^2 - \mathcal{W}_2^2 - 2\mathcal{W}_0\mathcal{W}_2 + 4\mathcal{W}_1\mathcal{W}_2\right).$$

Proof: We prove this formula with the strong induction over n. We see that the identity is true for $n = 1$ as follows:

$$\begin{aligned}
&\begin{vmatrix} G\mathcal{W}_3 & G\mathcal{W}_2 & G\mathcal{W}_1 \\ G\mathcal{W}_2 & G\mathcal{W}_1 & G\mathcal{W}_0 \\ G\mathcal{W}_1 & G\mathcal{W}_0 & G\mathcal{W}_{-1} \end{vmatrix} \\
= \;& \det \begin{pmatrix} 3\mathcal{W}_2 - \mathcal{W}_1 - \mathcal{W}_0 + i\mathcal{W}_2 & \mathcal{W}_2 + i\mathcal{W}_1 & \mathcal{W}_1 + i\mathcal{W}_0 \\ \mathcal{W}_2 + i\mathcal{W}_1 & \mathcal{W}_1 + i\mathcal{W}_0 & \mathcal{W}_0 + i(-\mathcal{W}_2 + 3\mathcal{W}_1 - \mathcal{W}_0) \\ \mathcal{W}_1 + i\mathcal{W}_0 & \mathcal{W}_0 + i(-\mathcal{W}_2 + 3\mathcal{W}_1 - \mathcal{W}_0) & -(1-4i)\mathcal{W}_0 + (3-4i)\mathcal{W}_1 - (1-i)\mathcal{W}_2 \end{pmatrix} \\
= \;& 4(\mathcal{W}_0 + 2\mathcal{W}_1 - \mathcal{W}_2)\left(\mathcal{W}_0^2 - 2\mathcal{W}_1^2 - \mathcal{W}_2^2 - 2\mathcal{W}_0\mathcal{W}_2 + 4\mathcal{W}_1\mathcal{W}_2\right)
\end{aligned}$$

Next, we assume that the identity is true for $n = 1, 2, ..., k$, i.e.,

$$\begin{vmatrix} G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ G\mathcal{W}_{k+1} & G\mathcal{W}_k & G\mathcal{W}_{k-1} \\ G\mathcal{W}_k & G\mathcal{W}_{k-1} & G\mathcal{W}_{k-2} \end{vmatrix} = 4(-1)^{k+1}(\mathcal{W}_0 + 2\mathcal{W}_1 - \mathcal{W}_2)\left(\mathcal{W}_0^2 - 2\mathcal{W}_1^2 - \mathcal{W}_2^2 - 2\mathcal{W}_0\mathcal{W}_2 + 4\mathcal{W}_1\mathcal{W}_2\right).$$

At last, we need to show the identity holds for also $n = k + 1$.

$$\begin{aligned}
\begin{vmatrix} G\mathcal{W}_{k+3} & G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} \\ G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ G\mathcal{W}_{k+1} & G\mathcal{W}_k & G\mathcal{W}_{k-1} \end{vmatrix} &= \begin{vmatrix} 3G\mathcal{W}_{k+2} - G\mathcal{W}_{k+1} - G\mathcal{W}_k & G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} \\ 3G\mathcal{W}_{k+1} - G\mathcal{W}_k - G\mathcal{W}_{k-1} & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ 3G\mathcal{W}_k - G\mathcal{W}_{k-1} - G\mathcal{W}_{k-2} & G\mathcal{W}_k & G\mathcal{W}_{k-1} \end{vmatrix} \\
&= \begin{vmatrix} 3G\mathcal{W}_{k+2} & G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} \\ 3G\mathcal{W}_{k+1} & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ 3G\mathcal{W}_k & G\mathcal{W}_k & G\mathcal{W}_{k-1} \end{vmatrix} - \begin{vmatrix} G\mathcal{W}_{k+1} & G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} \\ G\mathcal{W}_k & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ G\mathcal{W}_{k-1} & G\mathcal{W}_k & G\mathcal{W}_{k-1} \end{vmatrix} \\
&\quad - \begin{vmatrix} G\mathcal{W}_k & G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} \\ G\mathcal{W}_{k-1} & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ G\mathcal{W}_{k-2} & G\mathcal{W}_k & G\mathcal{W}_{k-1} \end{vmatrix} \\
&= -\begin{vmatrix} G\mathcal{W}_k & G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} \\ G\mathcal{W}_{k-1} & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ G\mathcal{W}_{k-2} & G\mathcal{W}_k & G\mathcal{W}_{k-1} \end{vmatrix} = -\begin{vmatrix} G\mathcal{W}_{k+2} & G\mathcal{W}_{k+1} & G\mathcal{W}_k \\ G\mathcal{W}_{k+1} & G\mathcal{W}_k & G\mathcal{W}_{k-1} \\ G\mathcal{W}_k & G\mathcal{W}_{k-1} & G\mathcal{W}_{k-2} \end{vmatrix} \\
&= 4(-1)^{k+2}(\mathcal{W}_0 + 2\mathcal{W}_1 - \mathcal{W}_2)\left(\mathcal{W}_0^2 - 2\mathcal{W}_1^2 - \mathcal{W}_2^2 - 2\mathcal{W}_0\mathcal{W}_2 + 4\mathcal{W}_1\mathcal{W}_2\right).
\end{aligned}$$

Therefore, it is also true for $n = k + 1$. □

We can obtain the Simson's Formulas of Gaussian John and Gaussian John-Lucas numbers as special case of Theorem 6.

Corollary 6. *For every integer n, the Simson's Formulas of Gaussian John and Gaussian John-Lucas numbers are given by*

$$\begin{vmatrix} G\mathcal{J}_{n+2} & G\mathcal{J}_{n+1} & G\mathcal{J}_n \\ G\mathcal{J}_{n+1} & G\mathcal{J}_n & G\mathcal{J}_{n-1} \\ G\mathcal{J}_n & G\mathcal{J}_{n-1} & G\mathcal{J}_{n-2} \end{vmatrix} = 4(-1)^n \text{ and } \begin{vmatrix} G\mathcal{H}_{n+2} & G\mathcal{H}_{n+1} & G\mathcal{H}_n \\ G\mathcal{H}_{n+1} & G\mathcal{H}_n & G\mathcal{H}_{n-1} \\ G\mathcal{H}_n & G\mathcal{H}_{n-1} & G\mathcal{H}_{n-2} \end{vmatrix} = -128(-1)^{n+1}$$

respectively.

6 Matrix Formulation of $G\mathcal{W}_n$

One of the fruitful method for obtaining the some identities for particular sequences of which we fix the initial values is the matrix method. Let us describe the square matrix D of order 3 as:

$$D = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

If we define

$$B_n = \begin{pmatrix} \mathcal{J}_{n+1} & -\mathcal{J}_n - \mathcal{J}_{n-1} & -\mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{J}_{n-1} - \mathcal{J}_{n-2} & -\mathcal{J}_{n-1} \\ \mathcal{J}_{n-1} & -\mathcal{J}_{n-2} - \mathcal{J}_{n-3} & -\mathcal{J}_{n-2} \end{pmatrix}$$

then we know from [28] that

$$B_n = D^n,$$

i.e.,

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \mathcal{J}_{n+1} & -\mathcal{J}_n - \mathcal{J}_{n-1} & -\mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{J}_{n-1} - \mathcal{J}_{n-2} & -\mathcal{J}_{n-1} \\ \mathcal{J}_{n-1} & -\mathcal{J}_{n-2} - \mathcal{J}_{n-3} & -\mathcal{J}_{n-2} \end{pmatrix}.$$

Also we have the following identity from [28]:

$$D^n = \frac{1}{8}\begin{pmatrix} 3\mathcal{H}_{n+3}-4\mathcal{H}_{n+2}-3\mathcal{H}_{n+1} & -6\mathcal{H}_{n+2}+10\mathcal{H}_{n+1}+4\mathcal{H}_{n} & -3\mathcal{H}_{n+2}+4\mathcal{H}_{n+1}+3\mathcal{H}_{n} \\ 3\mathcal{H}_{n+2}-4\mathcal{H}_{n+1}-3\mathcal{H}_{n} & -6\mathcal{H}_{n+1}+10\mathcal{H}_{n}+4\mathcal{H}_{n-1} & -3\mathcal{H}_{n+1}+4\mathcal{H}_{n}+3\mathcal{H}_{n-1} \\ 3\mathcal{H}_{n+1}-4\mathcal{H}_{n}-3\mathcal{H}_{n-1} & -6\mathcal{H}_{n}+10\mathcal{H}_{n-1}+4\mathcal{H}_{n-2} & -3\mathcal{H}_{n}+4\mathcal{H}_{n-1}+3\mathcal{H}_{n-2} \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{8}\begin{pmatrix} 3\mathcal{H}_{n+3}-4\mathcal{H}_{n+2}-3\mathcal{H}_{n+1} & -6\mathcal{H}_{n+2}+10\mathcal{H}_{n+1}+4\mathcal{H}_{n} & -3\mathcal{H}_{n+2}+4\mathcal{H}_{n+1}+3\mathcal{H}_{n} \\ 3\mathcal{H}_{n+2}-4\mathcal{H}_{n+1}-3\mathcal{H}_{n} & -6\mathcal{H}_{n+1}+10\mathcal{H}_{n}+4\mathcal{H}_{n-1} & -3\mathcal{H}_{n+1}+4\mathcal{H}_{n}+3\mathcal{H}_{n-1} \\ 3\mathcal{H}_{n+1}-4\mathcal{H}_{n}-3\mathcal{H}_{n-1} & -6\mathcal{H}_{n}+10\mathcal{H}_{n-1}+4\mathcal{H}_{n-2} & -3\mathcal{H}_{n}+4\mathcal{H}_{n-1}+3\mathcal{H}_{n-2} \end{pmatrix}.$$

Consider the matrices $N_{\mathcal{J}}, E_{\mathcal{J}}$ defined in a such a way:

$$N_{\mathcal{J}} = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ -i & 3i & 1-i \end{pmatrix} \text{ and } E_{\mathcal{J}} = \begin{pmatrix} G\mathcal{J}_{n+1} & -G\mathcal{J}_{n}-G\mathcal{J}_{n-1} & -G\mathcal{J}_{n} \\ G\mathcal{J}_{n} & -G\mathcal{J}_{n-1}-G\mathcal{J}_{n-2} & -G\mathcal{J}_{n-1} \\ G\mathcal{J}_{n-1} & -G\mathcal{J}_{n-2}-G\mathcal{J}_{n-3} & -G\mathcal{J}_{n-2} \end{pmatrix}.$$

The next theorem exhibits the relations between D^n, $N_{\mathcal{J}}$ and $E_{\mathcal{J}}$.

Theorem 7. *For all integers n, we have*

$$D^n N_{\mathcal{J}} = E_{\mathcal{J}}.$$

Proof: It is clear from the matrix multiplication. Note that

$$\begin{aligned} D^n N_{\mathcal{J}} &= E_{\mathcal{J}} \Rightarrow N_{\mathcal{J}} = D^{-n}E_{\mathcal{J}} \\ &\Rightarrow N_{\mathcal{J}} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-n} \begin{pmatrix} G\mathcal{J}_{n+1} & -G\mathcal{J}_{n}-G\mathcal{J}_{n-1} & -G\mathcal{J}_{n} \\ G\mathcal{J}_{n} & -G\mathcal{J}_{n-1}-G\mathcal{J}_{n-2} & -G\mathcal{J}_{n-1} \\ G\mathcal{J}_{n-1} & -G\mathcal{J}_{n-2}-G\mathcal{J}_{n-3} & -G\mathcal{J}_{n-2} \end{pmatrix}. \end{aligned}$$

□

Theorem 7 can also be shown by mathematical induction.
We then obtain the matrix formulation of Gaussian John-Lucas numbers as a corollary.

Corollary 7. *For every integer n, we have*

$$D^n N_{\mathcal{H}} = E_{\mathcal{H}}$$

where

$$E_{\mathcal{H}} = \frac{1}{8}\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & h_{33} \end{pmatrix}$$

with

$$\begin{aligned} Z_{11} &= 3G\mathcal{H}_{n+3}-4G\mathcal{H}_{n+2}-3G\mathcal{H}_{n+1}, \\ Z_{12} &= -3G\mathcal{H}_{n+2}+G\mathcal{H}_{n+1}+7G\mathcal{H}_{n}+3G\mathcal{H}_{n-1}, \\ Z_{13} &= -3G\mathcal{H}_{n+2}+4G\mathcal{H}_{n+1}+3G\mathcal{H}_{n}, \\ Z_{21} &= 3G\mathcal{H}_{n+2}-4G\mathcal{H}_{n+1}-3G\mathcal{H}_{n}, \\ Z_{22} &= -3G\mathcal{H}_{n+1}+G\mathcal{H}_{n}+7G\mathcal{H}_{n-1}+3G\mathcal{H}_{n-2}, \\ Z_{23} &= -3G\mathcal{H}_{n+1}+4G\mathcal{H}_{n}+3G\mathcal{H}_{n-1}, \\ Z_{31} &= 3G\mathcal{H}_{n+1}-4G\mathcal{H}_{n}-3G\mathcal{H}_{n-1}, \\ Z_{32} &= -3G\mathcal{H}_{n}+G\mathcal{H}_{n-1}+7G\mathcal{H}_{n-2}+3G\mathcal{H}_{n-3}, \\ Z_{33} &= -3G\mathcal{H}_{n}+4G\mathcal{H}_{n-1}+3G\mathcal{H}_{n-2} \end{aligned}$$

and

$$N_{\mathcal{H}} = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ -i & 3i & 1-i \end{pmatrix}.$$

The proof of this corollary can be seen from Lemma 5 and the matrix multiplication.

7 Conclusion

The important features of the Gaussian version of the generalized John sequences that have been newly introduced to the literature are examined. In this sense, it is considered as an important reference source for researchers who will work with special sequences. In the present work, matrices and important equations have been obtained and they contain important results that can be applied to daily life problems.

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DRBEM Solution of MHD Flow in a Rectangular Duct under Axially-changing External Magnetic Field

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Abstract: The laminar, fully-developed magnetohydrodynamic (MHD) pipe flow of a viscous and incompressible fluid has been considered between consecutive magnets placed on the pipe-axis. The flow is under the effect of an axial-dependent applied magnetic field $\vec{B} = (0, B_0(z), 0)$ and $B_0(z) = B_0 g(z)$ where B_0 denotes the external magnetic field intensity, and $g(z)$ is the function determining the strength of the applied magnetic field along the pipe-axis. The MHD flow equations are transformed to three nonlinear Poisson type equations in terms of velocity, induced magnetic field and electric potential, and they are solved by using the dual reciprocity boundary element method (DRBEM) with the fundamental solution of Laplace's equation. The study shows that, axially-changing magnetic field makes the flow to turn its direction at a certain position of the axis. The effects of the problem parameters, Hartmann number M and magnetic Reynolds number R_m on the flow behavior contrast with each other in the sense that, the lengths of the intervals on the pipe-axis on which the flow is reversed are increasing as M increases, however, they are getting shorter as R_m increases.

Keywords: Axially-changing magnetic field, DRBEM, MHD duct flow.

1 Introduction

Magnetohydrodynamic (MHD) investigates the behavior of electrically conducting fluids such as plasmas, liquid metals, electrolytes, etc. The MHD equations are derived from a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The MHD flow in channels has many applications in engineering, biology and industry such as nuclear fusion, geothermal energy extraction, blood flow pressure, MHD generators and accelerators, etc. The non-linear nature of the Navier-Stokes equations restricts one to find an analytic solution for the MHD channel flows. Thus, numerical approaches are used mostly for general geometry and applied magnetic field. Among these, Bozkaya and Tezer-Sezgin [1] have been used both the extended-domain-eigenfunction method (EDEM) and the boundary element method (BEM) to investigate the MHD pipe flow in annular-like domains with electrically conducting walls. For the solution of 3D MHD equations, a finite element method (FEM) has been developed by Salah et al. [2]. Also in this study, the stabilized finite element formulations have been used for the Navier-Stokes and magnetic equations to solve boundary layers and convection dominated flows. The biomagnetic fluid flow equations are solved under a point source magnetic field by using the numerical method based on a pressure-linked pseudotransient method in [3]. The numerical implementations are different in the above studies however they have a common property that the strength of the applied magnetic field B_0 is constant. However, there are gradients of the applied magnetic field varying in the streamwise direction in real life applications such as designing self cooled liquid-metal blankets which are used for fusion reactors. Kim [4] considered 3D liquid-metal MHD flow in a square duct under a non-uniform magnetic field. Sterl [5] studied MHD flow in rectangular ducts in 2D and 3D regions. He examined the effect of wall conductance and several Hartmann number values on the flow behavior and also axial-dependent applied magnetic field.

In this paper, the laminar MHD pipe flow of a viscous, incompressible and electrically conducting fluid in a rectangular duct is considered. The flow is under the influence of an axially-varying applied magnetic field $B_0(z)$. Some magnets are placed on the duct axis at fixed z-values and they are varying as a function of z. The flow is assumed to be fully-developed between those two fixed z-values. The MHD flow equations in terms of velocity, induced magnetic field and electric potential are solved by using the dual reciprocity boundary element method (DRBEM) with the fundamental solution of Laplace's equation [6]. The behavior of the fluid flow is examined for several values of Hartmann number M and magnetic Reynolds number R_m. It is seen that, axially-changing magnetic field makes the flow to turn its direction at a certain position of the axis giving the flow behavior along the pipe-axis as if it is 3D flow. The effects of the problem parameters, Hartmann number M and magnetic Reynolds number R_m on the flow behavior contrast with each other in the sense that, the lengths of the intervals on the pipe-axis on which the flow is reversed are increasing as M increases, however, they are getting shorter as R_m increases. The DRBEM implementation, discretizing only the boundary with constant elements, captures the well known behavior of the MHD flow. Thus, its computational cost is considerably small compared to the other domain type numerical methods.

2 Mathematical formulation

The laminar, fully-developed MHD flow of a viscous and incompressible fluid is considered between consecutive magnets placed on the pipe-axis. The flow is under the effect of an axial-dependent vertically applied magnetic field. The governing non-dimensional MHD flow equations in terms of the velocity, induced magnetic field and electric potential are

$$\begin{aligned}
&\nabla^2 V + Mg(z)\frac{\partial B}{\partial y} = -1 + \frac{M^2}{R_m} g(z)\frac{\partial g(z)}{\partial z} \\
&\nabla^2 B + Mg(z)\frac{\partial V}{\partial y} = 0 \qquad\qquad -1 \le x, y \le 1 \\
&\nabla^2 \Phi = -g(z)\frac{\partial V}{\partial x}.
\end{aligned} \tag{1}$$

where $M = B_0 L_0 \sqrt{\sigma}/\sqrt{\mu}$ is the Hartmann number and L_0, σ, μ are the characteristic length, electrical conductivity and viscosity of the fluid respectively. The walls of the duct are considered as insulated with no-slip velocity together with the Dirichlet type boundary condition for Φ which correspond to the following wall conditions

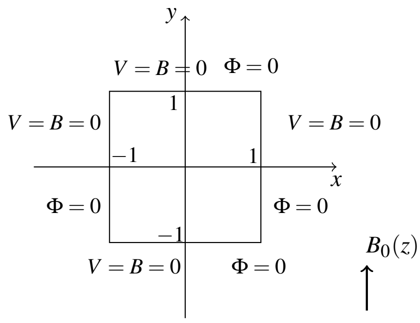

Fig. 1: The boundary conditions on duct walls.

3 The DRBEM formulation

The DRBEM is applied to the nonlinear Poisson type equations (1) with the fundamental solution of Laplace's equation, $u^* = \dfrac{ln(1/r)}{2\pi}$ given in [6]. The terms other than Laplacian are considered as inhomogeneities. Weighting the equations in (1) over Ω by u^* and applying Green's second identity two times, the following boundary-domain integral equations are obtained

$$\begin{aligned}
&c_i V_i + \int_\Gamma q^* V d\Gamma - \int_\Gamma u^* \frac{\partial V}{\partial n} d\Gamma = -\int_\Omega \Big(-Mg(z)\frac{\partial B}{\partial y} - 1 + \frac{M^2}{R_m} g(z)\frac{\partial g(z)}{\partial z}\Big) u^* d\Omega \\
&c_i B_i + \int_\Gamma q^* B d\Gamma - \int_\Gamma u^* \frac{\partial B}{\partial n} d\Gamma = -\int_\Omega \Big(-Mg(z)\frac{\partial V}{\partial y}\Big) u^* d\Omega \\
&c_i \Phi_i + \int_\Gamma q^* \Phi d\Gamma - \int_\Gamma u^* \frac{\partial \Phi}{\partial n} d\Gamma = -\int_\Omega \Big(-g(z)\frac{\partial V}{\partial x}\Big) u^* d\Omega
\end{aligned} \tag{2}$$

where $q^* = \dfrac{\partial u^*}{\partial n}$, the index i denotes the source point, $i = 1, ..., N$, N is the number of constant boundary elements. The constant c_i is $1/2$ and 1 when the source point is on the boundary and in the interior of the domain, respectively.

The integrands on the right hand sides in equations (2) are considered as inhomogeneities and these inhomogeneous terms can be expanded by a series of a radial basis functions as

$$\begin{aligned}
&b_1(x,y) = -\Big(-Mg(z)\frac{\partial B}{\partial y} - 1 + \frac{M^2}{R_m} g(z)\frac{\partial g(z)}{\partial z}\Big) = \sum_{j=1}^{N+L} \alpha_j f_j(x,y) \\
&b_2(x,y) = Mg(z)\frac{\partial V}{\partial y} = \sum_{j=1}^{N+L} \beta_j f_j(x,y) \\
&b_3(x,y) = g(z)\frac{\partial V}{\partial x} = \sum_{j=1}^{N+L} \gamma_j f_j(x,y)
\end{aligned} \tag{3}$$

where $f_j(x,y)$ are the radial basis functions which are connected to the particular solutions $\hat{u}_j$ with $\nabla^2 \hat{u}_j = f_j$, the coefficients α_j's, β_j's and

γ_j's are undetermined contants and L denotes the number of interior points.

Substituting $f_j = \nabla^2 \hat{u}_j$ in the equations (3) and applying Green's second identity two times again to the right hand sides of the equations (2), the following boundary integral equations are obtained

$$
\begin{aligned}
c_i V_i + \int_\Gamma q^* V d\Gamma - \int_\Gamma u^* \frac{\partial V}{\partial n} d\Gamma &= \sum_{j=1}^{N+L} \alpha_j \left(c_i \hat{u}_{ij} + \int_\Gamma q^* \hat{u}_j d\Gamma - \int_\Gamma u^* \hat{q}_j d\Gamma \right) \\
c_i B_i + \int_\Gamma q^* B d\Gamma - \int_\Gamma u^* \frac{\partial B}{\partial n} d\Gamma &= \sum_{j=1}^{N+L} \beta_j \left(c_i \hat{u}_{ij} + \int_\Gamma q^* \hat{u}_j d\Gamma - \int_\Gamma u^* \hat{q}_j d\Gamma \right) \\
c_i \Phi_i + \int_\Gamma q^* \Phi d\Gamma - \int_\Gamma u^* \frac{\partial \Phi}{\partial n} d\Gamma &= \sum_{j=1}^{N+L} \gamma_j \left(c_i \hat{u}_{ij} + \int_\Gamma q^* \hat{u}_j d\Gamma - \int_\Gamma u^* \hat{q}_j d\Gamma \right)
\end{aligned} \tag{4}
$$

where $\hat{q}_j = \dfrac{\partial \hat{u}_j}{\partial n} = \dfrac{\partial \hat{u}_j}{\partial x}\dfrac{\partial x}{\partial n} + \dfrac{\partial \hat{u}_j}{\partial y}\dfrac{\partial y}{\partial n}$, $i = 1, ..., N$.

Constructing the matrices $\hat{U}$, $\hat{Q}$ and coordinate matrix F by taking the vectors $\hat{u_j}$, $\hat{q}_j$ and $f_{ij} = 1 + r_{ij}$, r_{ij} being the distance from the point i to the point j, as columns respectively, and evaluating the values of b_1, b_2 and b_3 at $N+L$ points, a sets of linear equations as $b_1 = F\alpha$ $b_2 = F\beta$ and $b_3 = F\gamma$ are obtained. Thus, the equations (4) can be rewritten in matrix-vector form

$$
\begin{aligned}
HV - G\frac{\partial V}{\partial n} &= (H\hat{U} - G\hat{Q})F^{-1}\{-Mg(z)\frac{\partial B}{\partial y} - 1 + \frac{M^2}{R_m} g(z) \frac{\partial g(z)}{\partial z}\} \\
HB - G\frac{\partial B}{\partial n} &= (H\hat{U} - G\hat{Q})F^{-1}\{-Mg(z)\frac{\partial V}{\partial y}\} \\
H\Phi - G\frac{\partial \Phi}{\partial n} &= (H\hat{U} - G\hat{Q})F^{-1}\{-g(z)\frac{\partial V}{\partial x}\}.
\end{aligned} \tag{5}
$$

The components of the H and G matrices for a constant element are given as

$$
\begin{aligned}
H_{ij} &= c_i \delta_{ij} + \frac{1}{2\pi} \int_{\Gamma_j} \frac{\partial}{\partial n}\left(ln\left(\frac{1}{r}\right)\right) d\Gamma_j, \quad H_{ii} = -\sum_{j=1, j\neq i}^{N} H_{ij} \\
G_{ij} &= \frac{1}{2\pi} \int_{\Gamma_j} ln\left(\frac{1}{r}\right) d\Gamma_j, \quad G_{ii} = \frac{l}{2\pi}\left(ln\left(\frac{2}{l}\right) + 1\right)
\end{aligned} \tag{6}
$$

where r is the modulus of the distance vector from the point i to element j, δ_{ij} is the Kronecker delta function, l is the length of the elements.

The space derivatives for V and B are computed by using the coordinate function F as

$$
\frac{\partial V}{\partial x} = \frac{\partial F}{\partial x} F^{-1} V, \quad \frac{\partial V}{\partial y} = \frac{\partial F}{\partial y} F^{-1} V \quad \text{and} \quad \frac{\partial B}{\partial y} = \frac{\partial F}{\partial y} F^{-1} B.
$$

Then, the discrete equations in (5) become,

$$
\begin{aligned}
HV - G\frac{\partial V}{\partial n} + K\big(Mg(z)\frac{\partial F}{\partial y} F^{-1} B\big) &= K\{-1 + \frac{M^2}{R_m} g(z)\frac{\partial g(z)}{\partial z}\} \\
HB - G\frac{\partial B}{\partial n} + K\big(Mg(z)\frac{\partial F}{\partial y} F^{-1} V\big) &= 0 \\
H\Phi - G\frac{\partial \Phi}{\partial n} + K\big(g(z)\frac{\partial F}{\partial x} F^{-1} V\big) &= 0
\end{aligned} \tag{7}
$$

where $K = (H\hat{U} - G\hat{Q})F^{-1}$.

The solution of $\begin{Bmatrix} V \\ B \\ \Phi \end{Bmatrix}$ can be obtained from the following enlarged system of equations

$$
\begin{bmatrix} H_1 & H_2 & H_3 \\ H_4 & H_5 & H_6 \\ H_7 & H_8 & H_9 \end{bmatrix} \begin{Bmatrix} V \\ B \\ \Phi \end{Bmatrix} - \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \partial V/\partial n \\ \partial B/\partial n \\ \partial \Phi/\partial n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ 0 \\ 0 \end{Bmatrix} \tag{8}
$$

where the $(N+L)\times(N+L)$ matrices are

$$\begin{array}{ccc} H_1 = H & H_4 = H_2 & H_7 = K\big(g(z)\dfrac{\partial F}{\partial x}F^{-1}\big) \\ H_2 = K\big(Mg(z)\dfrac{\partial F}{\partial y}F^{-1}\big) & H_5 = H & H_8 = 0 \\ H_3 = 0 & H_6 = 0 & H_9 = H \end{array} \tag{9}$$

and $b_1 = K\big(-1+\dfrac{M^2}{R_m}g(z)\dfrac{\partial g(z)}{\partial z}\big)$.

Prescribing new matrices as

$$\mathbf{H}' = \begin{bmatrix} H_1 & H_2 & H_3 \\ H_4 & H_5 & H_6 \\ H_7 & H_8 & H_9 \end{bmatrix}, \quad G' = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \tag{10}$$

the enlarged system having dimensions $3(N+L)\times 3(N+L)$ becomes

$$H' \begin{Bmatrix} V \\ B \\ \Phi \end{Bmatrix} = G' \begin{Bmatrix} \partial V/\partial n \\ \partial B/\partial n \\ \partial \Phi/\partial n \end{Bmatrix} + \begin{Bmatrix} b_1 \\ 0 \\ 0 \end{Bmatrix}. \tag{11}$$

A linear system of equations such as $Ax = d$ can be obtained after inserting the given boundary conditions shown on Figure 1 into (11) and swapping the corresponding columns of H' and G'. The solution of $Ax = d$ gives the unknown values of V, B and Φ at the discretized points wherever they are unknown.

4 Numerical results

The ducts $\Omega_i = [-1,1]\times[-1,1]$ are discretized at the locations z_i on the pipe-axis by using $N = 200$ constant boundary elements and $L = 2500$ interior nodes. The pipe-axis dependent function in $B_0(z)$ is taken as $g(z) = \dfrac{1}{1+e^{-z/0.15}}$. The z_i values (positions of the magnets) are considered between $-2.13 \le z \le 2.13$. The velocity, induced magnetic field and electric potential values are obtained with Hartmann number values $M = 10$, 30 and magnetic Reynolds number values as $R_m = 2$ and 25 respectively. The solution of the resulting system $Ax = d$ is obtained by using a solver mldivide from matlab.

Figures 2 and 3 show the velocity and induced current contours and velocity level curves at several locations in $[-2.13, 2.13]$ along the pipe for increasing values of Hartmann number M as 10 and 30 by taking the magnetic Reynolds number fixed as $R_m = 2$. The fluid flows in the positive pipe-axis direction first and then it reverses its direction at a certain z-value, and then the flow becomes positive again in that interval. It can be seen from these figures that, as M increases the reversed flow occurs much earlier, and then the flow turns to the pipe-axis direction (positive z-axis) much later. That is, the length of the interval on the pipe-axis on which the flow is reversed is increasing (i.e. for $M = 10$ the length of the interval for reversed flow is 1.30, for $M = 30$ it is 1.80). The flattening tendency of the flow is also observed as M increases at the same location of the pipe (i.e. at $z = 2.13$). The current lines align in the direction of applied magnetic field in terms of two loops.

Figures 3 and 4 show the velocity and induced current contours and velocity level curves for increasing values of magnetic Reynolds number R_m as 2 and 25 by taking Hartmann number fixed as $M = 30$. It is observed that, as R_m increases the flow reverses much later however, reversing back to the pipe-axis direction occurs much earlier. That is, the length of the interval for the reverse flow is getting shorter as R_m increases. The lengths are 1.80 and 1.22 for R_m as 2 and 25, respectively. This opposite effects of the increase in the values of Hartmann number and magnetic Reynolds number, on the lengths of the sections of the pipes for the reversed flow can be explained physically. For the same fluid of constant viscosity μ and electric conductivity σ, Hartmann number increases when the intensity of the applied magnetic field B_0 is strong. Its effect is also strong on the fluid and the flow reversion occurs on a longer interval on the pipe-axis. But, the magnetic Reynolds number R_m increases when magnetic permeability μ_0 increases. Thus, the flow changes direction quickly on the pipe-axis.

Figure 5 shows the velocity, induced current and electric potential contours for $M = 30$, $R_m = 2$. V and B contours stay the same and electric potential Φ curves show the same behavior in opposite direction with induced current B curves but in different magnitude.

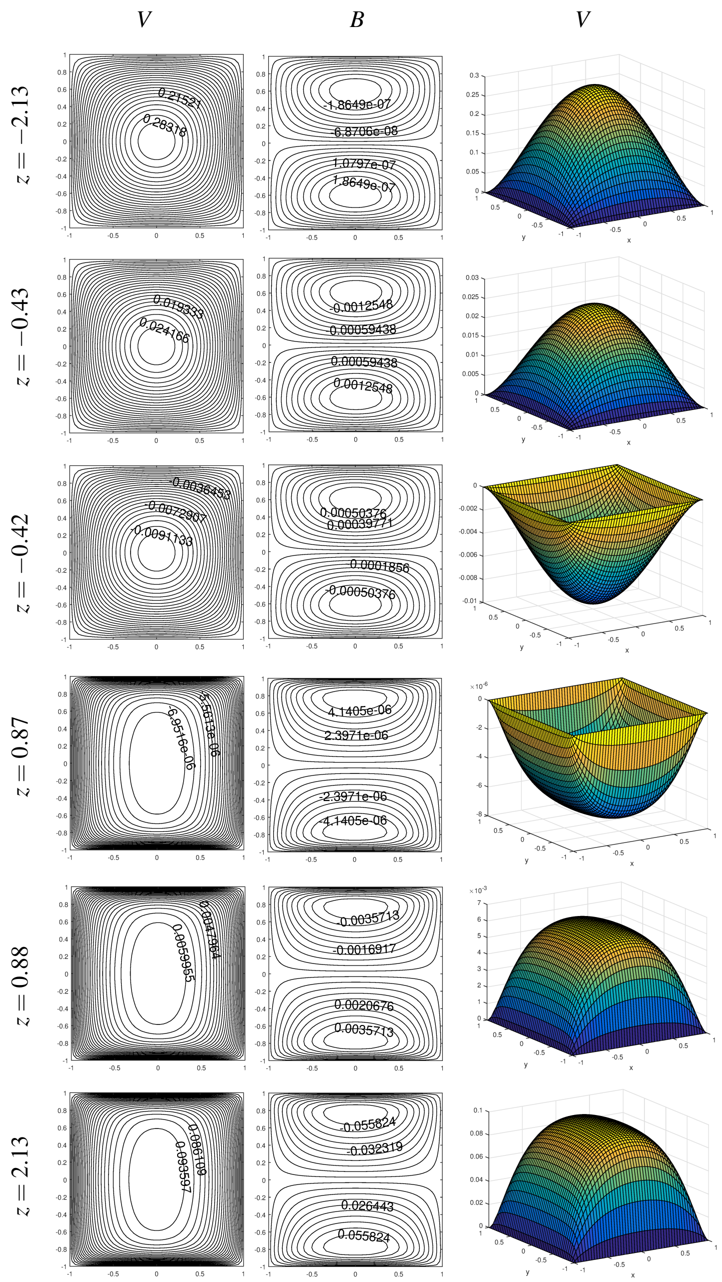

Fig. 2: Velocity and induced magnetic field, $M = 10$, $R_m = 2$.

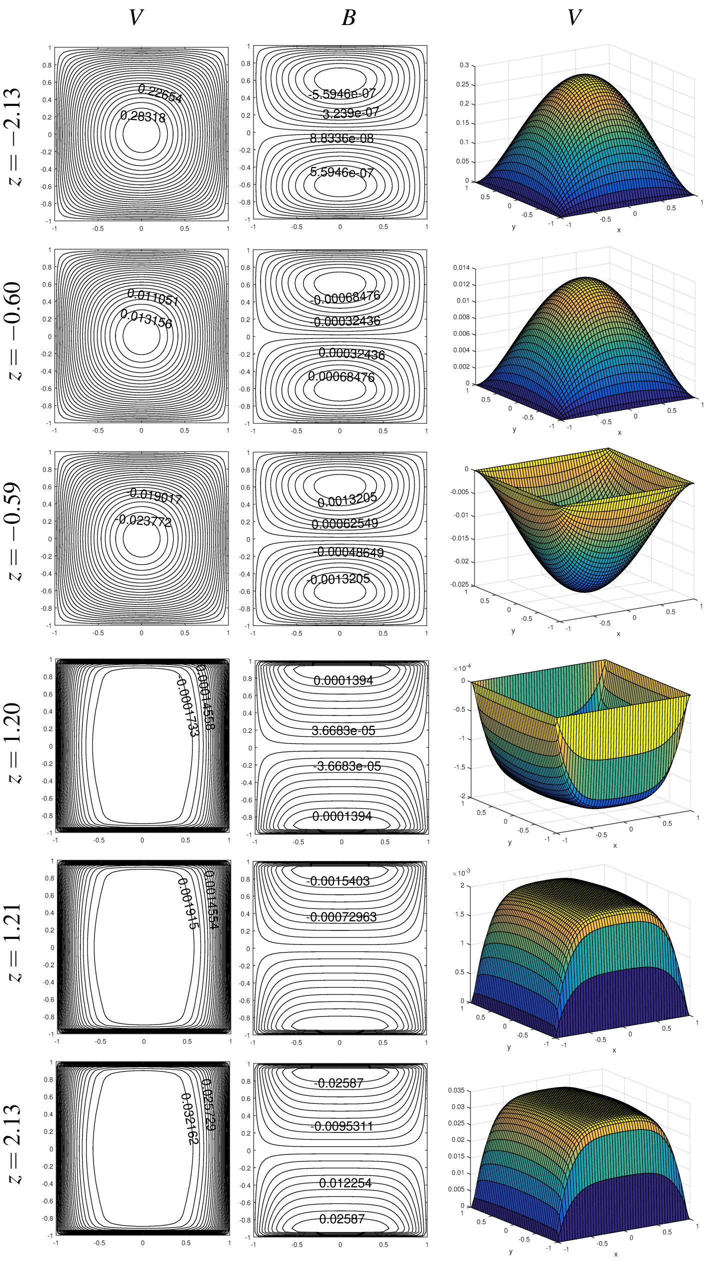

Fig. 3: Velocity and induced magnetic field, $M = 30$, $R_m = 2$.

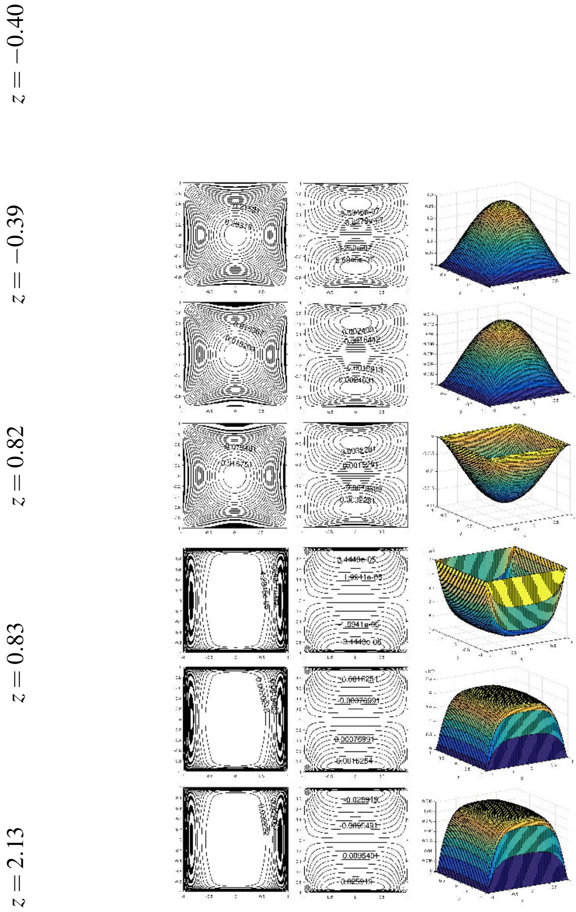

Fig. 4: Velocity and induced magnetic field, $M = 30$, $R_m = 25$.

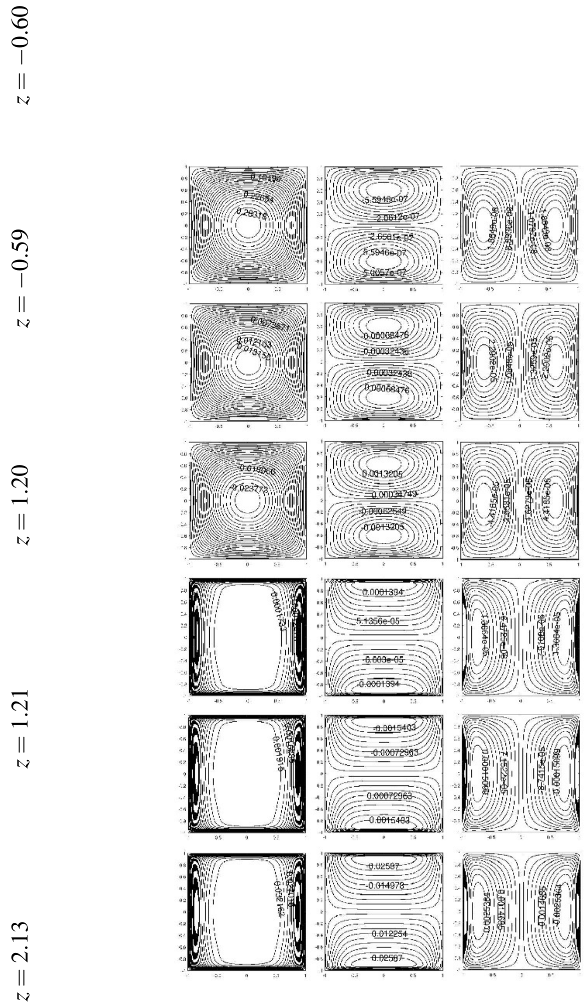

Fig. 5: Velocity, induced magnetic field and electric current, $M = 30$, $R_m = 2$.

5 Conclusion

In this study, the MHD duct flow under the effect of an axially changing magnetic field is considered in ducts where the magnets are located at several points through the pipe. The MHD flow equations are solved in terms of velocity, induced magnetic field and electric potential with the fully-developed flow assumption between these points. This way, the three-dimensional effects on the MHD flow are obtained throughout the pipe. The numerical results show that, axially changing magnetic field makes the flow to change its direction at a certain position of the pipe-axis. But then, the flow turns back to the pipe-axis direction after traveled a shorter distance for the case of increasing Hartmann number than the case of increasing magnetic Reynolds number. Thus, with the axially-dependent applied magnetic field, the 3D flow behavior of MHD flow is very well captured by using the DRBEM numerical approach.

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On certain bivariate generalized Bernstein operators

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Abstract: In approximation theory, positive linear operators are generalized in order to approximate continous functions with better convergence results. Quite recently, some new bivariate operators have been introduced to extend Bernstein operators and obtain more accurate and sensitive numerical results. In this study, we focus on some recent bivariate positive linear operators to approximate functions.

Keywords: Bersntein operators, Positive linear operators, Bivariate operators.

1 Introduction

Assume that $D = (d_{l,o,u,v})$ is a four-dimensional summability method. Given a double sequence $\varrho = (\varrho_{u,v})$, D transform of ϱ, denoted by $D\varrho := ((D\varrho)_{l,o})$, is defined as

$$(D\varrho)_{l,o} = \sum_{u,v=1}^{\infty} d_{l,o,u,v}\varrho_{u,v},$$

and the double series is $\varPi$-convergent for $(l, o) \in \mathbb{N}^2$. When four-dimensional matrix $D = (d_{l,o,u,v})$ maps every bounded $\varPi-$convergent sequence into a $\varPi-$convergent sequence with the same $\varPi-$limit, it is called $RH-$*regular* (shortly RHR). A four-dimensional matrix $D = (d_{l,o,u,v})$ is RHR if and only if

(a) $\varPi - \lim_{l,o} d_{l,o,u,v} = 0,$

(b) $\varPi - \lim_{l,o} \sum_{u,v=1}^{\infty} d_{l,o,u,v} = 1,$

(c) $\varPi - \lim_{l,o} \sum_{u=1}^{\infty} \left|d_{l,o,u,v}\right| = 0 \ (\forall v \in \mathbb{N}),$

(d) $\varPi - \lim_{l,o} \sum_{v=1}^{\infty} \left|d_{l,o,u,v}\right| = 0 \ (\forall u \in \mathbb{N}),$

(e) $\sum_{u,v=1}^{\infty} \left|d_{l,o,u,v}\right|$ is $\varPi-$convergent,

(f) The inequality $\sum_{u,v>E_2} \left|d_{l,o,u,v}\right| < E_1$ is satisfied for finite positive integers E_1 and E_2 and for each $(l, o) \in \mathbb{N}^2$.

These conditions are called Robison-Hamilton conditions. Assume that $D = (d_{l,o,u,v})$ is a nonnegative RHR matrix, and $S \subset \mathbb{N}^2$, then $D-$density of S is defined as

$$\delta_D^2(S) := \varPi - \lim_{l,o} \sum_{(u,v)\in S} d_{l,o,u,v}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $\varrho = (\varrho_{u,v})$ is called $D-$*statistically convergent* to Q and denoted by $st_D^2 - \lim_{u,v} \varrho_{u,v} = Q$ if, for every $\tau > 0$,

$$\delta_D^2(\left\{(u, v) \in \mathbb{N}^2 : |\varrho_{u,v} - Q| \geq \tau\right\}) = 0.$$

A $\varPi-$convergent double sequence is $D-$statistically convergent to the same number even if converse statement may not be true. When $D = C(1, 1)$, $C(1, 1)-$statistical convergence becomes statistical convergence for double sequences, where $C(1, 1) = (c_{l,o,u,v})$ is double Cesàro matrix, defined by $c_{l,o,u,v} = 1/lo$ if $1 \leq u \leq o$, $1 \leq v \leq l$, and $c_{l,o,u,v} = 0$ otherwise. Suppose that $(\xi_{u,v})$ is a double sequence of nonnegative numbers with condition $\xi_{0,0} > 0$, then power series

$$\xi(a,b) := \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v$$

has radius of convergence Θ, where $\Theta \in (0,\infty]$ and $a, b \in (0,\Theta)$. When following equality is satisfied

$$\lim_{a,b\to\Theta^-} \frac{1}{\xi(a,b)} \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v \varrho_{u,v} = Q$$

for each $a, b \in (0,\Theta)$, then double sequence $\varrho = (\varrho_{u,v})$ is said to be convergent to Q in the sense of PSM. PSM for double sequences is regular if and only if

$$\lim_{a,b\to\Theta^-} \frac{\sum\limits_{r=0}^{\infty} \xi_{r,v} a^u}{\xi(a,b)} = 0; \qquad \lim_{a,b\to\Theta^-} \frac{\sum\limits_{s=0}^{\infty} \xi_{\mu,s} b^v}{\xi(a,b)} = 0$$

are satisfied for any μ, υ. In this work, we assume that PSM is regular. When $\Theta = 1$ and $\xi_{u,v} = 1$ PSM becomes Abel summability method, and it becomes logarithmic summability method if $\xi_{u,v} = \frac{1}{(u+1)(v+1)}$.

2 Recent results on bivariate operators

In this section, the definitions of some recent bivariate operators and related polynomials are provided.
Let throughout the paper the binomial coefficients be given by the formula:

$$\binom{p}{i} = \begin{cases} \frac{p!}{i!(p-i)!}, & 0 \le i \le p, \\ 0, & \text{otherwise.} \end{cases}$$

The following polynomial functions

$$\begin{aligned} a_{u,0}(\rho;x) &= (1-x)^u (1-\rho_1 x), \\ a_{u,i}(\rho;x) &= x^i(1-x)^{u-i}\left(\binom{u}{v} + \rho_i - \rho_i x - \rho_{i+1}x\right), \; i = 1,2\ldots,[\frac{u}{2}]-1, \\ a_{u,[\frac{u}{2}]}(\rho;x) &= x^{[\frac{u}{2}]}(1-x)^{u-[\frac{u}{2}]}\left(\binom{u}{[\frac{u}{2}]} + \rho^{[\frac{u}{2}]} - \rho^{[\frac{u}{2}]}x + \rho^{[\frac{u}{2}]+1}x\right), \\ a_{u,i}(\rho;x) &= x^i(1-x)^{u-i}\left(\binom{u}{v} - \rho_i + \rho_i x + \rho_{i+1}x\right), \; i = [\frac{u}{2}]+1,\ldots,u-1, \\ a_{u,u}(\rho;x) &= x^u(1-\rho_u+\rho_u x) \end{aligned} \tag{1}$$

are called generalized Bernstein polynomials of degree u $(u \ge 2)$ and for $x \in [0,1]$ with shape parameters ρ_i, $i = 1,2,\ldots,u$, where

$$\begin{cases} \rho_i \in [-\binom{u}{i}, \binom{u}{i-1}] & ; i = 1,2,\ldots,[\frac{u}{2}] \\ \rho_i \in [-\binom{u}{i-1}, \binom{u}{i}] & ; i = [\frac{u}{2}]+1,\ldots,u \end{cases} \quad \text{with} \quad \begin{cases} [\frac{u}{2}] = \frac{u}{2} & ; \text{if u is even} \\ [\frac{u}{2}] = \frac{u-1}{2} & ; \text{if u is odd.} \end{cases} \tag{2}$$

These polynomials were introduced by Han et al. in. [1] and they are reduced to classical Bernstein basis functions $b_{u,i}(x)$ of degree u on $x \in [0,1]$ which is defined as

$$b_{u,i}(x) = \binom{u}{v} x^i (1-x)^{u-i}, \qquad i = 0,\ldots,u$$

when $\rho_i = 0$ $(i = 1,2,\ldots,u)$. Generalized Bernstein basis functions with parameters ρ_i $(i = 1,2,\ldots,u)$ are linearly independent (see [2]) and these basis functions are effectively and flexibly used in designing parametric curves and surfaces (see [1, 2]). These functions also have partition of unity, symmetry and nonnegativity properties (see [1]). In 2017, Hu et al. [2] have obtained the following equations to convert classical Bernstein polynomials of degree u to generalized Bernstein polynomials of degree u associated with shape parameters ρ_i:

$$
\begin{aligned}
a_{u,0}(\rho;x) &= b_{u+1,0}(x) + \frac{\binom{u}{0}-\rho_1}{\binom{u+1}{1}} b_{u+1,1}(x),\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}+\rho_i}{\binom{u+1}{i}} b_{u+1,i}(x) + \frac{\binom{u}{i}-\rho_{i+1}}{\binom{u+1}{i+1}} b_{u+1,i+1}(x),\ \ i=1,2\dots,\left[\frac{u}{2}\right]-1,\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}+\rho_v}{\binom{u+1}{i}} b_{u+1,i}(x) + \frac{\binom{u}{i}+\rho_{i+1}}{\binom{u+1}{i+1}} b_{u+1,i+1}(x),\ i=\left[\frac{u}{2}\right],\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}-\rho_i}{\binom{u+1}{i}} b_{u+1,i}(x) + \frac{\binom{u}{i}+\rho_{i+1}}{\binom{u+1}{i+1}} b_{u+1,i+1}(x),\ \ i=\left[\frac{u}{2}\right]+1,\dots,u-1,\\
a_{u,u}(\rho;x) &= \frac{\binom{u}{u}-\rho_u}{\binom{u+1}{u}} b_{u+1,u}(x) + b_{u+1,u+1}(x).
\end{aligned}
\tag{3}
$$

Let $C[0,1]=\mathbf{C}$ be the space of all continuous functions on unit interval $[0,1]$ and $C\left([0,1]\times[0,1]\right)=\bar{\mathbf{C}}$. The operators $\mathcal{B}_u^{\nu}, \mathcal{B}_v^{\mu} : \mathbf{C} \longrightarrow \mathbf{C}$ for any $u, v \in \mathbb{N}$ are given as follows, respectively,

$$
\mathcal{B}_u^{\nu}(f;y) = \sum_{i=0}^{u} f\left(\frac{i}{u}\right) a_{u,i}(\nu;y),
\tag{4}
$$

$$
\mathcal{B}_v^{\mu}(g;z) = \sum_{j=0}^{v} g\left(\frac{j}{v}\right) a_{v,j}(\mu;z),
\tag{5}
$$

where polynomials $a_{u,i}(\nu;y)$ and $a_{v,j}(\mu;z)$ are given in (3). The parametric extension of (4) and (5) for $u,v\in\mathbb{N}$ and $h\in\bar{\mathbf{C}}$ are the operators

$$
\mathcal{B}_u^{\nu,y}, \mathcal{B}_v^{\mu,z} : \bar{\mathbf{C}} \longrightarrow \bar{\mathbf{C}},
$$

where

$$
\mathcal{B}_u^{\nu,y}(h;y,z) = \sum_{i=0}^{u} a_{u,i}(\nu;y) h\left(\frac{i}{u},\frac{i}{u}\right),
\tag{6}
$$

$$
\mathcal{B}_v^{\mu,z}(h;y,z) = \sum_{j=0}^{v} a_{v,j}(\mu;z) h\left(\frac{j}{v},\frac{j}{v}\right).
\tag{7}
$$

The parametric extensions of bivariate operators commute on $\bar{\mathbf{C}}$. Their product establishes bivariate operators $\mathcal{B}_{u,v}^{\nu,\mu} : \bar{\mathbf{C}} \longrightarrow \bar{\mathbf{C}}$ defined for any $u,v\in\mathbb{N}$ and any $h\in\bar{\mathbf{C}}$ by the relation

$$
\mathcal{B}_{u,v}^{\nu,\mu}(h;y,z) \quad = \quad \sum_{i=0}^{u}\sum_{j=0}^{v} a_{u,i}(\nu;y) a_{v,j}(\mu;z) h\left(\frac{i}{u},\frac{j}{v}\right).
\tag{8}
$$

Let $z,y\in\mathcal{I}$, we define following operators

$$
\mathcal{K}_{c,d}^{p,q}(\vartheta;z,y) = (c+1)(d+1)\sum_{m=0}^{c}\sum_{n=0}^{d} a_{c,m}(p;z) a_{d,n}(q;y) \int_{\frac{m}{c+1}}^{\frac{m+1}{c+1}} \int_{\frac{n}{d+1}}^{\frac{n+1}{d+1}} \vartheta(t,s)\, dt ds,
\tag{9}
$$

where shape parameters p_m and q_n satisfy the conditions (2), and call them as generalized bivariate Bernstein-Kantorovich operators. We refer to certain recent papers about approximation of functions by positive linear operators [9–11, 30–32, 35, 36].

By the following theorem we give uniform convergence of some positive linear operators.

Theorem 1. *For any $\alpha\in[0,1]$, then $L(r)$ converge uniformly to r on $[0,1]$, that is,*

$$
\lim \|L(r)-r\| = 0,
$$

where $L = \mathcal{B}_{u,v}^{\nu,\mu},\ \ \mathcal{K}_{c,d}^{p,q}$.

Proof: Taking into account moments of Bernstein type operators we have

$$
L(e_0) = e_0 \text{ as } m\to\infty, \quad L(e_1;x) = e_1 \text{ as } m\to\infty
$$

and similarly $L_{m,\alpha}(e_2)=e_2$ if one applies the limit. Hence, by the Korovkin theorem, we obtain

$$
\lim \|L(f)-f\| = 0,
$$

where $L = \mathcal{B}_{u,v}^{\nu,\mu},\ \ \mathcal{K}_{c,d}^{p,q}$. □

3 Concluding Remarks

This paper is based on the results in [4, 37], this is why we refer these papers for further literature. We will study approximation properties of Stancu, Schurer, Kantorovich and some other modifications of focused bivariate operators in future.

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On some beta-type operators

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Abstract: One of the main ideas of approximate theory is to approximate functions via positive linear operators. Recently, some new beta-type operators have been introduced using some shape parameters and combining the current operators in order to obtain more accurate and sensitive numerical results. In this study, we focus on certain recent beta-type positive linear operators to approximate functions.

Keywords: Beta-type operators, Positive linear operators, Convergence.

1 Introduction

Karl Weierstrass focused on approximation of continuous functions by polynomials (see [25]). Bernstein demonstrated a differented way for the proof of well-known Weierstrass approximation theorem (see [8]).

There are several extentions of Bernstein operators, we refer to the following list for further research:

(a) λ-Bernstein operators [15] with $\tilde{b}_{n,i}(\lambda;x)$ Bézier bases and shape parameter λ (see [24]):

$$\begin{aligned}
\tilde{b}_{n,0}(\lambda;x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\
\tilde{b}_{n,i}(\lambda;x) &= b_{n,i}(x) + \frac{n-2i+1}{n^2-1}\lambda b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1}\lambda b_{n+1,i+1}(x), \quad i = 1,2\ldots,n-1, \\
\tilde{b}_{n,n}(\lambda;x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x).
\end{aligned} \tag{1}$$

(b) Bernstein type operators by using continuously differentiable ∞ times function τ on $[0,1]$ [16].
(c) New variant of Bernstein operators [21]
(d) (p,q)-Bernstein operators.
(e) Stancu-type λ-Bernstein operators [23].
(f) Modified U_n operators [14] and references therein.
(g) α-Bernstein operators [17, 19] $p^{(\alpha)}_{m,\gamma,j}(z)$ denotes the α-Bernstein-Schurer polynomials defined by

$$p^{(\alpha)}_{1,\gamma,0}(z) = 1-z, \quad p^{(\alpha)}_{1,\gamma,1}(z) = z$$

and

$$\begin{aligned}
p^{(\alpha)}_{m,\gamma,j}(z) = & \left[(1-\alpha)\, z \binom{m+\gamma-2}{j} + (1-\alpha)(1-z)\binom{m+\gamma-2}{j-2}\right. \\
& \left. +\alpha z (1-z)\binom{m+\gamma}{j}\right] z^{j-1}(1-z)^{m+\gamma-(j+1)} \qquad (m \geq 2).
\end{aligned} \tag{2}$$

(h) Bivariate extension of α-Bernstein-Durrmeyer operators [20].
(i) Kantorovich modifications of α-Bernstein operators.
(j) λ-Bernstein-Schurer operators [22].
(k) Bivariate λ-Bernstein operators [31].
(l) λ-Bernstein-Kantorovich operators [18].
(m) Univariate and bivariate λ-Bernstein-Kantorovich operators [12].
(n) Genuine modified Bernstein-Durrmeyer operators.
(p) Blending type Bernstein operators [6].

(r) Blending type Bernstein-Kantorovich operators [7].
(s) Baskakov operators based on $\alpha \in [0,1]$ [37]:

$$\mathcal{B}_m^{(\alpha)}(\zeta;x)=\sum_{j=0}^{\infty} b_{m,j}^{(\alpha)}(x)\zeta\left(\frac{j}{m}\right), \quad x\in[0,\infty), \tag{3}$$

where

$$b_{m,j}^{(\alpha)}(x)=\frac{x^{j-1}}{(1+x)^{m+j-1}}\left[\frac{\alpha x}{(1+x)}\binom{m+j-1}{j}-(1-\alpha)(1+x)\binom{m+j-3}{j-2}+(1-\alpha)x\binom{m+j-1}{j}\right].$$

These $\mathcal{B}_m^{(\alpha)}(\zeta;x)$ reduce to Baskakov operators [38] for $\alpha=1$.
(t) A Durrmeyer type generalization of the operators (3) is

$$\mathcal{B}_{m,\alpha}^{*}(\zeta;x) \;=\; \sum_{j=0}^{\infty} b_{m,j}^{(\alpha)}(x)\frac{1}{B(j+1,m)}\int_0^\infty \frac{t^j}{(1+t)^{m+j+1}}\zeta(t)dt, \tag{4}$$

where $B(j+1,m)$ is the beta function defined as

$$B(r,s)=\int_0^\infty \frac{w^{r-1}}{(1+w)^{r+s}}dw=\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \quad r,s>0.$$

(u) Bézier variant $\mathcal{B}_{m,\alpha,\theta}^{*}$ of the operators $\mathcal{B}_{m,\alpha}^{*}$ as follows:

$$\mathcal{B}_{m,\alpha,\theta}^{*}(\zeta;x)=\sum_{j=0}^{\infty} Q_{m,j,\alpha}^{(\theta)}(x)\frac{1}{B(j+1,m)}\int_0^\infty \frac{t^j}{(1+t)^{m+j+1}}\zeta(t)dt, \tag{5}$$

where $\theta\ge 1$ and $Q_{m,j,\alpha}^{(\theta)}(x)=\left[V_{m,j,\alpha}(x)\right]^\theta-\left[V_{m,j+1,\alpha}(x)\right]^\theta$ with $V_{m,j,\alpha}(x)=\sum_{v=j}^{\infty} b_{m,v}^{(\alpha)}(x)$.
Alternatively, (5) can be written as (see [3])

$$\mathcal{B}_{m,\alpha,\theta}^{*}(\zeta;x)=\int_0^\infty U_{m,\alpha,\theta}(x,t)\zeta(t)dt, \;\; x\in[0,\infty), \tag{6}$$

where

$$U_{m,\alpha,\theta}(x,t)=\sum_{j=0}^{\infty} Q_{m,j,\alpha}^{(\theta)}(x)\frac{1}{B(j+1,m)}\frac{t^j}{(1+t)^{m+j+1}}.$$

(v) For $m\in\mathbb{N}$ and $\rho>0$, the functional (see [4])

$$F_{m,i}^{\rho}: C[0,1]\to\mathbb{R}$$

is defined by

$$F_{m,i}^{\rho}(g)=\int_0^1 \mu_{m,i}^{\rho}(t)g(t)\,dt \qquad (i=1,2,\ldots,m-1), \tag{7}$$

$$F_{m,0}^{\rho}(g)=g(0), \quad F_{m,m}^{\rho}(g)=g(1),$$

where $\mu_{m,i}^{\rho}(t)$ in (7) is given by the formula

$$\mu_{m,i}^{\rho}(t)=\frac{t^{i\rho-1}(1-t)^{(m-i)\rho-1}}{B(i\rho,(m-i)\rho)}$$

and the Euler's beta function in the last equality is defined by

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt \quad (a,b>0).$$

(y) Assume that θ and β are two real parameters satisfying $0 \leq \theta \leq \beta$. Genuine (α, ρ)-Durrmeyer-Stancu operators $U_{m,\alpha}^{\beta,\theta,\rho}$ defined by (see [2])

$$U_{m,\alpha}^{\beta,\theta,\rho}(g;y) = \sum_{i=0}^{m} F_{m,i}^{\beta,\theta,\rho}(g)\, p_{m,i}^{(\alpha)}(y),$$

where

$$F_{m,i}^{\beta,\theta,\rho}(g) = \int_0^1 \mu_{m,i}^{\rho}(t) g\left(\frac{mt+\theta}{m+\beta}\right) dt$$

for $i = 1, 2, \ldots, m-1$, $F_{m,0}^{\beta,\theta,\rho}(g) = g\left(\frac{\theta}{m+\beta}\right)$ and $F_{m,1}^{\beta,\theta,\rho}(g) = g\left(\frac{m+\theta}{m+\beta}\right)$. Consequently, one may write the operators $U_{m,\alpha}^{\beta,\theta,\rho}$ as

$$\begin{aligned} U_{m,\alpha}^{\beta,\theta,\rho}(g;y) &= \sum_{i=1}^{m-1} \int_0^1 \left[\frac{t^{i\rho-1}(1-t)^{(m-i)\rho-1}}{B(i\rho,(m-i)\rho)} g\left(\frac{mt+\theta}{m+\beta}\right) dt\right] p_{m,i}^{(\alpha)}(y) \\ &\quad + g\left(\frac{\theta}{m+\beta}\right) p_{m,0}^{(\alpha)}(y) + g\left(\frac{m+\theta}{m+\beta}\right) p_{m,m}^{(\alpha)}(y). \end{aligned} \tag{8}$$

This paper is focused on the literature review of certain blending type Bernstein operators.

2 Convergence of beta-type operators

In this section, we study convergence of some beta-type operators that are given in the previous section.

The moments of $U_{m,\alpha}^{\beta,\theta,\rho}$ operators are given as below:

Let $e_i(y) = y^i$, $(i = 0, 1, 2)$. Then, the operators $U_{m,\alpha}^{\beta,\theta,\rho}$ satisfy

$$\begin{aligned} U_{m,\alpha}^{\beta,\theta,\rho}(e_0;y) &= 1, \\ U_{m,\alpha}^{\beta,\theta,\rho}(e_1;y) &= \frac{my+\theta}{m+\beta}, \\ U_{m,\alpha}^{\beta,\theta,\rho}(e_2;y) &= \frac{m^3\rho y^2 + (y-y^2)(m^2+2m\rho(1-\alpha)) + m^2 y}{(m+\beta)^2(m\rho+1)} + \frac{\theta^2+2m\theta y}{(m+\beta)^2}. \end{aligned}$$

By the following theorem, a uniform convergence theorem for some positive linear operators is given.

Theorem 1. *For any $\alpha \in [0,1]$, then $L(r)$ converge uniformly to r on $[0,1]$, that is,*

$$\lim_{m\to\infty} \|L(r) - r\| = 0,$$

where $L = U_{m,\alpha}^{\beta,\theta,\rho}, \mathcal{L}_{p,\lambda}^{(\alpha,s)}, \ \mathcal{B}_{p,\lambda}^{\alpha,s}, \ \mathcal{K}_{p,\lambda}^{\alpha,s}, \ \mathcal{B}_{m,\alpha,\theta}^{*}$.

Proof: Taking into account moments of Bernstein type operators we have

$$L(e_0) = e_0 \text{ as } m \to \infty, \quad L(e_1;x) = e_1 \text{ as } m \to \infty$$

and similarly $L_{m,\alpha}(e_2) = e_2$ as $m \to \infty$. Hence, by the Korovkin theorem, we obtain

$$\lim_{m\to\infty} \|L(f) - f\| = 0,$$

where $L = U_{m,\alpha}^{\beta,\theta,\rho}, \mathcal{L}_{p,\lambda}^{(\alpha,s)}, \ \mathcal{B}_{p,\lambda}^{\alpha,s}, \ \mathcal{K}_{p,\lambda}^{\alpha,s}, \ \mathcal{B}_{m,\alpha,\theta}^{*}$. □

3 Concluding Remarks

This paper is based on the results in [2, 3], this is why we refer these papers for further literature. We will study approximation properties some related operators in close future.

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A New Function Type in Grill Minimal Spaces, Its Properties and A New Supra Topology

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Abstract: .

In this study, the minimal grill semi-closure ($\mathcal{MGSC}$) function is defined using the concepts of minimal structure, grill and m-semi-closure. The properties of this function are examined. Using this function, a new set operator is defined. Moreover, with the help of this operator, a new supra topology is obtained.
Keywords: Minimal structure, Minimal space, Minimal semi-closure, Grill, Grill minimal space.

1 Introduction and Preliminaries

Choquet introduced the concept of grill in [1]. There have been many studies in the literature on minimal spaces [2]-[4] . In minimal structures, the concepts m-semi-open, m-semi-closed and m-semi-closure were introduced to the literature by Won Keun Min [5] . Modak obtained a new topology using the concepts of grill and minimal structure [4].

In this study, a new supra topology is obtained by defining a new function type and a new set operator in grill minimal spaces. Moreover, the properties of these is examined.

Definition 1 ([6]). *Let X be nonempty set and $\mathcal{M} \subseteq \mathcal{P}(X)$. If $\{X, \emptyset\} \subseteq \mathcal{M}$, then $\mathcal{M}$ is called minimal structure on X and $(X, \mathcal{M})$ is called minimal space. Each element of $\mathcal{M}$ is called minimal open (briefly m-open) and the complement of a m-open is called minimal closed (m-closed).*

Definition 2 ([7]). *Let $(X, \mathcal{M})$ be a minimal space and $A \subseteq X$. m-interior of A and m-closure of A are defined as follows:*

1. $m\text{-}Int(A) = \bigcup\{U : U \in \mathcal{M} \text{ and } U \subseteq A\}$
2. $m\text{-}Cl(A) = \bigcap\{F : X \setminus F \in \mathcal{M} \text{ and } A \subseteq F\}$

Definition 3 ([5]). *Let $(X, \mathcal{M})$ be a minimal space. A subset A of X is called a m-semi-open set if $A \subseteq mCl(mInt(A))$. The complement of a m-semi-open set is called a m-semi-closed set. The family of all m-semi-open sets in X is denoted by $mSO(X)$.*

Definition 4 ([5]). *Let $(X, \mathcal{M})$ be a minimal space and $A \subseteq X$. m-semi-interior of A and m-semi-closure of A are defined as follows:*

1. $m\text{-}sInt(A) = \bigcup\{U : U \text{ is } m\text{-semiopen and } U \subseteq A\}$
2. $m\text{-}sCl(A) = \bigcap\{F : X \setminus F \text{ is semi open and } A \subseteq F\}$

Lemma 5 ([5]). *Every m-open set is m-semi open set.*

Lemma 6 ([5]). *Let $(X, \mathcal{M})$ be a minimal space and $A \subseteq X$. Then $x \in m\text{-}sCl(A)$ if and only if $A \cap V \neq \emptyset$ for every m-semi open set V containing x.*

Definition 7 ([1]). *Let $X \neq \emptyset$ and $\mathcal{G} \subseteq \mathcal{P}(X)$. If $\mathcal{G}$ satifies the following conditions, it is called a grill on X :*

1. $\emptyset \notin \mathcal{G}$.
2. *If $A \in \mathcal{M}$ and $A \subseteq B$, then $B \in \mathcal{G}$.*
3. *If $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $A \in \mathcal{G}$.*

If $(X, \mathcal{M})$ is a minimal space, then the triplet $(X, \mathcal{M}, \mathcal{G})$ is called a grill minimal space.

Definition 8 ([4]). *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $A \subseteq X$. An operator $()^{*\mathcal{M}} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by*

$$()^{*\mathcal{M}}(A)(\mathcal{G}) = \{x \in X : U \cap A \in \mathcal{G} \textit{ for every } U \in \mathcal{M}(x)\}$$

where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$.

Theorem 9. *[8] Let X be nonempty subset. $\tau^s \subseteq \mathcal{P}(X)$ is called a supra topology on X if τ^s satisfies the following conditions:*

1. $\emptyset$, $X \in \tau^s$
2. *The arbitrary union of the sets belonging to τ^s belongs to τ^s.*

2 $\gamma^{\mathcal{M}}$ Function in Grill Minimal Spaces

Definition 10. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $A \subseteq X$. An operator $\gamma^{\mathcal{M}} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by*

$$\gamma^{\mathcal{M}}(A)(\mathcal{G}) = \{x \in X : m\text{-}sCl(U) \cap A \in \mathcal{G} \textit{ for every } U \in \mathcal{M}(x)\}$$

and is called the minimal grill semi-closure ($\mathcal{MGSC}$) function of A with respect to $\mathcal{G}$ and $\mathcal{M}$. Sometimes, we write briefly $\gamma^{\mathcal{M}}(A)$ instead of $\gamma^{\mathcal{M}}(A)(\mathcal{G})$.

Theorem 11. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $A \subseteq X$. Then,*

$$(A)^{*\mathcal{M}} \subseteq \gamma^{\mathcal{M}}(A)(\mathcal{G})$$

.

Proof: Let $x \in (A)^{*\mathcal{M}}$. Then $U \cap A \in \mathcal{G}$ for every $U \in \mathcal{M}(x)$. Therefore $(U \cap A) \subseteq (m\text{-}sCl(U) \cap A) \in \mathcal{G}$ from the definition of grill. Consequently, $x \in \gamma^{\mathcal{M}}(A)(\mathcal{G})$. □

Let us show that the inclusion in the above theorem is strictly hold.

Example 12. *Let $\mathcal{M} = \{X, \emptyset, \{a, b\}, \{b, c\}\}$ be a minimal space on $X = \{a, b, c, d\}$ with a grill $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$. Consider the subset $A = \{d\}$. Then,*

$$(A)^{*\mathcal{M}} = \{d\} \subsetneq \gamma^{\mathcal{M}}(A) = X$$

Theorem 13. *Let $(X, \mathcal{M})$ be a minimal space, $A, B \subseteq X$ and $\mathcal{G}$, $\mathcal{L}$ be two grills on X.*

1. *If $A \notin \mathcal{G}$, then $\gamma^{\mathcal{M}}(A) = \emptyset$.*
2. $\gamma^{\mathcal{M}}(\emptyset) = \emptyset$.
3. *If $A \subseteq B$, then $\gamma^{\mathcal{M}}(A) \subseteq \gamma^{\mathcal{M}}(B)$.*
4. *$\gamma^{\mathcal{M}}(A)$ is minimal closed. That is, $\gamma^{\mathcal{M}}(A) = m\text{-}Cl(\gamma^{\mathcal{M}}(A))$.*
5. *If $\mathcal{G} \subseteq \mathcal{L}$, then $\gamma^{\mathcal{M}}(A)(\mathcal{G}) \subseteq \gamma^{\mathcal{M}}(A)(\mathcal{L})$.*

Proof:

1. Let $A \notin \mathcal{G}$. From the definition of grill, $(m\text{-}sCl(U) \cap A) \notin \mathcal{G}$ for every $U \in \mathcal{M}$. Therefore $\gamma^{\mathcal{M}}(A) = \emptyset$.
2. It is obvious from 1.
3. Let $x \in \gamma^{\mathcal{M}}(A)$. So, $(m\text{-}sCl(U) \cap A) \in \mathcal{G}$ for every $U \in \mathcal{M}(x)$. From the definition of grill, $(m\text{-}sCl(U) \cap A) \subseteq (m\text{-}sCl(U) \cap B) \in \mathcal{G}$. Therefore $x \in \gamma^{\mathcal{M}}(B)$. That is, $\gamma^{\mathcal{M}}(A) \subseteq \gamma^{\mathcal{M}}(B)$.
4. We have $\gamma^{\mathcal{M}}(A) \subseteq m\text{-}Cl(\gamma^{\mathcal{M}}(A))$. Now, we must show that $m\text{-}Cl(\gamma^{\mathcal{M}}(A)) \subseteq \gamma^{\mathcal{M}}(A)$. Let $x \in m\text{-}Cl(\gamma^{\mathcal{M}}(A))$. Then, $(U \cap \gamma^{\mathcal{M}}(A)) \neq \emptyset$ for every $U \in \mathcal{M}(x)$. There exists $y \in X$ such that $y \in U$ and $y \in \gamma^{\mathcal{M}}(A)$. Therefore $U \in \mathcal{M}(y)$. Since $y \in \gamma^{\mathcal{M}}(A)$, $m\text{-}sCl(U) \cap A \in \mathcal{G}$. Consequently, $x \in \gamma^{\mathcal{M}}(A)$. That is, $\gamma^{\mathcal{M}}(A) = m\text{-}Cl(\gamma^{\mathcal{M}}(A))$.
5. Let $\mathcal{G} \subseteq \mathcal{L}$ and $x \in \gamma^{\mathcal{M}}(A)(\mathcal{G})$. Therefore $m\text{-}sCl(U) \cap A \in \mathcal{G}$ for every $U \in \mathcal{M}(x)$. Since $\mathcal{G} \subseteq \mathcal{L}$, $m\text{-}sCl(U) \cap A \in \mathcal{L}$ for every $U \in \mathcal{M}(x)$. Therefore $\gamma^{\mathcal{M}}(A)(\mathcal{G}) \subseteq \gamma^{\mathcal{M}}(A)(\mathcal{L})$.

□

A minimal space is called "minimal space with the property (I)" if the finite intersection of minimal open sets is minimal open.

Theorem 14. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space with the property (I) and $A, B \subseteq X$. Then,*

$$\gamma^{\mathcal{M}}(A \cup B) = \gamma^{\mathcal{M}}(A) \cup \gamma^{\mathcal{M}}(B)$$

.

Proof: From Theorem 13-3), $\gamma^{\mathcal{M}}(A) \subseteq \gamma^{\mathcal{M}}(A \cup B)$ and $\gamma^{\mathcal{M}}(B) \subseteq \gamma^{\mathcal{M}}(A \cup B)$. Therefore $\gamma^{\mathcal{M}}(A) \cup \gamma^{\mathcal{M}}(B) \subseteq \gamma^{\mathcal{M}}(A \cup B)$. Now, we must show that $\gamma^{\mathcal{M}}(A \cup B) \subseteq \gamma^{\mathcal{M}}(A) \cup \gamma^{\mathcal{M}}(B)$. Let $x \notin (\gamma^{\mathcal{M}}(A) \cup \gamma^{\mathcal{M}}(B))$. There exists $U, V \in \mathcal{M}(x)$ such that $(m\text{-}sCl(U) \cap A) \notin$

$\mathcal{G}$ and $(m\text{-}sCl(V) \cap B) \notin \mathcal{G}$. From the property (I), $(U \cap V) \in \mathcal{M}(x)$. Suppose that $m\text{-}sCl(U \cap V) \cap (A \cup B) \in \mathcal{G}$. Then,

$$\begin{aligned} m\text{-}sCl(U \cap V) \cap (A \cup B) &\subseteq ((m\text{-}sCl(U) \cap m\text{-}sCl(V)) \cap (A \cup B) \\ &\subseteq (m\text{-}sCl(U) \cap A) \cup (m\text{-}sCl(V) \cap B) \in \mathcal{G} \\ &\Rightarrow (m\text{-}sCl(U) \cap A) \in \mathcal{G} \text{ or } (m\text{-}sCl(V) \cap B) \in \mathcal{G} \end{aligned}$$

. This is a contradiction. Therefore $m\text{-}sCl(U \cap V) \cap (A \cup B) \notin \mathcal{G}$. That is, $x \notin \gamma^{\mathcal{M}}(A \cup B)$. Consequently, $\gamma^{\mathcal{M}}(A \cup B) \subseteq \gamma^{\mathcal{M}}(A) \cup \gamma^{\mathcal{M}}(B)$. ☐

Definition 15. *[3] The minimal structure $(X, \mathcal{M})$ is said to be m-regular if for each m-closed set F and each $x \notin F$ there exist disjoint m-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 16. *[3] If the minimal structure $(X, \mathcal{M})$ is m-regular, for each $x \in X$ and each m -open set U containing x, there exists a m-open set W such that $x \in W \subseteq m\text{-}Cl(W) \subseteq U$.*

Theorem 17. *Let $(X, \mathcal{M}, \mathcal{G})$ be a m-regular grill minimal space with the property (I) and $A \subseteq X$. If $U \in \mathcal{M}$, then*

$$U \cap \gamma^{\mathcal{M}}(A) = U \cap \gamma^{\mathcal{M}}(U \cap A)$$

.

Proof: Since $\gamma^{\mathcal{M}}(U \cap A) \subseteq \gamma^{\mathcal{M}}(A)$, $U \cap \gamma^{\mathcal{M}}(U \cap A) \subseteq U \cap \gamma^{\mathcal{M}}(A)$. Conversely, let $x \in U \cap \gamma^{\mathcal{M}}(A)$. Therefore $x \in U$ and $x \in \gamma^{\mathcal{M}}(A)$. Since $(X, \mathcal{M})$ is m-regular, there exists $W \in \mathcal{M}(x)$ such that $x \in W \subseteq m\text{-}Cl(W) \subseteq U$. Let V be any m-open containing x. Then, $V \cap W \in \mathcal{M}(x)$. Since $x \in \gamma^{\mathcal{M}}(A)$, $m\text{-}sCl(W \cap V) \cap A \in \mathcal{G}$. Moreover, the definition of grill,

$$\begin{aligned} m\text{-}sCl(V \cap W) \cap A &\subseteq (m\text{-}sCl(V) \cap m\text{-}sCl(W)) \cap A \\ &\subseteq (m\text{-}sCl(V) \cap m\text{-}Cl(W)) \cap A \\ &\subseteq m\text{-}sCl(V) \cap (U \cap A) \in \mathcal{G} \end{aligned}$$

. Therefore $x \in \gamma^{\mathcal{M}}(U \cap A)$. Consequently, $U \cap \gamma^{\mathcal{M}}(A) \subseteq U \cap \gamma^{\mathcal{M}}(U \cap A)$.

☐

Theorem 18. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space with the property (I) and $A, B \subseteq X$. Then,*

$$\gamma^{\mathcal{M}}(A) \setminus \gamma^{\mathcal{M}}(B) = \gamma^{\mathcal{M}}(A \setminus B) \setminus \gamma^{\mathcal{M}}(B)$$

.

Proof: Since $\gamma^{\mathcal{M}}(A \setminus B) \subseteq \gamma^{\mathcal{M}}(A)$, $\gamma^{\mathcal{M}}(A \setminus B) \setminus \gamma^{\mathcal{M}}(B) \subseteq \gamma^{\mathcal{M}}(A) \setminus \gamma^{\mathcal{M}}(B)$. Conversely,

$$\begin{aligned} \gamma^{\mathcal{M}}(A) &= \gamma^{\mathcal{M}}((A \setminus B) \cup (A \cap B)) \\ &= \gamma^{\mathcal{M}}(A \setminus B) \cup \gamma^{\mathcal{M}}(A \cap B) \\ &\subseteq \gamma^{\mathcal{M}}(A \setminus B) \cup \gamma^{\mathcal{M}}(B) \end{aligned}$$

and therefore $\gamma^{\mathcal{M}}(A) \setminus \gamma^{\mathcal{M}}(B) \subseteq \gamma^{\mathcal{M}}(A \setminus B) \setminus \gamma^{\mathcal{M}}(B)$. The desired result is obtained. ☐

Theorem 19. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space with the property (I). If $B \notin \mathcal{G}$, then*

$$\gamma^{\mathcal{M}}(A \cup B) = \gamma^{\mathcal{M}}(A) = \gamma^{\mathcal{M}}(A \setminus B)$$

.

Proof: From Theorem 13-1., Theorem 14. and Theorem 18., $\gamma^{\mathcal{M}}(A \cup B) = \gamma^{\mathcal{M}}(A) \cup \gamma^{\mathcal{M}}(B) = \gamma^{\mathcal{M}}(A) = \gamma^{\mathcal{M}}(A \setminus B)$. ☐

3 $\psi^{\gamma}_{\mathcal{MG}}$-Operator

Definition 20. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $A \subseteq X$. An operator $\psi^{\gamma}_{\mathcal{MG}} : \mathcal{P}(X) \to \mathcal{M}$ is defined as*

$$\begin{aligned} \psi^{\gamma}_{\mathcal{MG}}(A) &= \{x \in X : \textit{there exist a } U \in \mathcal{M} \textit{ such that } (m\text{-}sCl(U) \setminus A) \notin \mathcal{G}\} \\ &= X \setminus \gamma^{\mathcal{M}}(X \setminus A) \end{aligned}$$

.

Theorem 21. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $A, B \subseteq X$.*

1. $\psi^{\gamma}_{\mathcal{MG}}(A)$ *is m-open.*
2. *If $A \subseteq B$, then $\psi^{\gamma}_{\mathcal{MG}}(A) \subseteq \psi^{\gamma}_{\mathcal{MG}}(B)$.*
3. $\psi^{\gamma}_{\mathcal{MG}}(\psi^{\gamma}_{\mathcal{MG}}(A)) = X \setminus \gamma^{\mathcal{M}}(\gamma^{\mathcal{M}}(X \setminus A))$
4. $\psi^{\gamma}_{\mathcal{MG}}(A) = \psi^{\gamma}_{\mathcal{MG}}(\psi^{\gamma}_{\mathcal{MG}}(A))$ *if and only if* $\gamma^{\mathcal{M}}(X \setminus A) = \gamma^{\mathcal{M}}(\gamma^{\mathcal{M}}(X \setminus A))$

Proof:

1. From Theorem 13-4. and the definition $\psi^{\gamma}_{\mathcal{MG}}(A)$, it is clear.
2. From Theorem 13-3., it is obvious.
3.

$$\begin{aligned}\psi^{\gamma}_{\mathcal{MG}}(\psi^{\gamma}_{\mathcal{MG}}(A)) &= \psi^{\gamma}_{\mathcal{MG}}(X \setminus \gamma^{\mathcal{M}}(X \setminus A)) \\ &= X \setminus \gamma^{\mathcal{M}}(X \setminus (X \setminus \gamma^{\mathcal{M}}(X \setminus A)) \\ &= X \setminus \gamma^{\mathcal{M}}(\gamma^{\mathcal{M}}(X \setminus A))\end{aligned}$$

4. From 3),

$$\begin{aligned}\psi^{\gamma}_{\mathcal{MG}}(\psi^{\gamma}_{\mathcal{MG}}(A)) = \psi^{\gamma}_{\mathcal{MG}}(A) &\Leftrightarrow X \setminus \gamma^{\mathcal{M}}(\gamma^{\mathcal{M}}(X \setminus A)) = X \setminus \gamma^{\mathcal{M}}(X \setminus A) \\ &\Leftrightarrow \gamma^{\mathcal{M}}(\gamma^{\mathcal{M}}(X \setminus A)) = \gamma^{\mathcal{M}}(X \setminus A).\end{aligned}$$

□

Theorem 22. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space with the property (I) and $A, B \subseteq X$.*

1. $\psi^{\gamma}_{\mathcal{MG}}(A \cap B) = \psi^{\gamma}_{\mathcal{MG}}(A) \cap \psi^{\gamma}_{\mathcal{MG}}(B)$
2. *If $A \notin \mathcal{G}$, then $\psi^{\gamma}_{\mathcal{MG}}(A) = X \setminus \gamma^{\mathcal{M}}(X)$*
3. *If $B \notin \mathcal{G}$, then $\psi^{\gamma}_{\mathcal{MG}}(A \setminus B) = \psi^{\gamma}_{\mathcal{MG}}(A)$*
4. *If $B \notin \mathcal{G}$, then $\psi^{\gamma}_{\mathcal{MG}}(A \cup B) = \psi^{\gamma}_{\mathcal{MG}}(A)$*

Proof:

1. Using Theorem 14.,

$$\begin{aligned}\psi^{\gamma}_{\mathcal{MG}}(A \cap B) &= X \setminus \gamma^{\mathcal{M}}(X \setminus (A \cap B)) \\ &= X \setminus [\gamma^{\mathcal{M}}(X \setminus A) \cup \gamma^{\mathcal{M}}(X \setminus B)] \\ &= [X \setminus \gamma^{\mathcal{M}}(X \setminus A)] \cap [X \setminus \gamma^{\mathcal{M}}(X \setminus B)] \\ &= \psi^{\gamma}_{\mathcal{MG}} \cap \psi^{\gamma}_{\mathcal{MG}}(B)\end{aligned}$$

2. Let $A \notin \mathcal{G}$. Then, $X \setminus \gamma^{\mathcal{M}}(X \setminus A) = X \setminus \gamma^{\mathcal{M}}(X)$ from Theorem 19. Therefore $\psi^{\gamma}_{\mathcal{MG}}(A) = X \setminus \gamma^{\mathcal{M}}(X)$.
3. From Theorem 14. and 19.,

$$\begin{aligned}\psi^{\gamma}_{\mathcal{MG}}(A \setminus B) &= X \setminus \gamma^{\mathcal{M}}(X \setminus (A \setminus B)) \\ &= X \setminus \gamma^{\mathcal{M}}((X \setminus A) \cup B) \\ &= X \setminus [\gamma^{\mathcal{M}}(X \setminus A) \cup \gamma^{\mathcal{M}}(B)] \\ &= X \setminus \gamma^{\mathcal{M}}(X \setminus A) \\ &= \psi^{\gamma}_{\mathcal{MG}}(A)\end{aligned}$$

.

4. Using Theorem 19.,

$$\begin{aligned}\psi^{\gamma}_{\mathcal{MG}}(A \cup B) &= X \setminus \gamma^{\mathcal{M}}(X \setminus (A \cup B)) \\ &= X \setminus \gamma^{\mathcal{M}}((X \setminus A) \cap (X \setminus B)) \\ &= X \setminus \gamma^{\mathcal{M}}((X \setminus A) \setminus B) \\ &= X \setminus \gamma^{\mathcal{M}}(X \setminus A) \\ &= \psi^{\gamma}_{\mathcal{MG}}(A)\end{aligned}$$

.

□

Theorem 23. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $\tau^{\gamma}_{\mathcal{M}} = \{A \subseteq X : A \subseteq \psi^{\gamma}_{\mathcal{MG}}(A)\}$.*

1. $\emptyset, X \in \tau^{\gamma}_{\mathcal{M}}$.
2. $\tau^{\gamma}_{\mathcal{M}}$ *is closed under arbitrary union.*

That is, $\tau^{\gamma}_{\mathcal{M}}$ is a supra topology on X.

Proof: It is obvious that $\emptyset, X \in \tau^{\gamma}_{\mathcal{M}}$. Let $\{A_\alpha\}_{\alpha \in I}$ be a family of subsets of $\tau^{\gamma}_{\mathcal{M}}$ for any index set I. Since $A_\alpha \subseteq \psi^{\gamma}_{\mathcal{MG}}(A_\alpha)$ for every $\alpha \in I$, $A_\alpha \subseteq \psi^{\gamma}_{\mathcal{MG}}(A_\alpha) \subseteq \psi^{\gamma}_{\mathcal{MG}}(\cup_{\alpha \in I} A_\alpha)$. Then, $\cup_{\alpha \in I} A_\alpha \subseteq \psi^{\gamma}_{\mathcal{MG}}(\cup_{\alpha \in I} A_\alpha)$. Hence $\cup_{\alpha \in I} A_\alpha \in \tau^{\gamma}_{\mathcal{M}}$. □

4 Conclusion

The properties of both the minimal grill semi-closure ($\mathcal{MGSC}$) function and the operator $\psi^{\gamma}_{\mathcal{MG}}(A)$ have been examined. A new supra topology has been created with the help of the operator $\psi^{\gamma}_{\mathcal{MG}}(A)$.

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Blow up solutions for a fourth-order parabolic equation with variable exponents

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Abstract: In this work, we consider the following fourth-order parabolic equation with variable exponents

$$u_t + \Delta^2 u = u^{q(x)}.$$

We investigate the blow up of solutions with positive initial energy

Keywords: Blow up, parabolic equation, variable exponent.

1 Introduction

In this work, we study following fourth-order parabolic equation with variable exponents

$$\begin{cases} u_t + \Delta^2 u = u^{q(x)}, & x \in \Omega,\ t > 0, \\ u(x,t) = \frac{\partial u}{\partial v}(x,t) = 0, & x \in \partial\Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, and $u_0(x) \geq 0$.

The problem (1) occurs in many mathematical models of applied science, such as electro-rheological fluids, heat transfer, chemical reactions, population dynamics, etc., The interested readers may refer to [1, 2] and the references therein.

Wu et al. [3] considered the following semilinear parabolic equation with variable exponent

$$u_t - \Delta u = u^{q(x)}.$$

They proved the blow up of solutions. Later, many authors studied the blow up of solutions the same problem under different conditions (see [4–6]). Recently, some authors studied the partial diffirential equations with variable exponents (see [7–10]).

2 Preliminaries

Let $q(x)$ satisfy the following condition:

$$1 < q^- := \inf_{x\in\Omega} q(x) \leq q(x) \leq q^+ := \sup_{x\in\Omega} q(x) < \infty, \tag{2}$$

$$\forall z,\xi \in \Omega,\ |z-\xi| < 1,\ |q(z) - q(\xi)| \leq \omega(z-\xi), \tag{3}$$

where

$$\overline{\lim_{\tau\to 0^+}} \omega(\tau) \ln \frac{1}{\tau} = C < \infty.$$

By $L^{q(.)}(\Omega)$ we denote the space of measurable functions $u(x)$ on Ω such that

$$A_{q(.)}(f) = \int_\Omega |u(x)|^{q(x)} < \infty.$$

The space $L^{q(.)}(\Omega)$ is a Banach space in [11]. It follows directly from the definition that

$$\min\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\} \leq A_{q(.)}(u) \leq \max\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\}. \tag{4}$$

By Corollary 3.34 in [11], we know $L^{q^+ +1}(\Omega) \hookrightarrow L^{q(x)+1}(\Omega)$. And then according to the embedding $H_0^2(\Omega) \hookrightarrow L^{q^+ +1}(\Omega)$ and Poincàre inequality, we have

$$\|u\|_{q(.)+1} \leq B\|\Delta u\|_2, \tag{5}$$

where $1 < q^- \leq q(.) \leq q^+ \leq \frac{K+4}{K-4}\,(K > 4)$ and B is the embedding constant. Set

$$E_1 = \frac{1}{q^- + 1}\left[\frac{q^+ - 1}{2} B^{q^+ +1} \alpha_1^{\frac{q^+ +1}{2}} + \frac{q^- - 1}{2} B^{q^- +1} \alpha_1^{\frac{q^- +1}{2}}\right], \tag{6}$$

where α_1 satisfies

$$\frac{1}{q^- + 1}\left[B^{q^+ +1}\left(q^+ + 1\right)\alpha_1^{\frac{q^+ -1}{2}} + B^{q^- +1}\left(q^- + 1\right)\alpha_1^{\frac{q^- -1}{2}}\right] = 1. \tag{7}$$

Set

$$\overline{E_1} = \left(\frac{q^+ - 1}{q^- - 1}\right)^{\frac{2}{q^+ +1}}\left\{\frac{\alpha_1}{2} - \frac{1}{q^- + 1}\left[B^{q^+ +1}\left(\frac{q^+ - 1}{q^- - 1}\right)^{\frac{q^+ -1}{q^+ +1}}\alpha_1^{\frac{q^+ +1}{2}} + B^{q^- +1}\left(\frac{q^+ - 1}{q^- - 1}\right)^{\frac{q^- -1}{q^+ +1}}\alpha_1^{\frac{q^- +1}{2}}\right]\right\} \tag{8}$$

and

$$E(t) = \frac{1}{2}\|\Delta u\|^2 - \int_\Omega \frac{1}{q(x)+1} u^{q(x)+1}(x,t)\,dx. \tag{9}$$

We have the following result:

Theorem 1. *Suppose that $q(x)$ satisfies the conditions (2)-(3), and the following assumptions hold:*

(H_1) $E(0) < \overline{E_1}$, $\|\Delta u_0\|^2 > \alpha_1$
(H_2) $\sqrt{2q^+ - 1} < q^- \leq q^+ \leq \frac{K+2}{K-2}$.

Then the solution of problem (1) blows up in finite time.

3 Main Result

To prove Theorem 1, we require the following lemmas.

Lemma 1. *For $t \geq 0$, $E(t)$ is a nonincreasing function.*

Proof: Multiplying the first equation of (1) by u_t and integrating over Ω, by using integrating by parts, we get

$$E^{'}(t) = \|u_t\|^2 \leq 0.$$

□

Lemma 2. *Assume that u is a solution of problem (1). If $E(0) < \overline{E_1}$ and $\|\Delta u_0\|^2 > \alpha_1$, then there exists a positive constant $\alpha_2 > \overline{\alpha} = \left(\frac{q^+-1}{q^--1}\right)^{\frac{2}{q^+ +1}}\alpha_1$, such that*

$$\|\Delta u\|^2 \geq \alpha_2,\ \forall t \geq 0, \tag{10}$$

and

$$\int_\Omega \frac{1}{q(x)+1} u^{q(x)+1} dx \geq \frac{1}{q^- + 1} B^{q^+ +1}\alpha_2^{\frac{q^+ +1}{2}} + B^{q^- +1}\alpha_2^{\frac{q^- +1}{2}}. \tag{11}$$

Proof: By the (5) and (9) we have

$$\begin{aligned}
E(t) &\geq \frac{1}{2}\|\Delta u\|^2 - \frac{1}{q^- + 1}\int_\Omega u^{q(x)+1} dx. \\
&\geq \frac{1}{2}\|\Delta u\|^2 - \frac{1}{q^- + 1}\max\left\{\|u\|_{q(.)+1,\Omega}^{q^+ +1}, \|u\|_{q(.)+1,\Omega}^{q^- +1}\right\} \\
&= \frac{1}{2}\alpha - \frac{1}{q^- + 1}\left(B^{q^+ +1}\alpha^{\frac{q^+ +1}{2}} + B^{q^+ +1}\alpha^{\frac{q^- +1}{2}}\right) = h(\alpha),
\end{aligned} \tag{12}$$

where $\alpha = \|\Delta u\|^2$.

It is easy to verify that h is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$; $h(\alpha) \to -\infty$ as $\alpha \to +\infty$ and $h(\alpha_1) = E_1$, where E_1 and α_1 are given respectively in (6) and (7). Since $E(0) < E_1$, there exists an $\alpha_2 > \overline{\alpha} > \alpha_1$ such that $h(\alpha_2) = E(0)$. Let $\alpha_0 = \| \Delta u_0 \|$, then we have $h(\alpha_0) \leq E(0) = h(\alpha_2)$ by (12), which implies that $\alpha_0 \geq \alpha_2$ since $\alpha_0, \alpha_2 > \alpha_1$. To prove (10), we suppose on the contrary that $\| \Delta u(\cdot, t) \|^2 < \alpha_2$ for some $t_0 > 0$. By the continuity of $\| \Delta u(\cdot, t) \|^2$, we may choose

$$E(0) = h(\alpha_2) < h\left(\| \Delta u(\cdot, t) \|^2\right) \leq E(t_0),$$

this contradicts the conclusion of Lemma 1. From (9) we get

$$\frac{1}{2} \|\Delta u\|^2 \leq E(0) + \frac{1}{q(x)+1} \int_\Omega u^{q(x)+1} dx,$$

which implies that

$$\begin{aligned} \frac{1}{q(x)+1} \int_\Omega u^{q(x)+1} dx &\geq \frac{1}{2} \|\Delta u\|^2 - E(0) \geq \frac{\alpha_2}{2} - h(\alpha_2) \\ &= \frac{1}{q^- + 1} \left(B^{q^+ + 1} \alpha_2^{\frac{q^+ + 1}{2}} + B^{q^- + 1} \alpha_2^{\frac{q^- + 1}{2}} \right). \end{aligned}$$

□

Lemma 3. *For all* $t > 0$,

$$0 < H(0) < H(t) \leq \int_\Omega \frac{u^{q(x)+1}}{q(x)+1} dx \tag{13}$$

where

$$H(t) = E_1 - E(t), \ t \geq 0. \tag{14}$$

Proof: By Lemma 1, we have $H'(t) \geq 0$, that is $H(t) \geq H(0) > 0$, $t \geq 0$. (9) and (14) yield

$$H(t) = E_1 - \frac{1}{2} \|\Delta u\|^2 + \int_\Omega \frac{u^{q(x)+1}}{q(x)+1} dx.$$

Also, (10)-(12) we have

$$E_1 - \frac{1}{2} \|\Delta u\|^2 \leq E_1 - \frac{\alpha_2}{2} \leq E_1 - \frac{\alpha_1}{2} \leq 0, \ t > 0.$$

□

Proof of Theorem 1. Set

$$G(t) = \frac{1}{2} \int_\Omega u^2 dx,$$

then

$$G'(t) = \int_\Omega u u_t = \int_\Omega u^{q(x)+1} dx - \int_\Omega |\Delta u|^2 dx. \tag{15}$$

From (9), (14) and (15), we have

$$G'(t) = \int_\Omega \frac{q(x)-1}{q(x)+1} u^{q(x)+1} dx - 2E_1 + 2H(t). \tag{16}$$

Moreover, by (6) and (11), we get

$$2E_1 \leq \frac{\left(q^+ - 1\right) B^{q^+ + 1} \alpha_1^{\frac{q^+ + 1}{2}} + \left(q^- - 1\right) B^{q^- + 1} \alpha_1^{\frac{q^- + 1}{2}}}{B^{q^+ + 1} \alpha_2^{\frac{q^+ + 1}{2}} + B^{q^- + 1} \alpha_2^{\frac{q^- + 1}{2}}} \int_\Omega \frac{1}{q(x)+1} u^{q(x)+1} dx. \tag{17}$$

By (13), (16) and (17), we can deduce the following inequality

$$G'(t) \geq C \int_\Omega u^{q(x)+1} dx,$$

where

$$C = \frac{B^{q^{+}+1}\left[\left(q^{-}-1\right)\alpha_2^{\frac{q^{+}+1}{2}} - \left(q^{+}-1\right)\alpha_1^{\frac{q^{+}+1}{2}}\right]}{\left(B^{q^{+}+1}\alpha_2^{\frac{q^{+}+1}{2}} + B^{q^{-}+1}\alpha_2^{\frac{q^{-}+1}{2}}\right)\left(q^{+}+1\right)} > 0.$$

By (4) and embedding $L^{q(.)+1}(\Omega) \hookrightarrow L^{2}(\Omega)$ in [11], we get

$$G^{'}(t) \geq C \min\left\{\|u\|_2^{q^{-}+1}, \|u\|_2^{q^{-}+1}\right\}. \tag{18}$$

Then, inequality (18) and Gronwall's inequality yield

$$G^{\frac{q^{-}-1}{2}}(t) \geq \frac{1}{G^{\frac{1-q^{-}}{2}}(0) - \frac{q^{-}-1}{2}C_2 t},$$

where $C_2 = 2C\min\left\{\left(\frac{1}{\widetilde{C}}\right)^{q^{-}+1}, \left(\frac{1}{\widetilde{C}}\right)^{q^{+}+1}\right\}\min\left\{1, G^{\frac{q^{+}-q^{-}}{2}}(0)\right\}$, $\widetilde{C}$ is a positive constant depending on Ω, q^{+}. Then $G(t)$ blows up at a finite time $T^{*} \leq \frac{G^{\frac{1-q^{-}}{2}}(0)}{\frac{q^{-}-1}{2}C_2}$, and so does $u(x,t)$.

4 Conclusion

We studied a nonlinear fourth-order parabolic equation with variable exponents, we proved the blow up of solutions with positive initial energy.

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Some Improvement of Berezin Number Inequalities via Specht's Ratio

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Abstract: The Berezin symbol $\widetilde{X}$ and the Berezin number of an operator X on the reproducing kernel Hilbert space $\mathcal{H}(\Lambda)$ over some set Λ with the reproducing kernel are defined, respectively, by

$$\widetilde{X}(\varrho) = \left\langle X\frac{k_\varrho}{\|k_\varrho\|}, \frac{k_\varrho}{\|k_\varrho\|}\right\rangle, \ \varrho \in \Lambda \text{ and } \mathrm{ber}(X) = \sup_{\varrho\in\Lambda} |\widetilde{X}(\varrho)|.$$

By using this bounded function $\widetilde{X}$, we discover a few inequalities pertaining to Berezin number inequalities of functional Hilbert space operators. There are also some conclusions drawn using Hermite-Hadamard inequality. We strengthen and broaden a few inequalities in relation to Specht's ratio. With these enhancements, we also demonstrate a number of new inequalities for the Berezin norm and Berezin radius of operators.

Keywords: Berezin number, Hermite-Hadamard inequality, Specht's ratio, positive operator.

1 Introduction

Inequalities are used in mathematical analysis to analyze the properties of operators in the form of upper and lower bounds. Mathematical inequalities are the most successful means of describing and offering solutions to real-world issues in practically all fields of science and engineering. The boundedness property of many types of operators studied in analysis courses, including mathematical and functional analysis, is a significant consideration when developing theory and applications. Upper and lower bounds, for example, are used for developing the operator norm, which is essential when dealing with related difficulties. Many researchers in mathematics and mathematical physics are interested in the Berezin transform of an operator defined on the reproducing kernel Hilbert space. In this context, several mathematicians have conducted substantial research on the Berezin radius inequality given in (2) (see [17–19]). In fact, it is of interest to academics to get refinements and extensions of this disparity [8, 9, 26]. The purpose of this research is to improve and generalize some inequalities with respect to Specht's ratio using the Berezin transform for operators on the reproducing kernel Hilbert space. Furthermore, we used the previously described refinements to show several additional inequalities for the Berezin norm and Berezin radius of operators. Related results are contained in [30, 31]. We will now outline the preliminary concepts needed to proceed with the findings of this investigation.

Recall that reproducing kernel Hilbert space (shortly, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Lambda)$ of complex-valued functions on some set Λ such that the evaluation functionals $\varphi_\varrho(f) = f(\varrho)$, $\varrho \in \Lambda$, are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem, for each $\varrho \in \Lambda$ there exists a unique function $k_\varrho \in \mathcal{H}$ such that $f(\varrho) = \langle f, k_\varrho\rangle$ for all $f \in \mathcal{H}$. The family $\{k_\varrho : \varrho \in \Lambda\}$ is called the reproducing kernel of the space $\mathcal{H}$. The Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in C : |z| < 1\}$ is the unit disc, the Bergman space $L_a^2(\mathbb{D})$, the Dirichlet space $D^2(\mathbb{D})$ and the Fock space $F(\mathbb{C})$ are example of RKHSs. Aronzajn [1], for example, provides a comprehensive treatment of the theory of RKHSs and reproducing kernels.

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions.

Definition 1. *Let $\mathcal{H}$ be an RKHS on a set Λ and let T be a bounded linear operator on $\mathcal{H}$.*
(i) For $\varrho \in \Lambda$, the Berezin transform of X at ϱ (or Berezin symbol of X) is

$$\widetilde{X}(\varrho) := \left\langle X\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle_{\mathcal{H}}.$$

(ii) The Berezin range of X (or Berezin set of X) is

$$\mathrm{Ber}(X) := \mathrm{Range}(\widetilde{X}) = \left\{\widetilde{X}(\varrho) : \varrho \in \Lambda\right\}.$$

(iii) The Berezin radius of X (or Berezin number of X) is

$$\mathrm{ber}(X) := \sup_{\varrho\in\Lambda} \left|\widetilde{X}(\varrho)\right|.$$

We also define the following so-called Berezin norm of operators $X \in \mathcal{L}(\mathcal{H})$:

$$\|X\|_{\mathrm{Ber}} := \sup_{\varrho \in \Lambda} \left\| X \widehat{k}_{\varrho} \right\|.$$

It is easy to see that actually $\|X\|_{\mathrm{Ber}}$ determines a new operator norm in $\mathcal{L}(\mathcal{H}(\Lambda))$ (since the set of reproducing kernels $\{k_{\varrho} : \varrho \in \Lambda\}$ span the space $\mathcal{H}(\Lambda)$). It is also trivial that $\mathrm{ber}(X) \leq \|X\|_{\mathrm{Ber}} \leq \|X\|$ (for more facts about reproducing kernel Hilbert spaces and Berezin symbol, see, Aronzajn [1] and Berezin [4]).

Berezin range and Berezin radius of operators are new numerical characteristics of operators on the RKHS which are presented by Karaev in [28]. For the basic properties and facts on these new concepts, see [19, 29]

For each bounded operator X on $\mathcal{H}$, the Berezin transform $\widetilde{X}$ is a bounded real-analytic function on Λ. Properties of the operator X are often reflected in properties of the Berezin transform $\widetilde{X}$. The Berezin transform itself was introduced by F. Berezin in [4] and has proven to be a importal tool in operator theory, as many foundational properties of significant operators are encoded in their Berezin transforms. The Berezin set and number, also denoted by $\mathrm{Ber}(X)$ and $\mathrm{ber}(X)$, respectively, were purportedly first formally introduced by Karaev in [28].

In an RKHS, the Berezin range of an operator X is a subset of the numerical range of X,

$$W(X) := \{\langle Xx, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}.$$

Hence

$$\mathrm{ber}(X) \leq w(X) := \sup\{|\langle Xx, x\rangle| : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

(the numerical radius of operator X). The numerical range of an operator has some interesting properties. For example, it is well known that the spectrum of an operator is contained in the closure of its numerical range. For basic properties of the numerical radius, we refer to [12, 33, 34, 39].

It is well-known that

$$\frac{1}{2}\|X\| \leq w(X) \leq \|X\| \tag{1}$$

and

$$\mathrm{ber}(X) \leq w(X) \leq \|X\| \tag{2}$$

for any $X \in \mathcal{L}(\mathcal{H}(\Lambda))$.

Suppose that $(\mathcal{H}, \langle ., .\rangle)$ is a complex Hilbert space and that $\mathcal{L}(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. We recall some definitions and concepts from [37].

An operator $X \in \mathcal{L}(\mathcal{H})$ is positive, defined by $X \geq 0$, if X is self-adjoint ($X = X^*$) and $\langle Xx, x\rangle \geq 0$ or equivalently, X is positive if and only if $X = Y^*Y$ for some operator $Y \in \mathcal{L}(\mathcal{H})$. In particular, for some scalar m and M, we can write $mI \leq X \leq MI$ if $m \leq \langle Xx, x\rangle \leq M$ for every $x \in \mathcal{H}$, where I denote the identity operator of $\mathcal{L}(\mathcal{H})$. The absolute value of X is defined by $|X| = (X^*X)^{\frac{1}{2}}$. Note that for a self-adjoint operator X, $mI \leq X \leq MI$ if and only if $sp(X) \subset [m, M]$. Also the set of all positive invertible operators is defined by $\mathcal{L}^+(\mathcal{H})$.

Let $X \in \mathcal{L}^+(\mathcal{H})$ and let Y be a positive operator $\mathcal{L}(\mathcal{H})$. The operator $\upsilon-$weighted geometric mean of X and Y for $\upsilon \in [0, 1]$ is denoted by

$$X \natural_{\upsilon} Y = X^{\frac{1}{2}} \left(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}} \right) X^{\frac{1}{2}}.$$

Recall that a linear map $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is positive, if it keeps things positive. It is normalized if $\varphi(I_{\mathcal{H}}) = I_{\mathcal{K}}$. The Specht's ratio [13, 38] was denoted by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}\ (h \neq 1)$$

for a positive real number h, and it has some properties as follows:

(i) $S(1) = 1$ and $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1, \infty)$.

(iii) $S(h)$ is a monotone decreasing function on $(0, 1)$.

Some results have given in the following with related to Specht's ratio:

Lemma 1 ([15]). *For $a, b > 0$ and $\upsilon \in [0, 1]$, it follows that $(1-\upsilon)a + \upsilon b \geq S\left(\left(\frac{b}{a}\right)^r\right) a^{1-\upsilon} b^{\upsilon}$, where $r = \min\{\upsilon, 1-\upsilon\}$ and $S(.)$ is the Specht's ratio.*

Theorem 1 ([15]). *Let X and R be two positive operators and let m, m', M, M' be positive real numbers satisfying the following conditions (i) $0 \leq m'I \leq X \leq mI \leq MI \leq Y \leq M'I$ or (ii) $0 \leq m'I \leq Y \leq mI \leq MI \leq X \leq M'I$ with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then*

$$\begin{aligned} (1-\upsilon)X + \upsilon Y &\geq S(h^r)\, X \natural_{\upsilon} Y \geq X \natural_{\upsilon} Y \\ &\geq S(h^r) \left\{ (1-\upsilon) X^{-1} + \upsilon Y^{-1} \right\}^{-1} \\ &\geq \left\{ (1-\upsilon) X^{-1} + \upsilon Y^{-1} \right\}^{-1} \end{aligned}$$

where $\upsilon \in [0, 1]$, $r = \min\{\upsilon, 1-\upsilon\}$ and $S(.)$ is the Specht's ratio.

Remark 1. *If $X = aI$, $R = bI$, $\upsilon = \frac{1}{2}$ and $r = \frac{1}{2}$ in Theorem 1, then*

$$S\left(\sqrt{h}\right)\sqrt{ab} \leq \frac{a+b}{2}, \tag{3}$$

where $S(.)$ is the Specht's ratio.

In 1994, Furuta [14] showed the following inequallity:

$$\left|\left\langle X_1 \left|X_1\right|^{\alpha+\beta-1} x_1, x_2\right\rangle\right|^2 \leq \left\langle \left|X_1\right|^{2\alpha} x_1, x_1\right\rangle \left\langle \left|X_1\right|^{2\beta} x_2, x_2\right\rangle \tag{4}$$

for any $x_1, x_2 \in \mathcal{H}$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$.

In Dragomir's [11] a work, Dragomir provides a useful extension of Furuta's inequality, as follow:

$$\left|\left\langle X_4 X_3 X_2 X_1 x_1, x_2\right\rangle\right|^2 \leq \left\langle X_1^* \left|X_2\right|^2 X_1 x_1, x_1\right\rangle \left\langle X_4 \left|X_3\right|^2 X_4^* x_2, x_2\right\rangle \tag{5}$$

for any $X_4, X_3, X_2, X_1 \in \mathcal{L}(\mathcal{H})$ and any vectors $x_1, x_2 \in \mathcal{H}$. The inequality in (5) holds if and only if the vectors $X_2 X_1 x_1$ and $X_4^* X_3^* x_2$ are linearly depended in $\mathcal{H}$.

In [27], Huban et al. proved the following results,

$$\operatorname{ber}(X) \leq \frac{1}{2}\left\|\left|X\right| + \left|X^*\right|\right\|_{\text{ber}} \leq \frac{1}{2}\left(\|X\|_{\text{ber}} + \left\|X^2\right\|_{\text{ber}}^{\frac{1}{2}}\right) \leq \|X\|_{\text{ber}}, \tag{6}$$

and

$$\operatorname{ber}^r(X) \leq \frac{1}{2}\left\|\left|X\right|^{2r\zeta} + \left|X^*\right|^{2r(1-\zeta)}\right\|_{\text{ber}}, r \geq 1, 0 < \zeta < 1. \tag{7}$$

Başaran and Gürdal in [7, Theorem 2.3] improved the left hand of inequality (7) with help of improvement of Hölder-McCarthy's inequality. Huban et al. in [26, Theorem 3.11] showed the following inequality by the product of two operators

$$\operatorname{ber}^r\left(Y^* X\right) \leq \frac{1}{2}\left\|\left|X\right|^{2r} + \left|Y\right|^{2r}\right\|_{\text{ber}}, r \geq 1. \tag{8}$$

2 Auxiliary Theorems

In this section, we present some useful lemmas that we need them for improving and generalizing some inequalities.

Lemma 2. *([32]) Let $X \in \mathcal{L}(\mathcal{H})$ and for any $x, y \in \mathcal{H}$.*
(i) If $0 \leq \alpha \leq 1$, then

$$\left|\langle Xx, y\rangle\right| \leq \left\langle \left|X\right|^{2\alpha} x, x\right\rangle^{\frac{1}{2}} \left\langle \left|X^*\right|^{2(1-\alpha)} y, y\right\rangle^{\frac{1}{2}}. \tag{9}$$

(ii) If f and g is non-negative continuos functions on $[0,\infty)$ satisfying $f(X)\, g(X) = X$, then

$$\left|\langle Xx, y\rangle\right| \leq \sqrt{\left\|f\left(\left|X\right|\right) x\right\| \left\|g\left(\left|X^*\right|\right) y\right\|}. \tag{10}$$

For a convex function $f : J \to \mathbb{R}$ and for any $a, b \in J$, the well-known Hermite-Hadamard inequality (for more information on the Hermite-Hadamard inequality see the relevant reference [10]) obtain the following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\left(ta + (1-t)\, b\right) dt \leq \frac{f(a) + f(b)}{2} \tag{11}$$

Mond and Pečarić [36] proved the following result.

Lemma 3. *Let $X \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator with spectrum contained in the interval J, and $x \in \mathcal{H}$ be a unit vector. If f is a convex function on J, then*

$$f\left(\langle Xx, x\rangle\right) \leq \left\langle f(X)\, x, x\right\rangle. \tag{12}$$

If f is concave the above inequality is reversed.

The third lemma in the this section is a direct result of [3].

Lemma 4. *([35, page 5]) Let f be a twice differentiable on $[a,b]$. If f is a convex such that $f'' \geq \mu = \min_{x \in [a,b]} f(x) > 0$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2} - \frac{1}{8}\mu\,(b-a)^2. \tag{13}$$

3 Main results

Now, let's prove the first theorem.

Theorem 2. *Let $\mathcal{H} = \mathcal{H}(\Lambda)$ be a RKHS. If $X, Y, T \in \mathcal{L}(\mathcal{H})$, let f and g be nonnegative continous function on $[0,\infty)$ satisfying the relation $f(t)\, g(t) = t$ for all $t \in [0,\infty)$, and let τ be a nonnegative increasing convex function on $[0,\infty)$ and twice differentiable such that $\tau'' \geq \mu > 0$, with $\tau(0) = 0$. Also let the positive real numbers m, m', M, M' satisfy the following conditions (i) $0 \leq m'I \leq Y^* \widetilde{f^2(|T|)\, Y}(\varrho) \leq$*

$mI \leq MI \leq X^* g^2 \widetilde{(|T^*|)\, X}(\varrho) \leq M'I$ or (ii) $0 \leq m'I \leq X^* g^2 \widetilde{(|T^*|)\, X}(\varrho) \leq mI \leq MI \leq Y^* \widetilde{f^2 (|X|)\, Y}(\varrho) \leq M'I$, with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$, then

$$\tau\left(\operatorname{ber}\left(X^* T R\right)\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| \tau\left(Y^* f^2\left(|T|\right) Y\right) + \tau\left(X^* g^2\left(\left|T^*\right|\right) X\right) \right\|_{\text{ber}} - \inf_{\varrho \in \Lambda} \zeta\left(\varrho\right), \tag{14}$$

whenever

$$\zeta\left(\varrho\right) = \frac{1}{8S\left(\sqrt{h}\right)} \mu\left(X^* g^2\left(|T^*|\right) \widetilde{X - Y}^* f^2\left(|T|\right) Y\left(\varrho\right)\right)^2$$

where $S(.)$ is the Specht's ratio.

Proof: Let $\widehat{k}_\varrho$ be a normalized reproducing kernel. From inequality (10), we have

$$\begin{aligned} \left|\left\langle X^* T Y \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right| &= \left|\left\langle T Y \widehat{k}_\varrho, X \widehat{k}_\varrho \right\rangle\right| \\ &\leq \sqrt{\left\langle Y^* f^2\left(|T|\right) Y \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle \left\langle X^* g^2\left(|T^*|\right) X \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle} \\ &\leq \frac{1}{2S\left(\sqrt{h}\right)} \left(\left\langle Y^* f^2\left(|T|\right) Y \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle + \left\langle X^* g^2\left(\left|T^*\right|\right) X \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right) \\ &= \frac{1}{2S\left(\sqrt{h}\right)} \left(\left\langle \left(Y^* f^2\left(|T|\right) Y + X^* g^2\left(\left|T^*\right|\right) X\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right). \end{aligned} \tag{15}$$

It follows from last inequality and (15) that

$$\left|\left\langle X^* T Y \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right| \leq \frac{1}{2S\left(\sqrt{h}\right)} \left(\left\langle \left(Y^* f^2\left(|T|\right) Y + X^* g^2\left(\left|T^*\right|\right) X\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right).$$

Then we get

$$\begin{aligned} \tau\left(\left|\left\langle X^* T Y \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right|\right) &\leq \tau\left(\frac{1}{2S\left(\sqrt{h}\right)} \left(\left\langle \left(Y^* f^2\left(|T|\right) Y + X^* g^2\left(\left|T^*\right|\right) X\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right)\right) \\ &\leq \frac{1}{S\left(\sqrt{h}\right)} \tau\left(\frac{1}{2} \left(\left\langle \left(Y^* f^2\left(|T|\right) Y + X^* g^2\left(\left|T^*\right|\right) X\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right)\right) \\ &\leq \frac{1}{S\left(\sqrt{h}\right)} \left[\frac{\tau\left(\left\langle Y^* f^2\left(|T|\right) Y \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right) + \tau\left(\left\langle X^* g^2\left(|T^*|\right) X \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right)}{2}\right] \\ &- \frac{1}{8} \mu\left(\left\langle X^* g^2\left(\left|T^*\right|\right) X \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle - \left\langle \left(Y^* f^2\left(|T|\right) Y\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right)^2 \text{ (by (13))} \\ &\leq \frac{1}{2S\left(\sqrt{h}\right)} \left\langle \tau\left(Y^* f^2\left(|T|\right) Y\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle + \left\langle \tau\left(X^* g^2\left(\left|T^*\right|\right) X\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle \\ &- \frac{1}{8S\left(\sqrt{h}\right)} \mu\left(\left\langle X^* g^2\left(\left|T^*\right|\right) X \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle - \left\langle \left(Y^* f^2\left(|T|\right) Y\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right)^2 \text{ (by (12))} \\ &= \frac{1}{2S\left(\sqrt{h}\right)} \left[\left\langle \left(\tau\left(Y^* f^2\left(|T|\right) Y\right) + \tau\left(X^* g^2\left(\left|T^*\right|\right) X\right)\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right] \\ &- \frac{1}{8S\left(\sqrt{h}\right)} \mu\left(\left\langle X^* g^2\left(\left|T^*\right|\right) X \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle - \left\langle \left(Y^* f^2\left(|T|\right) Y\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right)^2, \end{aligned} \tag{16}$$

where inequality (16) follows from $\tau\left(\alpha t\right) \leq \alpha \tau\left(t\right) \left(\alpha = \frac{1}{S\left(\sqrt{h}\right)} \leq 1\right)$. Hence

$$\tau\left(\left|\left\langle X^* T Y \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right|\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left[\left\langle \left(\tau\left(Y^* f^2\left(|T|\right) Y\right) + \tau\left(X^* g^2\left(\left|T^*\right|\right) X\right)\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right] - \zeta\left(\varrho\right),$$

whenever

$$\zeta\left(\varrho\right) = \frac{1}{8S\left(\sqrt{h}\right)} \mu\left(X^* g^2\left(|T^*|\right) \widetilde{X - Y}^* f^2\left(|T|\right) Y\left(\varrho\right)\right)^2.$$

Taking the supremum over $\varrho \in \Lambda$ in the above inequality, we get

$$\sup_{\varrho\in\Lambda}\left(\tau\left(\left|\left\langle X^{*}TY\widehat{k}_{\varrho},\widehat{k}_{\varrho}\right\rangle\right|\right)\right)\leq\sup_{\varrho\in\Lambda}\left[\frac{1}{2S\left(\sqrt{h}\right)}\left[\left\langle\left(\tau\left(Y^{*}f^{2}\left(|T|\right)Y\right)+\tau\left(X^{*}g^{2}\left(|T^{*}|\right)X\right)\right)\widehat{k}_{\varrho},\widehat{k}_{\varrho}\right\rangle\right]-\zeta\left(\varrho\right)\right]$$

Thus

$$\tau\left(\operatorname{ber}\left(X^{*}TY\right)\right)\leq\frac{1}{2S\left(\sqrt{h}\right)}\left\|\tau\left(Y^{*}f^{2}\left(|T|\right)Y\right)+\tau\left(X^{*}g^{2}\left(|T^{*}|\right)X\right)\right\|_{\operatorname{ber}}-\inf_{\varrho\in\Lambda}\zeta\left(\varrho\right),$$

whenever

$$\zeta\left(\varrho\right)=\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(X^{*}g^{2}\left(|T^{*}|\right)\widetilde{X-Y^{*}}f^{2}\left(|T|\right)Y\left(\varrho\right)\right)^{2}$$

where $S\left(.\right)$ is the Specht's ratio.

This completes the proof and implies inequality (14). □

Remark 2. *Gürdal and Başaran [21, Theorem 3] showed the inequality:*

$$\tau\left(\operatorname{ber}\left(X^{*}TY\right)\right)\leq\frac{1}{2S\left(\sqrt{h}\right)}\left\|\tau\left(Y^{*}f^{2}\left(|T|\right)Y\right)+\tau\left(X^{*}g^{2}\left(|T^{*}|\right)X\right)\right\|_{\operatorname{ber}}. \tag{17}$$

Inequality (14) is improvement the inequality (17).

Remark 3. *From Bakherad and Garayev [5, Thorem 3.5] and function $f\left(t\right)=t^{r}$, for each $X,Y,T\in\mathcal{L}\left(\mathcal{H}\right)$, proved the following general Berezin radius inequality:*

$$\operatorname{ber}^{r}\left(X^{*}TY\right)\leq\frac{1}{2}\left\|\left(X^{*}\left|T^{*}\right|X\right)^{r}+\left(Y^{*}\left|T\right|Y\right)^{r}\right\|_{\operatorname{ber}},\ r\geq1. \tag{18}$$

From inequality (18) and Theorem 2, we imply the following inequalities.

Corollary 1. *We know that $\tau\left(t\right)=t^{r}$, $r\geq1$, is an increasing convex function on $[0,\infty)$. Let the assumption of Theorem 2 be satisfied. Then*

(i) If (i) $0<m'I<Y^{*}\left|T\right|Y\leq mI<MI\leq X^{*}\left|T^{*}\right|X<M'I$ or (ii) $0<m'I<X^{*}\left|T^{*}\right|X\leq mI<MI\leq Y^{*}\left|X\right|Y<M'I$, with $h=\frac{M}{m}$ and $h'=\frac{M'}{m'}$, for positive real number m, m', M, M', then

$$\operatorname{ber}^{r}\left(X^{*}TY\right)\leq\frac{1}{2S\left(\sqrt{h}\right)}\left\|\left(Y^{*}\left|T\right|Y\right)^{r}+\left(X^{*}\left|T^{*}\right|X\right)^{r}\right\|_{\operatorname{ber}}$$
$$-\inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(X^{*}\left|T^{*}\right|\widetilde{X-Y^{*}}\left|T\right|Y\left(\varrho\right)\right)^{2}.$$

(ii) If $T=I$ holds in conditions of (i), then

$$\operatorname{ber}^{r}\left(X^{*}Y\right)\leq\frac{1}{2S\left(\sqrt{h}\right)}\left\||X|^{2r}+|Y|^{2r}\right\|_{\operatorname{ber}}-\inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(|X|\widetilde{^{2}-|Y|}^{2}\left(\varrho\right)\right)^{2},$$

which refines inequality (8).

(iii) If $X=Y=I$ holds in conditions of (i), then

$$\operatorname{ber}^{r}\left(T\right)\leq\frac{1}{2S\left(\sqrt{h}\right)}\left\||T|^{r}+\left|T^{*}\right|^{r}\right\|_{\operatorname{ber}}-\inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\widetilde{|T^{*}|-|T|}\left(\varrho\right)\right)^{2},$$

where $S\left(.\right)$ is the Specht's ratio.

Remark 4. *Corollary 1 is improvement of [21, Corollary 1].*

Special case of corollary 1 is as follow.

Corollary 2. *Let the assumption of Theorem 2 be satisfied. By taking $\tau\left(t\right)=t^{2}$, on $[0,\infty)$, hence the required μ would be "2".*

(i) If (i) $0<m'I<Y^{*}\left|T\right|Y\leq mI<MI\leq X^{*}\left|T^{*}\right|X<M'I$ or (ii) $0<m'I<X^{*}\left|T^{*}\right|X\leq mI<MI\leq Y^{*}\left|T\right|Y<M'I$, with $h=\frac{M}{m}$ and $h'=\frac{M'}{m'}$, for positive real number m, m', M, M', then

$$\operatorname{ber}^{2}\left(X^{*}TY\right)\leq\frac{1}{2S\left(\sqrt{h}\right)}\left\|\left(Y^{*}\left|T\right|Y\right)^{2}+\left(X^{*}\left|T^{*}\right|X\right)^{2}\right\|_{\operatorname{ber}}$$
$$-\inf_{\varrho\in\Lambda}\frac{1}{4S\left(\sqrt{h}\right)}\left(X^{*}\left|T^{*}\right|\widetilde{X-Y^{*}}\left|T\right|Y\left(\varrho\right)\right)^{2},$$

which refines inequality (18).

(ii) *If* $T = I$ *holds in conditions of* (i)*, then*

$$\mathrm{ber}^2\left(X^*Y\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| |X|^4 + |Y|^4 \right\|_{\mathrm{ber}} - \inf_{\varrho\in\Lambda} \frac{1}{4S\left(\sqrt{h}\right)} \left(\widetilde{|X|^2 - |Y|^2} (\varrho) \right)^2,$$

which refines inequality (8) *in special conditions.*

(iii) *If* $X = Y = I$ *holds in conditions of* (i)*, then*

$$\mathrm{ber}^2(T) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| |T|^2 + |T^*|^2 \right\|_{\mathrm{ber}} - \inf_{\varrho\in\Lambda} \frac{1}{4S\left(\sqrt{h}\right)} \left(\widetilde{|T^*| - |T|} (\varrho) \right)^2,$$

where $S(.)$ *is the Specht's ratio.*

Remark 5. *Gürdal and Yücel [23]demonstrated the following disparities:*

$$f\left(\mathrm{ber}\left(X_4X_3X_2X_1\right)\right) \leq \left\| f\left(X_1^* |X_2|^2 X_1\right) + f\left(X_4 |X_3|^2 X_4^*\right) \right\|_{\mathrm{ber}} - \frac{1}{8}\mu \left(\widetilde{X_1^* |X_2|^2 X_1 - X_4 |X_3|^2 X_4^*} (\varrho) \right), \tag{19}$$

$$f\left(\mathrm{ber}\left(X |X|^{\alpha+\beta-1}\right)\right) \leq \left\| f\left(|X|^{2\alpha}\right) + f\left(|X^*|^{2\beta}\right) \right\|_{\mathrm{ber}} - \frac{1}{8}\mu \left(\widetilde{|X|^{2\alpha} - |X^*|^{2\beta}} (\varrho) \right), \tag{20}$$

where $X, X_4, X_3, X_2, X_1 \in \mathcal{L}(\mathcal{H})$*,* $\mu \geq 0$ *and* $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta \geq 1$*.*

We use the inequality (5) to prove the following theorem.

Theorem 3. *Let* $\mathcal{H} = \mathcal{H}(\Lambda)$ *be a RKHS. Let* $X_4, X_3, X_2, X_1 \in \mathcal{L}(\mathcal{H})$ *, let* f *be a non-negative increasing convex function on* $\mathbb{R}$ *and also that* f *is twice differentiable such that* $f'' \geq \mu > 0$*, with* $f(0) = 0$*. Let the positive real numbers* m*,* m'*,* M*,* M' *satisfy one of the following conditions* (i) $0 < m'I \leq X_1^* |X_2|^2 X_1 \leq mI \leq MI \leq X_4 |X_3|^2 X_4^* \leq M'I$ *or* (ii) $0 < m' \leq \left(X_4 |X_3|^2 X_4^*\right) \leq mI \leq MI \leq X_1^* |X_2|^2 X_1 \leq M'I$*, with* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$*. Then we have*

$$f\left(\left|\left\langle X_4X_3X_2X_1\widehat{k}_\varrho, \widehat{k}_\omega\right\rangle\right|\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left[\left\langle f\left(X_1^* |X_2|^2 X_1\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle + \left\langle f\left(X_4 |X_3|^2 X_4^*\right) \widehat{k}_\omega, \widehat{k}_\omega \right\rangle \right] - \frac{1}{8S\left(\sqrt{h}\right)} \mu \left(\left\langle X_1^* |X_2|^2 X_1 \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle - \left\langle X_4 |X_3|^2 X_4^* \widehat{k}_\omega, \widehat{k}_\omega \right\rangle \right)^2 \tag{21}$$

where $S(.)$ *is the Specht's ratio.*

Proof: Let $\varrho, \omega \in \Lambda$ be arbitrary number. Using the monotoncity and convexity of increasing function f for the inequality (5), we reach

$$f\left(\left|\left\langle X_4X_3X_2X_1\widehat{k}_\varrho, \widehat{k}_\omega\right\rangle\right|\right) \leq \sqrt{f\left(\left\langle X_1^* |X_2|^2 X_1 \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle \left\langle X_4 |X_3|^2 X_4^* \widehat{k}_\omega, \widehat{k}_\omega \right\rangle\right)}.$$

Now inequality (3) implies that

$$\sqrt{f\left(\left\langle X_1^* |X_2|^2 X_1 \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle \left\langle X_4 |X_3|^2 X_4^* \widehat{k}_\omega, \widehat{k}_\omega \right\rangle\right)} \leq f\left(\frac{1}{2S\left(\sqrt{h}\right)} \left(\left\langle X_1^* |X_2|^2 X_1 \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle + \left\langle X_4 |X_3|^2 X_4^* \widehat{k}_\omega, \widehat{k}_\omega \right\rangle \right) \right).$$

Hence,

$$\begin{aligned} f\left(\left|\left\langle X_4X_3X_2X_1\widehat{k}_\varrho,\widehat{k}_\omega\right\rangle\right|\right) &\leq f\left(\frac{1}{2S\left(\sqrt{h}\right)}\left(\left\langle X_1^*\left|X_2\right|^2X_1\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle+\left\langle X_4\left|X_3\right|^2X_4^*\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right)\right) \\ &\leq \frac{1}{S\left(\sqrt{h}\right)}f\left(\frac{\left(\left\langle X_1^*\left|X_2\right|^2X_1\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle+\left\langle X_4\left|X_3\right|^2X_4^*\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right)}{2}\right) \\ &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left[f\left(\left\langle X_1^*\left|X_2\right|^2X_1\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\right)+f\left(\left\langle X_4\left|X_3\right|^2X_4^*\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right)\right] \\ &-\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\left\langle X_1^*\left|X_2\right|^2X_1\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle-\left\langle X_4\left|X_3\right|^2X_4^*\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right)^2 \\ &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left[\left\langle f\left(X_1^*\left|X_2\right|^2X_1\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle+\left\langle f\left(X_4\left|X_3\right|^2X_4^*\right)\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right] \\ &-\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(X_1^*\left|X_2\right|^2X_1\widehat{k}_\varrho,\widehat{k}_\varrho-\left\langle X_4\left|X_3\right|^2X_4^*\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right)^2, \end{aligned}$$

where the second inequality follows from inequality $\tau(\alpha t)\leq\alpha\tau(t)$ $\left(\alpha=\frac{1}{S(\sqrt{h})}\leq 1\right)$; the third inequality follows from inequality (13); the last inequality followa from inequality (12). We have the desired result. $\square$

Remark 6. *Theorem 3 is refinement of the [2, Lemma 6].*

Corollary 3. *Let $X\in\mathcal{L}(\mathcal{H})$, let f be a non-negative increasing convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f''\geq\mu>0$, with $f(0)=0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0<m'I\leq|X|^{2\alpha}\leq mI\leq MI\leq|X^*|^{2\beta}\leq M'I$ or (ii) $0<m'\leq|X^*|^{2\beta}\leq mI\leq MI\leq|X|^{2\alpha}\leq M'I$, with $h=\frac{M}{m}$ and $h'=\frac{M'}{m'}$. Then we have*

$$\begin{aligned} f\left(\left|\left\langle X\left|X\right|^{\alpha+\beta-1}\widehat{k}_\varrho,\widehat{k}_\omega\right\rangle\right|\right) &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left[\left\langle f\left(\left|X\right|^{2\alpha}\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle+\left\langle f\left(\left|X^*\right|^{2\beta}\right)\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right] \\ &-\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\left\langle\left|X\right|^{2\alpha}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle-\left\langle\left|X^*\right|^{2\beta}\widehat{k}_\omega,\widehat{k}_\omega\right\rangle\right)^2 \end{aligned} \tag{22}$$

where $S(.)$ is the Specht's ratio.

Proof: Replacing X_4 by U, X_2 by $1_\mathcal{H}$, X_3 by $|X|^\beta$ and X_1 by $|X|^\alpha$ in (21), we have

$$X_4X_3X_2X_1=U\left|X\right|^\beta\left|X\right|^\alpha=X\left|X\right|^{\alpha+\beta-1}.$$

Then, by using

$$X_1^*\left|X_2\right|^2X_1=\left|X\right|^{2\alpha}\text{ and }X_4\left|X_3\right|^2X_4^*=\left|X^*\right|^{2\beta},$$

we get inequality (22). $\square$

Remark 7. *Corollary 3 is refinement of the [2, Corollary 1].*

Theorem 4. *Let $\mathcal{H}=\mathcal{H}(\Lambda)$ be a RKHS. Let $X_4,X_3,X_2,X_1\in\mathcal{L}(\mathcal{H})$, let f be a non-negative increasing convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f''\geq\mu>0$, with $f(0)=0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0<m'I\leq X_1^*|X_2|^2X_1\leq mI\leq MI\leq X_4|X_3|^2X_4^*\leq M'I$ or (ii) $0<m'\leq X_4|X_3|^2X_4^*\leq mI\leq MI\leq X_1^*|X_2|^2X_1\leq M'I$, with $h=\frac{M}{m}$ and $h'=\frac{M'}{m'}$. Then we have*

$$\begin{aligned} f\left(\operatorname{ber}\left(X_4X_3X_2X_1\right)\right) &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left\|f\left(X_1^*\left|X_2\right|^2X_1\right)+f\left(X_4\left|X_3\right|^2X_4^*\right)\right\|_{\text{ber}} \\ &-\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(X_1^*\left|X_2\right|^2\widetilde{X_1-X_4}\left|X_3\right|^2X_4^*(\varrho)\right)^2, \end{aligned} \tag{23}$$

where $S(.)$ is the Specht's ratio.

Proof: Let $\varrho, \omega \in \Lambda$ be arbitrary number. By taking $\widehat{k}_\varrho = \widehat{k}_\omega$ in inequality (21), then we get

$$\begin{aligned} f\left(\left|\left\langle X_4X_3X_2X_1\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right|\right) &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left[\left\langle f\left(X_1^*\left|X_2\right|^2 X_1\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle + \left\langle f\left(X_4\left|X_3\right|^2 X_4^*\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right] \\ &- \frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\left\langle X_1^*\left|X_2\right|^2 X_1\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle - \left\langle X_4\left|X_3\right|^2 X_4^*\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right)^2 \\ &= \frac{1}{2S\left(\sqrt{h}\right)}\left\langle\left(f\left(X_1^*\left|X_2\right|^2 X_1\right) + f\left(X_4\left|X_3\right|^2 X_4^*\right)\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle \\ &- \frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\left\langle\left(X_1^*\left|X_2\right|^2 X_1 - X_4\left|X_3\right|^2 X_4^*\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right)^2 \end{aligned}$$

and

$$\begin{aligned} \sup_{\varrho\in\Lambda} f\left(\left|\left\langle X_4X_3X_2X_1\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right|\right) &\leq \sup_{\varrho\in\Lambda}\frac{1}{2S\left(\sqrt{h}\right)}\left\langle\left(f\left(X_1^*\left|X_2\right|^2 X_1\right) + f\left(X_4\left|X_3\right|^2 X_4^*\right)\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle \\ &- \inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(X_1^*\left|X_2\right|^2 \widetilde{X_1 - X_4}\left|X_3\right|^2 X_4^*\left(\varrho\right)\right)^2, \end{aligned}$$

which equivalent to

$$\begin{aligned} f\left(\mathrm{ber}\left(X_4X_3X_2X_1\right)\right) &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left\|f\left(X_1^*\left|X_2\right|^2 X_1\right) + f\left(X_4\left|X_3\right|^2 X_4^*\right)\right\|_{\mathrm{ber}} \\ &- \inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(X_1^*\left|X_2\right|^2 \widetilde{X_1 - X_4}\left|X_3\right|^2 X_4^*\left(\varrho\right)\right)^2, \end{aligned}$$

and completes the proof of the theorem. □

Remark 8. *Inequality (23) is better than inequality (19).*

If $f(t) = t^n$ and $\mu = n$ is taken, the following corollary is an easy consequence of Theorem 4.

Corollary 4. *Let $X_4, X_3, X_2, X_1 \in \mathcal{L}(\mathcal{H})$. Then*

$$\begin{aligned} \mathrm{ber}^n\left(X_4X_3X_2X_1\right) &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left\|\left(X_1^*\left|X_2\right|^2 X_1\right)^n + \left(X_4\left|X_3\right|^2 X_4^*\right)^n\right\|_{\mathrm{ber}} \\ &- \inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}n\left(X_1^*\left|X_2\right|^2 \widetilde{X_1 - X_4}\left|X_3\right|^2 X_4^*\left(\varrho\right)\right)^2, \end{aligned} \tag{24}$$

where $S(.)$ is the Specht's ratio.

If we take $n = 2$ in inequality (24), then we have the following corollary.

Corollary 5. *Let $X_4, X_3, X_2, X_1 \in \mathcal{L}(\mathcal{H})$. Then*

$$\begin{aligned} \mathrm{ber}^2\left(X_4X_3X_2X_1\right) &\leq \frac{1}{2S\left(\sqrt{h}\right)}\left\|\left(X_1^*\left|X_2\right|^2 X_1\right)^2 + \left(X_4\left|X_3\right|^2 X_4^*\right)^2\right\|_{\mathrm{ber}} \\ &- \inf_{\varrho\in\Lambda}\frac{1}{4S\left(\sqrt{h}\right)}\left(X_1^*\left|X_2\right|^2 \widetilde{X_1 - X_4}\left|X_3\right|^2 X_4^*\left(\varrho\right)\right)^2, \end{aligned}$$

where $S(.)$ is the Specht's ratio.

Corollary 6. *Let $X \in \mathcal{L}(\mathcal{H})$, let f be a non-negative increasing convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f'' \geq \mu > 0$, with $f(0) = 0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0 < m'I \leq |X|^{2\alpha} \leq$*

$mI \leq MI \leq |X^*|^{2\beta} \leq M'I$ or (ii) $0 < m' \leq |X^*|^{2\beta} \leq mI \leq MI \leq |X|^{2\alpha} \leq M'I$, with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then we have

$$f\left(\mathrm{ber}\left(X|X|^{\alpha+\beta-1}\right)\right) \leq \frac{1}{2S\left(\sqrt{h}\right)}\left\|f\left(|X|^{2\alpha}\right)+f\left(|X^*|^{2\beta}\right)\right\|_{\mathrm{ber}} - \inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\widetilde{|X|^{2\alpha}-|X^*|^{2\beta}}(\varrho)\right)^2, \tag{25}$$

where $S(.)$ is the Specht's ratio and $\alpha, \beta \in [0,1]$ with $\alpha+\beta \geq 1$.

Proof: Let $\varrho, \omega \in \Lambda$ be arbitrary number. By taking $\widehat{k}_\varrho = \widehat{k}_\omega$ in inequality (22), then we get

$$f\left(\left|\left\langle X|X|^{\alpha+\beta-1}\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right|\right) \leq \frac{1}{2S\left(\sqrt{h}\right)}\left(\left\langle f\left(|X|^{2\alpha}\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle + \left\langle f\left(|X^*|^{2\beta}\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right) - \frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\left\langle|X|^{2\alpha}\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle - \left\langle|X^*|^{2\beta}\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right)^2.$$

Equvalently, we can write

$$f\left(\left|X\widetilde{|X|^{\alpha+\beta-1}}(\varrho)\right|\right) \leq \frac{1}{2S\left(\sqrt{h}\right)}\left\langle\left(f\left(|X|^{2\alpha}\right)+f\left(|X^*|^{2\beta}\right)\right)\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle - \inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\widetilde{|X|^{2\alpha}-|X^*|^{2\beta}}(\varrho)\right)^2.$$

By taking the supremum over $\varrho \in \Lambda$ in the above inequality, we have

$$f\left(\mathrm{ber}\left(X|X|^{\alpha+\beta-1}\right)\right) \leq \frac{1}{2S\left(\sqrt{h}\right)}\left\|f\left(|X|^{2\alpha}\right)+f\left(|X^*|^{2\beta}\right)\right\|_{\mathrm{ber}} - \inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\widetilde{|X|^{2\alpha}-|X^*|^{2\beta}}(\varrho)\right)^2,$$

which completes the proof. □

The following corollary is result of Corollary 6.

Corollary 7. *Let $X \in \mathcal{L}(\mathcal{H})$ and $\alpha, \beta \in [0,1]$ such that $\alpha+\beta \geq 1$.If the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0 < m'I \leq |X|^{2\alpha} \leq mI \leq MI \leq |X^*|^{2\beta} \leq M'I$ or (ii) $0 < m' \leq |X^*|^{2\beta} \leq mI \leq MI \leq |X|^{2\alpha} \leq M'I$, with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then we have*

$$\mathrm{ber}^n\left(X|X|^{\alpha+\beta-1}\right) \leq \frac{1}{2S\left(\sqrt{h}\right)}\left\||X|^{2\alpha n}+|X^*|^{2\beta n}\right\|_{\mathrm{ber}} - \inf_{\varrho\in\Lambda}\frac{1}{8S\left(\sqrt{h}\right)}\mu\left(\widetilde{|X|^{2\alpha}-|X^*|^{2\beta}}(\varrho)\right)^2,$$

where $S(.)$ is the Specht's ratio. In particular, for $n = \mu = 2$, we obtain

$$\mathrm{ber}^2\left(X|X|^{\alpha+\beta-1}\right) \leq \frac{1}{2S\left(\sqrt{h}\right)}\left\||X|^{4\alpha}+|X^*|^{4\beta}\right\|_{\mathrm{ber}} - \inf_{\varrho\in\Lambda}\frac{1}{4S\left(\sqrt{h}\right)}\left(\widetilde{|X|^{2\alpha}-|X^*|^{2\beta}}(\varrho)\right)^2,$$

where $S(.)$ is the Specht's ratio.

Considering $X_4 = X_1 = X$ and $X_2 = X_3 = Y$ in the inequality (23), we have the following inequality.

Corollary 8. *Let $X, Y \in \mathcal{L}(\mathcal{H})$, let f be a non-negative increasing convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f'' \geq \mu > 0$, with $f(0) = 0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0 < m'I \leq X^*|Y|^2X \leq$*

$mI \leq MI \leq X^* |Y^*|^2 X \leq M'I$ *or* (ii) $0 < m' \leq X^* |R^*|^2 X \leq mI \leq MI \leq X^* |R|^2 X \leq M'I$, *with* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then we have*

$$f\left(\operatorname{ber}\left((XY)^2\right)\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| f\left(X^* |Y|^2 X\right) + f\left(X^* |Y^*|^2 X\right) \right\|_{\operatorname{ber}} - \inf_{\varrho \in \Lambda} \frac{1}{8S\left(\sqrt{h}\right)} \mu \left(X^* |Y|^2 \widetilde{X - X^*} |Y^*|^2 X(\varrho) \right)^2,$$

where $S(.)$ *is the Specht's ratio.*

If we put $f(t) = t^n$ in the Corollary 8, then we have the following inequality.

Corollary 9. *Let* $X, Y \in \mathcal{L}(\mathcal{H})$. *If the positive real numbers* m, m', M, M' *satisfy one of the following conditions* (i) $0 < m'I \leq X^* |Y|^2 X \leq mI \leq MI \leq X^* |Y^*|^2 X \leq M'I$ *or* (ii) $0 < m' \leq X^* |Y^*|^2 X \leq mI \leq MI \leq X^* |Y|^2 X \leq M'I$, *with* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$, *then we have*

$$\operatorname{ber}^n\left((XY)^2\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| \left(X^* |Y|^2 X\right)^n + \left(X^* |Y^*|^2 X\right)^n \right\|_{\operatorname{ber}} - \inf_{\varrho \in \Lambda} \frac{1}{8S\left(\sqrt{h}\right)} \mu \left(X^* |Y|^2 \widetilde{X - X^*} |Y^*|^2 X(\varrho) \right)^2,$$

where $S(.)$ *is the Specht's ratio. In particular, for* $n = \mu = 2$, *we get*

$$\operatorname{ber}^2\left((XY)^2\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| \left(X^* |Y|^2 X\right)^2 + \left(X^* |Y^*|^2 X\right)^2 \right\|_{\operatorname{ber}} - \inf_{\varrho \in \Lambda} \frac{1}{4S\left(\sqrt{h}\right)} \mu \left(X^* |Y|^2 \widetilde{X - X^*} |Y^*|^2 X(\varrho) \right)^2,$$

where $S(.)$ *is the Specht's ratio.*

Considering $X_4 = X_1 = X$ and $X_2 = X_3 = Y$ in the inequality (23), we have the following corollary..

Corollary 10. *Let* $X, Y \in \mathcal{L}(\mathcal{H})$, *let* f *be a non-negative increasing convex function on* $\mathbb{R}$ *and also that* f *is twice differentiable such that* $f'' \geq \mu > 0$, *with* $f(0) = 0$. *Let the positive real numbers* m, m', M, M' *satisfy one of the following conditions* (i) $0 < m'I \leq X^* |Y|^2 X \leq mI \leq MI \leq X^* |Y^*|^2 X \leq M'I$ *or* (ii) $0 < m' \leq X^* |Y^*|^2 X \leq mI \leq MI \leq X^* |Y|^2 X \leq M'I$, *with* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then we have*

$$f\left(\operatorname{ber}\left(X^* Y^2 X\right)\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| f\left(X^* |Y|^2 X\right) + f\left(X^* |Y^*|^2 X\right) \right\|_{\operatorname{ber}} - \inf_{\varrho \in \Lambda} \zeta(\varrho),$$

where $\zeta(\varrho) = \frac{1}{8S(\sqrt{h})} \mu \left(X^* |Y|^2 \widetilde{X - X^*} |Y^*|^2 X \ (\varrho) \right)^2$ *and* $S(.)$ *is the Specht's ratio.*

If we take $f(t) = t^n$ in the Corollary 10, then we have the following inequality.

Corollary 11. *Let* $X, Y \in \mathcal{L}(\mathcal{H})$, *let* f *be a non-negative increasing convex function on* $\mathbb{R}$ *and also that* f *is twice differentiable such that* $f'' \geq \mu > 0$, *with* $f(0) = 0$. *Let the positive real numbers* m, m', M, M' *satisfy one of the following conditions* (i) $0 < m'I \leq X^* |Y|^2 X \leq mI \leq MI \leq X^* |Y^*|^2 X \leq M'I$ *or* (ii) $0 < m' \leq X^* |Y^*|^2 X \leq mI \leq MI \leq X^* |Y|^2 X \leq M'I$, *with* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then we have*

$$f\left(\operatorname{ber}\left(X^* Y^2 X\right)\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| \left(X^* |Y|^2 X\right)^n + \left(X^* |Y^*|^2 X\right)^n \right\|_{\operatorname{ber}} - \inf_{\varrho \in \Lambda} \zeta(\varrho),$$

where $\zeta(\varrho) = \frac{1}{8S(\sqrt{h})} \mu \left(X^* |Y|^2 \widetilde{X - X^*} |Y^*|^2 X(\varrho) \right)^2$ *and* $S(.)$ *is the Specht's ratio. In particular, for* $n = \mu = 2$, *we have*

$$\operatorname{ber}^2\left(X^* Y^2 X\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| \left(X^* |Y|^2 X\right)^2 + \left(X^* |Y^*|^2 X\right)^2 \right\|_{\operatorname{ber}} - \inf_{\varrho \in \Lambda} \zeta(\varrho),$$

where $\zeta(\varrho) = \frac{1}{4S(\sqrt{h})} \mu \left(X^* |Y|^2 \widetilde{X - X^*} |Y^*|^2 X(\varrho) \right)^2$ *and* $S(.)$ *is the Specht's ratio.*

Setting $X_4 = X_3 = X_2 = X_1 = X$ in the inequality (23), we obtain the following result.

Corollary 12. *Let $X \in \mathcal{L}(\mathcal{H})$. If the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0 < m'I \leq X^* |X|^2 X \leq mI \leq MI \leq X^* |X^*|^2 X \leq M'I$ or (ii) $0 < m' \leq X^* |X^*|^2 X \leq mI \leq MI \leq X^* |X|^2 X \leq M'I$, with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then we have*

$$f\left(\mathrm{ber}^4(X)\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| f\left(X^* |X|^2 X\right) + f\left(X^* |X^*|^2 X\right) \right\|_{\mathrm{ber}} - \frac{1}{8S\left(\sqrt{h}\right)} \mu \left(\widetilde{X^* |X|^2 X - X^* |X^*|^2 X}(\varrho) \right)^2,$$

where $S(.)$ is the Specht's ratio.

If we take $f(t) = t^n$ in the Corollary 12, then we have the following inequality.

Corollary 13. *Let $X \in \mathcal{L}(\mathcal{H})$. If the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0 < m'I \leq X^* |X|^2 X \leq mI \leq MI \leq X^* |X^*|^2 X \leq M'I$ or (ii) $0 < m' \leq X^* |X^*|^2 X \leq mI \leq MI \leq X^* |X|^2 X \leq M'I$, with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then we have*

$$\mathrm{ber}^{4n}(X) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| \left(X^* |X|^2 X\right)^n + \left(X^* |X^*|^2 X\right)^n \right\|_{\mathrm{ber}} - \frac{1}{8S\left(\sqrt{h}\right)} \mu \left(\widetilde{X^* |X|^2 X - X^* |X^*|^2 X}(\varrho) \right)^2,$$

where $S(.)$ is the Specht's ratio. In particular, for $n = \mu = 2$, we have

$$\mathrm{ber}^{8}(X) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| \left(X^* |X|^2 X\right)^2 + \left(X^* |X^*|^2 X\right)^2 \right\|_{\mathrm{ber}} - \frac{1}{4S\left(\sqrt{h}\right)} \mu \left(\widetilde{X^* |X|^2 X - X^* |X^*|^2 X}(\varrho) \right)^2,$$

Theorem 5. *Let $\mathcal{H} = \mathcal{H}(\Lambda)$ be a RKHS. Let $X \in \mathcal{L}(\mathcal{H})$, let f be a non-negative increasing convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f'' \geq \mu > 0$, with $f(0) = 0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0 < m'I \leq |X|^{2s} \leq mI \leq MI \leq |X^*|^{2t} \leq M'I$ or (ii) $0 < m'I \leq |X^*|^{2t} \leq mI \leq MI \leq |X|^{2s} \leq M'I$, with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$. Then we have*

$$f\left(\left|\left\langle X\widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle\right|\right) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left[\left\langle f\left(|X|^{2s}\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle + \left\langle f\left(|X^*|^{2t}\right) \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle \right] - \frac{1}{8S\left(\sqrt{h}\right)} \mu \left(\left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle - \left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho \right\rangle \right)^2. \tag{26}$$

Moreover, for every $n \geq 1$, it follows that

$$\mathrm{ber}^n(X) \leq \frac{1}{2S\left(\sqrt{h}\right)} \left\| |X|^{2sn} + |X^*|^{2tn} \right\|_{\mathrm{ber}}, \tag{27}$$

where $S(.)$ is the Specht's ratio.

Proof: Let $\widehat{k}_\varrho$ be a normalized reproducing kernel. Let $X = U|X|$ is the polar decomposition of X. Using the Schwarz inequality in the Hilbert space, Remark 2 and convexity of the function $h(t) = t^r$ for $r \geq 1$ imply that

$$\left|\left\langle X\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right| \leq \left|\left\langle |X|^s \widehat{k}_\varrho, |X|^t U^* \widehat{k}_\varrho\right\rangle\right| \tag{28}$$

$$\leq \left\||X|^s \widehat{k}_\varrho\right\| \left\||X|^t U^* \widehat{k}_\varrho\right\|$$

$$\leq \left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^{\frac{1}{2}}$$

$$\leq \frac{\left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle + \left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle}{2S\left(\sqrt{h}\right)}$$

$$\leq \left(\frac{\left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^n + \left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^n}{2S\left(\sqrt{h}\right)}\right)^{\frac{1}{n}}. \tag{29}$$

Also, by using Remark 2, we have

$$f\left(\left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2X} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^{\frac{1}{2}}\right) \leq f\left(\frac{\left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle + \left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle}{2S\left(\sqrt{h}\right)}\right)$$

$$\leq \frac{1}{S\left(\sqrt{h}\right)} f\left(\frac{\left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle + \left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle}{2}\right) \tag{30}$$

$$\leq \frac{1}{2S\left(\sqrt{h}\right)} \left(f\left(\left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right) + f\left(\left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right)\right)$$

$$- \frac{1}{8}\mu\left(\left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle - \left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right)^2 \text{ (by (13))}$$

$$\leq \frac{1}{2S\left(\sqrt{h}\right)} \left(\left\langle f\left(|X|^{2s}\right) \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle + \left\langle f\left(|X^*|^{2t}\right) \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right) \tag{31}$$

$$- \frac{1}{8}\mu\left(\left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle - \left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right)^2.$$

Hence, by combining (28) and (31), we obtain the desired inequality (26).

From (30) and applying hölder-McCarthy inequality for the positive operator $|X|^{2s}$ and $|X^*|^{2t}$ and the convexity of the function $f(t) = t^n$ for $n \geq 1$ imply that

$$\left(\frac{\left\langle |X|^{2s} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^n + \left\langle |X^*|^{2t} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle^n}{2S\left(\sqrt{h}\right)}\right)^{\frac{1}{r}} \leq \left(\frac{\left\langle |X|^{2sn} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle + \left\langle |X^*|^{2tn} \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle}{2S\left(\sqrt{h}\right)}\right)^{\frac{1}{n}} \tag{32}$$

$$= \left(\frac{\left\langle \left(|X|^{2sn} + |X^*|^{2tn}\right) \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle}{2S\left(\sqrt{h}\right)}\right)^{\frac{1}{n}}.$$

By (28) and (32) implies that

$$\left|\left\langle X\widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle\right|^n \leq \frac{\left\langle \left(|X|^{2sn} + |X^*|^{2tn}\right) \widehat{k}_\varrho, \widehat{k}_\varrho\right\rangle}{2S\left(\sqrt{h}\right)}.$$

By taking the supremum over $\varrho \in \Lambda$ in the above inequality and this fact operator $|X|^{2sn} + |X^*|^{2tn}$ is self-adjoint, we have desired inequality (27), which this refines (7). □

Theorem 6. *Let $\mathcal{H} = \mathcal{H}(\Lambda)$ be a RKHS. Let $X \in \mathcal{L}(\mathcal{H})$, let f be a non-negative increasing convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f_J \geq \mu > 0$, with $f(0) = 0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions (i) $0 < m'I \leq |X|^{\frac{2s}{\alpha}} \leq mI \leq MI \leq |X^*|^{\frac{2t}{1-\alpha}} \leq M'I$ or (ii) $0 < m'I \leq |X|^{\frac{2t}{1-\alpha}} \leq mI \leq MI \leq |X|^{\frac{2s}{\alpha}} \leq M'I$, with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$, $0 < \alpha < 1$ and $s + t = 1$ Then we have*

$$\text{ber}^{2n}(X) \leq \frac{1}{S(h^n)} \left\|\alpha |X|^{\frac{2sn}{\alpha}} + (1-\alpha) |X^*|^{\frac{2tn}{1-\alpha}}\right\|_{\text{ber}}. \tag{33}$$

where $S(.)$ is the Specht's ratio.

Proof: Let $\widehat{k}_\varrho$ be a normalized reproducing kernel. Let $X = U\left|X\right|$ is the polar decomposition of X. By utilizing the Schwarz inequality, we have

$$\begin{aligned}\left|\left\langle X\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\right|^2 &\le \left\langle \left|X\right|^{2s}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\left\langle \left|X^*\right|^{2t}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle = \left\langle \left(\left|X\right|^{\frac{2sn}{\alpha}}\right)^{\alpha}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\left\langle \left(\left|X\right|^{\frac{2tn}{1-\alpha}}\right)^{1-\alpha}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\\ &\le \left\langle \left|X\right|^{\frac{2sn}{\alpha}}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle^{\alpha}\left\langle \left(\left|X^*\right|^{\frac{2tn}{1-\alpha}}\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle^{1-\alpha} \text{ (by Hölder-McCarthy inequalities)}\\ &\le \frac{1}{S\left(h\right)}\left(\alpha\left\langle \left|X\right|^{\frac{2sn}{\alpha}}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle + \left(1-\alpha\right)\left\langle \left(\left|X^*\right|^{\frac{2tn}{1-\alpha}}\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\right) \text{ (by Theorem 1).}\end{aligned} \tag{34}$$

On the other hand, we get the elementary inequality from the convexity of $h\left(t\right) = t^n$ (for $n \ge 1$) in the following:

$$\alpha a + \left(1-\alpha\right)b \le \left(\alpha a^n + \left(1-\alpha\right)b^n\right)^{\frac{1}{n}},\ \alpha \in \left(0,1\right),\ a,b \ge 0.$$

Utilizing this inequality leads to

$$\begin{aligned}&\frac{1}{S\left(h\right)}\left(\alpha\left\langle \left|X\right|^{\frac{2s}{\alpha}}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle + \left(1-\alpha\right)\left\langle \left(\left|X^*\right|^{\frac{2t}{1-\alpha}}\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\right)\\ &\le \left(\frac{1}{S\left(h^n\right)}\left(\alpha\left\langle \left|X\right|^{\frac{2s}{\alpha}}\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle^{n} + \left(1-\alpha\right)\left\langle \left(\left|X^*\right|^{\frac{2t}{1-\alpha}}\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle^{n}\right)\right)^{\frac{1}{n}}\\ &\le \left(\frac{1}{S\left(h^n\right)}\left\langle\left(\alpha\left|X\right|^{\frac{2sn}{\alpha}} + \left(1-\alpha\right)\left|X^*\right|^{\frac{2tn}{1-\alpha}}\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\right)^{\frac{1}{n}}.\end{aligned} \tag{35}$$

From inequalities (34) and (35), we have

$$\left|\left\langle X\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle\right|^{2n} \le \frac{1}{S\left(h^n\right)}\left\langle\left(\alpha\left|X\right|^{\frac{2sn}{\alpha}} + \left(1-\alpha\right)\left|X^*\right|^{\frac{2tn}{1-\alpha}}\right)\widehat{k}_\varrho,\widehat{k}_\varrho\right\rangle.$$

Taking the supremum over $\varrho \in \Lambda$ in the above inequality, we have

$$ber^{2n}\left(X\right) \le \frac{1}{S\left(h^n\right)}\left\|\alpha\left|X\right|^{\frac{2sn}{\alpha}} + \left(1-\alpha\right)\left|X^*\right|^{\frac{2tn}{1-\alpha}}\right\|_{\text{ber}}.$$

□

For more recent results concerning Berezin radius inequalities for operators and other related results, we suggest [6, 16–18, 20, 22, 24, 25].

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Effects of Musics Composed Using Mathematical Methods and DNA On the EEG Frequency Bands of Healthy Individuals

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Abstract: Researches about the mathematical techniques that famous musicians used are still being debated by musicians and mathematicians. In this paper, composing new musics using new mathematical techniques or DNA was aimed to remove the lack of literature on this topic. Totally five musics were composed for this project by using DNA, Golden Ratio, and Fibonacci numbers. After the composing process, the EEG tests of the healthy volunteers will be taken while they were listening to composed musics. After that, the results will be evaluated (by comparing them), and new techniques for music therapy is aimed to be found.

Keywords: DNA, EEG, Fibonacci Numbers, Golden Ratio, Music, Mathematics

1 Introduction

1.1 The Relationship between Music and Mathematics

Music and mathematics share a strong relationship, with mathematics playing a crucial role in music theory and inspiring further research. For instance, Pythagoras emphasized the harmonious relationships between the frequencies of notes, leading to the development of the Pythagorean tuning system. Similarly, Bach explored the mathematical aspects of instrument tuning. Throughout history, mathematics has significantly contributed to the structural framework of music, showcasing its influence and potential for investigating new sound systems based on mathematical ratios (Ayata, 2020; Panti, 2020).

1.2 The Impact of Music on Psychology

The influence of music on human psychology is a captivating subject that continues to be extensively studied. Researchers often employ music, particularly classical compositions, as stimuli to investigate its effects (Okay & Ece, 2019). Studies have demonstrated that music has a profound impact on various physiological responses, such as changes in blood oxygen saturation, heart rate, and respiration, particularly observed in infants (Cassidy & Stanley, 1995). Leonard Meyer's book, "Emotion and Meaning in Music," serves as a seminal work that highlights the intricate connection between music and psychology (Spitzer, 2009).

1.3 About Music Therapy

Music therapy has a long history, with early references found in the works of Plato, who advocated for the inclusion of music in education for the holistic development of individuals (Plato, 2006). Islamic scholars, like Farabi, also recognized the impact of musical notes on emotional states (Yılmaz et al., 2019).

Throughout the Seljuk and Ottoman periods, music therapy was utilized as a complementary treatment in bimarhanes (mental hospitals) and darüşşifas (hospitals), and its practice has now expanded globally (Ersoy et al., 2018). Moreover, the sound system which is newly developed in this paper holds potential for contributing to the existing literature on music therapy.

1.4 Explanation of Brain Waves

The EEG (Electroencephalography) test is a measurement tool used to assess the electrical activity in the brain (Memorial, 2022). Brain waves, which are essential for our project, can be obtained through this test. They are classified based on their frequency and carry different meanings.

Delta waves, with a frequency up to 3Hz, are observed during deep sleep. Theta waves, ranging from 4Hz to 7Hz, occur during drowsiness, deep thinking, and meditation. Alpha waves, with frequencies between 8Hz and 12Hz, are present during wakefulness. Beta waves, in the range of 13Hz to 30Hz, are associated with focused thinking and concentration. Gamma waves, with frequencies above 30Hz, are important for learning and memory (Sisode, 2016; Koudelková et al., 2018).

Considering the distinct characteristics and implications of alpha, beta, delta, gamma, and theta waves, they will play a central role in analyzing the music compositions created in our project. For instance, the implications of alpha waves for relaxation and rest, as well as the logical and analytical thinking associated with beta waves, can provide valuable insights for the examination of music compositions (Koudelková et al., 2018).

1.5 The Use of Mathematical Structures in Music

Mathematical structures are widely used in music, serving as a foundation for various musical compositions. J.S. Bach's compositions, for example, exhibit structural and artistic features that can be analyzed and divided into distinct musical pieces, such as "prelude" and "recitative." This approach reveals the significant utilization of mathematics in music, including the exploration of values converging towards the golden ratio (Mutver, 2007).

In addition to Bach's works, studies have been conducted to investigate the mathematical and musical harmonies in different pieces of music. Frequency tables have been constructed, showcasing the mathematical and musical relationships between various measures. These studies have shown the consistent use of the golden ratio across different sections of music, providing further evidence of Bach's extensive application of mathematics (Mutver, 2007).

Despite these endeavors, the relationship between mathematics and music remains a topic of ongoing debate, and its full elucidation has not yet been achieved. In light of this, our project aims to examine this relationship objectively by relying on medical and scientific data, thus contributing to a deeper understanding of the subject.

1.6 Literature of DNA Music

DNA, the fundamental governing molecule of living organisms, consists of four nucleotides: adenine (A), thymine (T), guanine (G), and cytosine (C). RNA, on the other hand, includes uracil (U) instead of thymine. DNA and RNA are also referred to as nucleic acids.

The conversion of DNA into music, also known as *protein music*, *DNA music*, or *genetic music*, has a historical development. The initial work in this field is attributed to Gena et al. (1995) where researchers presented a mathematical algorithm to explain the process. However, the algorithm did not address the differentiation between cancer or tumor cells and did not specify the objective of observing differences between cancerous and healthy cells.

1.7 Psychological Analysis of Generated Music

In our project, music has been created using both mathematical structures and DNA transformation. At this point, it is necessary to analyze the music in a systematic manner. To enable systematic analysis, the decision has been made to use an EEG (Electroencephalography) device. All generated music will be played for individuals, and during this process, data on brain waves will be obtained using the EEG device. The collected data will be interpreted in light of the characteristics of alpha, beta, delta, gamma, and theta waves described in the introduction section.

2 Material and Method

2.1 Creating Music with Mathematical Structures

The methods used in the creation of music can be examined under two headings. The first method involves finding the remainder of each element of the Fibonacci Number Sequence when divided by 7 (mod 7 operation), and then each number is paired with a different note. The corresponding algorithm has been implemented in Python code, allowing for the determination of the number of Fibonacci numbers to be processed and the generation of a music file accordingly.

The second method is based on Zeckendorf's Theorem, which states that every positive integer can be expressed as the sum of one or more Fibonacci numbers. In this case, a Fibonacci number is chosen (e.g., 34), and the selected Fibonacci number is represented as the sum of different Fibonacci numbers, with each resulting number assigned to different notes. The assignment of notes is performed through a cyclical process, ensuring equal representation of all notes. Similarly to the first method, a Python code is created and the cyclical process is implemented in Python. An example composition generated using this method is provided in 'Figure 1'.

Fig. 1: The first period of music

Notes	Number of the notes in the first period	Fibonacci Numbers
C	1	F_1
E	5	F_5
G	5	F_5
B	5	F_5
D	5	F_5
F	8	F_6
A	5	F_5
Total	34	F_9

Table 1 Numbers Corresponding to Notes in 'Figure 1'

Notes	Distances	Fibonacci Distances
C	-	-
D	3-3-13-5	F_4-F_4-F_7-F_5
E	3-3-5-5	F_4-F_4-F_5-F_5
F	3-5-3-3-5-5-0	F_4-F_5-F_4-F_4-F_5-F_5-F_0
G	3-3-3-3-3	F_4-F_4-F_4-F_4-F_4
A	3-8-3-5	F_4-F_6-F_4-F_5
B	8-2-2-2	F_6-F_3-F_3-F_3

Table 2 Distances Between the Notes in the First Period with Fibonacci Numbers

2.2 *Studies on the Pythagorean Diatonic Scale*

The Pythagorean Diatonic Scale plays a crucial role in modern frequency generation, and thus studying the Pythagorean sequence is important. However, there is a lack of exploration regarding redesigning the Pythagorean sequence using the golden ratio. The current method utilizes $(3/2)^n$ as the basis for constructing the Pythagorean sequence and obtaining frequencies. The original ratios and frequencies of the Pythagorean sequence are presented in Tables 3 and 4, respectively.

C	D	E	F	G	A	B	C
1/1	9/8	81/64	4/3	3/2	27/16	243/128	2/1

Table 3 Original Pythagorean Diatonic Scale ratios

C	D	E	F	G	A	B	C
260 Hz	292 Hz	329 Hz	346 Hz	390 Hz	440 Hz	493 Hz	520 Hz

Table 4 Frequencies Generated According to the Original Pythagorean Diatonic Scale

The Pythagorean sequence, generated using the formula $(3/2)^n$, has been recalculated using the formula for ϕ^n. Furthermore, a new sequence based on the golden ratio has been created, with its ratios and frequencies presented in Tables 5 and 6, respectively.

Do	Do#	Re	Re#	Mi	Fa	Fa#	Sol	Sol#	La	La#	Si
1.0	1.05	1.12	1.18	1.25	1.30	1.38	1.46	1.55	1.61	1.71	1.81

Table 5 Ratios of the Diatonic Sequence Generated with the Golden Ratio (Displayed up to two decimal places)

Note	Do	Do#	Re	Re#	Mi	Fa
Frequency (Hz)	271.9	288.0	304.9	322.9	342.0	356.0
Note	Fa#	Sol	Sol#	La	La#	Si
Frequency (Hz)	376.9	399.2	422.7	440.0	465.9	493.4

Table 6 Frequencies of the Diatonic Sequence Generated with the Golden Ratio

In music theory, the following formula is used to compare the frequencies of two notes:

$$f(x, y) = 12 \cdot \log_2 \left(\frac{x}{y} \right) \tag{1}$$

Aralık	Oran	Aralık	Oran
Do - Do#	0.99	Do# - Re	0.98
Re - Re#	0.99	Re# - Mi	0.99
Mi - Fa	0.69	Fa - Fa#	0.98
Fa# - Sol	0.99	Sol - Sol#	0.99
Sol# - La	0.69	La - La#	0.99
La# - Si	0.99	-	-

Table 7 Ratios between Consecutive Notes

The ratios between consecutive notes in both sequences are presented in Table 7, based on the formula 1. The aim is to apply the newly created sound system based on the golden ratio to an existing composition.

2.3 *Creating Music With DNA*

One part of this project involves transforming DNA samples from cancerous and healthy cells into music. To achieve this, a mathematical algorithm based on base arithmetic and ASCII codes is used to obtain the musical values of each organic base in the DNA samples. The operation of the algorithm can be summarized as follows: (1) The notes are numbered starting from "C" as 0, "D" as 1, and continuing until "B" as 6, (2) The ASCII codes in base-10 are obtained for the letters A, T, G, and S used in the representation of organic bases, (3) The corresponding ASCII codes are written in base-7 (as there are a total of 7 notes), (4) The numerical values of the digits in the resulting 3-digit numbers are matched with notes, and (5) A Python software is developed based on this method. The steps of this method are illustrated in Table 8.

Steps
1. Number the notes starting from "C" as 0, "D" as 1, and continuing until "B" as 6. Example: C represents the note value 0, D represents the note value 1, E represents the note value 2, and so on until B represents the note value 6.
2. Obtain the ASCII codes (in base-10) for the letters A, T, G, and S used in the representation of organic bases. Example: The ASCII code for A is 65, T is 84, G is 71, and S is 83.
3. Convert the corresponding ASCII codes to base-7 since there are a total of 7 notes. Example: In base-7, the ASCII code 65 for A becomes 111, the ASCII code 84 for T becomes 144, the ASCII code 71 for G becomes 110, and the ASCII code 83 for S becomes 131.
4. Match the numerical values of the digits in the resulting 3-digit numbers with the corresponding notes. Example: The digit value 111 corresponds to the note C, the digit value 144 corresponds to the note D, the digit value 110 corresponds to the note E, and the digit value 131 corresponds to the note F.
5. Develop a Python software application based on this method. Example: A Python software application is created to automate the process described in the previous steps.

Table 8 Table summarizing obtaining music from DNA

During the process of creating music from cancerous DNA sequences, an analysis was conducted on two DNA sections obtained from the same region, one being healthy and the other being cancerous (or containing a tumor, etc.). A program was developed using the Python programming language to calculate the sequence that should correspond to the healthy DNA sequence when it is entirely healthy. This calculated sequence was then compared to the cancerous sequence. Based on this comparison, a new sequence was generated by writing '1' at the positions where there is a correct match (A with T, G with C), and '0' at the positions where there is no match. An example of this process is provided in Figure 3.

Step	Description
1	Two DNA sections were obtained from the same region, one being healthy and the other being cancerous (or containing a tumor, etc.).
2	A program was developed using the Python programming language to calculate the sequence that should have corresponded to the healthy DNA sequence when it was entirely healthy.
3	The calculated sequence was compared with the cancerous sequence.
4	A new sequence was generated based on the comparison results by writing '1' at the positions where there was a correct match (A with T, G with C), and '0' at the positions where there was no match.

Table 9 Explanation of the Comparison Process of Cancerous Sequences

In the generated sequences, positions with frequent adjacent '0's were identified as areas with a higher number of errors. To address this, the octave values of notes to be played in these positions were increased, and note durations were determined accordingly. Function 2 and Function 3 were developed to handle octave values and note durations, respectively. Moreover, for music composition, the first '0' value was followed by utilizing 100 organic bases.

$$f(x) = \begin{cases} 6 & \text{if } x \geq 10 \\ \lfloor \frac{x}{5} \rfloor + 4 & \text{if } x < 10 \end{cases} \tag{2}$$

$f(x)$: The f Function Used in Calculating Octave Values.

$$v(x) = \begin{cases} \frac{1}{2} & \text{if } x \leq 15 \\ \frac{3}{4} & \text{if } 15 < x < 20 \\ 1 & \text{if } 20 \leq x \end{cases} \tag{3}$$

$v(x)$: The v Function Used in Calculating Beat Counts.

The code to generate the music, incorporating the octave and beat count functions given as Function 1 and Function 2, respectively, has been written in the Python programming language. The Python code allows for obtaining the results using the provided healthy and cancerous DNA sequences.

3 Conclusion

Music therapy techniques have been used and developed since ancient times. In this paper, combining these techniques with DNA and mathematical methods and observing changes in the EEG frequency bands of healthy individuals were aimed.
First of all, mathematical structures like Golden Ratio and Fibonacci Sequence were used for composing the mathematical music. First mathematical music was composed by matching the Fibonacci numbers with notes, and second, music was composed by determining the number of the notes and the distances between them with Fibonacci numbers. Furthermore, one piece of music was composed by changing the Pythagorean Diatonic Scale by using the Golden Ratio (ϕ).
Secondly, two pieces of music were composed using healthy and cancerous DNA samples. In addition to this, two purposive functions were used to determine the beats and octaves of the cancerous DNA samples, and Python codes for all these processes were written.

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A New Note on Summability of Infinite Series and Fourier Series

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Abstract: In this paper, we have generalized two main theorems dealing with absolute Riesz summability factors of infinite series and Fourier series to the $|A,p_n,\beta;\delta|_k$ summability method. Also, some results related to the new theorem are obtained.

Keywords: Absolute matrix summability,Fourier series, Infinite series, Summability factors.

1 Introduction

Let $\sum a_n$ be an infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v \to \infty \quad as \quad n\to\infty, \quad (P_{-m}=p_{-m}=0, m\geq 1).$$

The sequence-to-sequence transformation

$$\sigma_n=\frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines the sequence (σ_n) of the $\left(\bar{N},p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [1]). If we write $X_n=\sum_{v=0}^{n}\frac{p_v}{P_v}$, then (X_n) is a positive increasing sequence tending to infinity as $n\to\infty$. The series $\sum a_n$ is said to be summable $\left|\bar{N},p_n\right|_k$, $k\geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\sigma_n-\sigma_{n-1}|^k<\infty.$$

Let $A=(a_{nv})$ be a normal matrix, i.e. a lower triangular matrix of non-zero diagonal entries. The series $\sum a_n$ is said to be summable $|A,p_n|_k$, $k\geq 1$, if (see[3])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|A_n(s)-A_{n-1}(s)|^k<\infty.$$

The series $\sum a_n$ is said to be summable $|A,p_n,\beta;\delta|_k$, $k\geq 1$, $\delta\geq 0$ and β is a real number, if (see [4])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|A_n(s)-A_{n-1}(s)|^k<\infty,$$

where

$$A_n(s)=\sum_{v=0}^{n} a_{nv}s_v, \quad n=0,1,\ldots$$

If we take $p_n=\frac{1}{n+1}$ and $k=1$, then we obtain $|R,\log n,1|$ summability (see [5]).) If we take $\beta=1$, then $|A,p_n,\beta;\delta|_k$ summability reduces to $|A,p_n;\delta|_k$ summability method (see [6]). If we take $\beta=1$ and $\delta=0$, then $|A,p_n,\beta;\delta|_k$ summability reduces to $|A,p_n|_k$ summability method.

For any sequence (λ_n) we write that $\Delta^2\lambda_n=\Delta\lambda_n-\Delta\lambda_{n+1}$ and $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$. The sequences (λ_n) is said to be of bounded variation, denoted by $\lambda_n\in BV$, if $\sum_{n=1}^{\infty}|\Delta\lambda_n|<\infty$.

2 Known Result

In [7] Bor has proved the following theorem dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 1. *Let (X_n) be a positive increasing sequence. If the sequences (X_n),(λ_n) and (p_n) satisfy the conditions*

$$\lambda_m = o(1) \quad as \quad m \to \infty, \tag{1}$$

$$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}\frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

and

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty,$$

where $t_n = \frac{1}{n+1}\sum_{v=0}^{n} va_v$, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3 Main Result

There are many papers on absolute matrix summability [8]-[21]. This study provides a generalization of above mentioned theorem to $|A, p_n, \beta; \delta|_k$ summability method under some suitable condition. Now, let us mention some notations.

Given a normal matrix $A = (a_{nv})$ be a normal matrix, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are given as follows.

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \quad \text{and} \quad \hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots$$

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{4}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{5}$$

Now, we shall prove the following theorem.

Theorem 2. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ...,$$

$$a_{n-1,v} \geq a_{nv} \quad for \quad n \geq v + 1,$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right),$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| = O\left(\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}\right) \quad as \quad m \to \infty,$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} |\hat{a}_{n,v+1}| = O(1) \quad as \quad m \to \infty,$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

and

$$\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v} = O(a_{nn}), \tag{6}$$

where $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$. If the conditions (1)-(3) of Theorem 1 are satisfied, then the series $\sum a_n\lambda_n$ is summable $|A, p_n, \beta; \delta|_k$, $k \geq 1$, $\delta \geq 0$ and $-\beta(\delta k + k - 1) + k > 0$.

Lemma 1. *[22] Under the conditions of Theorem* 1*, we get*

$$X_n|\lambda_n| = O(1) \quad as \quad n \to \infty. \tag{7}$$

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty,$$

Proof: Let (Θ_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (4) and (5), we have

$$\bar{\Delta}\Theta_n = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v.$$

By Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta}\Theta_n &= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big)\sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big)(v+1)t_v + \frac{\hat{a}_{nn}\lambda_n}{n}(n+1)t_n \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v}\Delta_v(\hat{a}_{nv})\lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v}\hat{a}_{n,v+1}\Delta\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v} + \frac{n+1}{n}a_{nn}\lambda_n t_n \\
&= \Theta_{n,1} + \Theta_{n,2} + \Theta_{n,3} + \Theta_{n,4}.
\end{aligned}$$

To prove Theorem 2, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} |\Theta_{n,r}|^k < \infty \quad for \quad r = 1, 2, 3, 4.$$

Firstly, using Hölder's inequality and $|\lambda_n| = O(1/X_n)$ by (7), we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} \left|\Theta_{n,1}\right|^k &= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right) \times \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v||\lambda_v|^{k-1}|t_v|^k\right) \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|\frac{|t_v|^k}{X_v^{k-1}}\right) \\
&= O(1)\sum_{v=1}^{m} |\lambda_v|\frac{|t_v|^k}{X_v^{k-1}} \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k} |\lambda_v|\frac{|t_v|^k}{X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \left(\frac{P_r}{p_r}\right)^{\beta(\delta k+k-1)-k} \frac{|t_r|^k}{X_r^{k-1}} \\
&+ O(1)|\lambda_m| \sum_{r=1}^{m} \left(\frac{P_r}{p_r}\right)^{\beta(\delta k+k-1)-k} \frac{|t_r|^k}{X_r^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} |\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

Now, since $v|\Delta\lambda_v| = O(1/X_v)$ by (8), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left|\Theta_{n,2}\right|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|v|\Delta\lambda_v|\frac{|t_v|}{v}\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|(v|\Delta\lambda_v|)^k\frac{|t_v|^k}{v}\right) \\
&\times\left(\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,v+1}|}{v}\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|(v|\Delta\lambda_v|)(v|\Delta\lambda_v|)^{k-1}\frac{|t_v|^k}{v}\right) \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|(v|\Delta\lambda_v|)\frac{|t_v|^k}{vX_v^{k-1}}\right) \\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{|t_v|^k}{vX_v^{k-1}}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{|t_v|^k}{vX_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{|t_r|^k}{rX_r^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{r=1}^{m}\frac{|t_r|^k}{rX_r^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|X_v+\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1}+O(1)m|\Delta\lambda_m|X_m \\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

Again using Hölder's inequality and by (6), we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left|\Theta_{n,3}\right|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{|t_v|^k}{v}\right)\times\left(\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,v+1}|}{v}\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}||\lambda_{v+1}|^{k-1}\frac{|t_v|^k}{v}\right) \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|^k}{vX_v^{k-1}}\right) \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{vX_v^{k-1}}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{vX_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v}\frac{|t_r|^k}{rX_r^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{r=1}^{m}\frac{|t_r|^k}{rX_r^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1}+O(1)m|\Delta\lambda_m|X_m \\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.
Finally, we get

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|\Theta_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} a_{nn}^k|\lambda_n|^k|t_n|^k \\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} a_{nn}^k|\lambda_n||\lambda_n|^{k-1}|t_n|^k \\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k} |\lambda_n|\frac{|t_n|^k}{X_n^{k-1}} \\ &= O(1) \quad as \quad m\to\infty,\end{aligned}$$

as in $\Theta_{n,1}$.
Therefore, we obtain that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|\Theta_{n,r}|^k = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.$$

This completes the proof of theorem. □

4 An Application to Trigonometric Fourier Series

Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x)$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx, \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx \quad and \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx.$$

Write $\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$, and $\phi_\alpha(t)=\frac{\alpha}{t^\alpha}\int_0^t(t-u)^{\alpha-1}\phi(u)du, (\alpha>0)$.
It is known that if $\phi_1(t)\in BV(0,\pi)$, then $t_n(x)=O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nA_n(x))$(see[23]).

Theorem 3. *(see[7]) If $\phi_1(t)\in BV(0,\pi)$, and sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 1, then the series $\sum A_n(x)\lambda_n$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.*

Now, we generalize Theorem 3 for $|A,p_n,\beta;\delta|_k$ summability method in the following form.

Theorem 4. *Let $A=(a_{nv})$ be a positive normal matrix as in Theorem 2. If $\phi_1(t)\in BV(0,\pi)$, and sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 2, then the series $\sum A_n(x)\lambda_n$ is summable $|A,p_n,\beta;\delta|_k$, $k\geq 1$, $\delta\geq 0$ and $-\beta(\delta k+k-1)+k>0$.*

5 Conclusion

If we take $\beta=1$, $\delta=0$ and $a_{nv}=\frac{p_v}{P_n}$ in Theorem 2, then we obtain Theorem 1 dealing with $|\bar{N},p_n|_k$ summability method. If we take $\beta=1$ in Theorem 2, then we have a result on $|A,p_n;\delta|_k$ summability method. Also, if we take $\beta=1$ and $\delta=0$ in Theorem 2, then we get a result on $|A,p_n|_k$ summability method. Finally, in the special cases of β, δ and (a_{nv}), we can obtain similar results from Theorem 4 for the trigonometric Fourier series.

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Some Properties of BLUPs in Constrained Multivariate Linear Model and its Reduced Models

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Abstract: In this study, we consider the best linear unbiased predictors (BLUPs) in the context of a constrained multivariate linear model with its reduced model. Some properties of the BLUPs and their analytical expressions are given under this reduced model that associates constrained multivariate linear model. Also, results for special cases are given.

Keywords: BLUP, constrained multivariate linear model, reduced model.

1 Introduction and preliminary results

We first introduce the notations that will be utilized throughout this study. Let $\mathbb{R}^{m\times n}$ stand for the set of all $m \times n$ real matrices. $\mathbf{A}'$, $r(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$, and $\mathbf{A}^{+}$ denote the transpose, the rank, the column space, and the Moore–Penrose generalized inverse of $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ denotes the identity matrix of order m. $\mathbf{A}^{\perp} = \mathbf{I}_m - \mathbf{A}\mathbf{A}^{+}$ stands for the orthogonal projectors. The inequality $\mathbf{A} \succcurlyeq \mathbf{0}$ means that symmetric matrix $\mathbf{A}$ is a positive semi-definite (psd) matrix in the Löwner partial ordering (LPO). Denote by $[a_1, \ldots, a_n]$ the columns of $\mathbf{A}$, the vectorization operation (vec operation) of a matrix $\mathbf{A} = [a_1, \ldots, a_n]$ is defined to be $\overrightarrow{\mathbf{A}} = [a_1', \ldots, a_n']'$. A well-known property on the vec operation of a triple matrix product is $\overrightarrow{\mathbf{A}_1\mathbf{\Lambda}\mathbf{A}_2} = (\mathbf{A}_2' \otimes \mathbf{A}_1)\overrightarrow{\mathbf{\Lambda}}$ for matrices $\mathbf{\Lambda}$, $\mathbf{A}_1$, and $\mathbf{A}_2$.

Consider a constrained multivariate linear model (CMLM)

$$\mathcal{M} : \mathbf{Y} = \mathbf{X}\mathbf{\Theta} + \mathbf{\Psi} = \mathbf{X}_1\mathbf{\Theta}_1 + \mathbf{X}_2\mathbf{\Theta}_2 + \mathbf{\Psi},\ \mathbf{C}\mathbf{\Theta} = \mathbf{C}_1\mathbf{\Theta}_1 + \mathbf{C}_2\mathbf{\Theta}_2 = \mathbf{D}, \tag{1}$$

where $\mathbf{Y} \in \mathbb{R}^{n\times m}$ is a matrix of observable dependent variables, $\mathbf{X} = [\mathbf{X}_1, \quad \mathbf{X}_2] \in \mathbb{R}^{n\times p}$ with $\mathbf{X}_i \in \mathbb{R}^{n\times p_i}$, $\mathbf{C} = [\mathbf{C}_1, \quad \mathbf{C}_2] \in \mathbb{R}^{s\times p}$ with $\mathbf{C}_i \in \mathbb{R}^{s\times p_i}$, and $\mathbf{D} \in \mathbb{R}^{s\times m}$ are known matrices of arbitrary ranks, $\mathbf{\Theta} = [\mathbf{\Theta}_1', \quad \mathbf{\Theta}_2']' \in \mathbb{R}^{p\times m}$ with $\mathbf{\Theta}_i \in \mathbb{R}^{p_i\times m}$ is a matrix of fixed but unknown parameters, $i = 1, 2$, $p_1 + p_2 = p$, $\mathbf{\Psi} \in \mathbb{R}^{n\times m}$ is a matrix of randomly distributed error terms with mean matrix $\mathrm{E}(\mathbf{\Psi}) = \mathbf{0}$ and dispersion matrix $\mathrm{D}(\overrightarrow{\mathbf{\Psi}}) = \boldsymbol{\sigma}^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)$, where $\mathbf{\Sigma}_1 = (\boldsymbol{\sigma}_{1ij}) \in \mathbb{R}^{n\times n}$ and $\mathbf{\Sigma}_2 = (\boldsymbol{\sigma}_{2ij}) \in \mathbb{R}^{m\times m}$ are psd matrices, and $\boldsymbol{\sigma}^2$ is an unknown positive number. Further, $\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1$ means that $\overrightarrow{\mathbf{\Psi}}$ has a Kronecker product structured covariance matrix.

Reduced linear models are obtained by using linear transformations on the model. They are one of the different forms of models to meet the analysis requirements. Especially, they can be considered when estimation/prediction problems on general parametric functions of partial parameters are considered. Premultiplying the model $\mathcal{M}$ by $\mathbf{X}_2^{\perp}$, we can consider the following reduced model of $\mathcal{M}$:

$$\mathcal{M}_1 : \mathbf{X}_2^{\perp}\mathbf{Y} = \mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{\Theta}_1 + \mathbf{X}_2^{\perp}\mathbf{\Psi},\ \mathbf{C}_1\mathbf{\Theta}_1 = \mathbf{D},\ \ \mathrm{E}(\mathbf{X}_2^{\perp}\mathbf{\Psi}) = \mathbf{0},\ \mathrm{D}(\mathbf{X}_2^{\perp}\mathbf{\Psi}) = \boldsymbol{\sigma}^2(\mathbf{\Sigma}_2 \otimes \mathbf{X}_2^{\perp}\mathbf{\Sigma}_1\mathbf{X}_2^{\perp}). \tag{2}$$

The two given equation parts in (1) and (2) can merge into the following combined form of matrices

$$\mathcal{R} :\ \mathbf{\Upsilon} = \widehat{\mathbf{X}}\mathbf{\Theta} + \widehat{\mathbf{\Psi}} = \widehat{\mathbf{X}}_1\mathbf{\Theta}_1 + \widehat{\mathbf{X}}_2\mathbf{\Theta}_2 + \widehat{\mathbf{\Psi}}, \tag{3}$$

$$\mathcal{R}_1 :\ \widehat{\mathbf{X}}_2^{\perp}\mathbf{\Upsilon} = \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\mathbf{\Theta}_1 + \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{\Psi}}, \tag{4}$$

respectively, and according to the expectation and covariance matrix assumptions in (1) and (2),

$$\mathrm{E}(\boldsymbol{\Upsilon}) = \widehat{\mathbf{X}}\boldsymbol{\Theta},\ \mathrm{E}(\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}) = \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\boldsymbol{\Theta}_1,\ \mathrm{D}(\overrightarrow{\boldsymbol{\Upsilon}}) = \sigma^2 \begin{bmatrix} \boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} := \sigma^2(\widehat{\boldsymbol{\Sigma}}_2 \otimes \widehat{\boldsymbol{\Sigma}}_1),$$
$$\mathrm{D}(\overrightarrow{\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}}) = \sigma^2(\widehat{\boldsymbol{\Sigma}}_2 \otimes \widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}) \tag{5}$$

are obtained, where

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{D} \end{bmatrix}, \quad \widehat{\mathbf{X}} = [\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2] = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}, \quad \widehat{\mathbf{X}}_1 = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{C}_1 \end{bmatrix}, \quad \widehat{\mathbf{X}}_2 = \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{C}_2 \end{bmatrix}, \quad \widehat{\boldsymbol{\Psi}} = \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{X}}_2^{\perp} = \begin{bmatrix} \mathbf{X}_2^{\perp} \\ \mathbf{0} \end{bmatrix}.$$

This merging operation in (3) and (4) is a well-known method of including equality restrictions in linear regression models.

We make statistical inferences of the models in (3) and (4) under the assumptions that the models are consistent, i.e., we assume that $\boldsymbol{\Upsilon} \in \mathcal{C}\left[\widehat{\mathbf{X}}, \quad \widehat{\boldsymbol{\Sigma}}_2 \otimes \widehat{\boldsymbol{\Sigma}}_1\right]$ holds with probability (wp) 1, corresponding to the consistency of $\mathcal{R}$, in this case, the model $\mathcal{R}_1$ in (4) is consistent, i.e., $\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon} \in \mathcal{C}\left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\boldsymbol{\Sigma}}_2 \otimes \widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\right]$ holds wp 1; see, e.g., [9].

To establish the results on predictors of all unknown matrices with partial parameters, we can consider the following matrix

$$\boldsymbol{\Phi}_1 = \mathbf{H}_1\boldsymbol{\Theta}_1 + \mathbf{J}\widehat{\boldsymbol{\Psi}} = \left[\mathbf{H}_1, \quad \mathbf{0}\right]\boldsymbol{\Theta} + \mathbf{J}\widehat{\boldsymbol{\Psi}}, \text{ or,}$$
$$\overrightarrow{\boldsymbol{\Phi}}_1 = (\mathbf{I}_m \otimes \mathbf{H}_1)\overrightarrow{\boldsymbol{\Theta}_1} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\widehat{\boldsymbol{\Psi}}}, \tag{6}$$

for given matrices $\mathbf{H}_1 \in \mathbb{R}^{t\times p_1}$ and $\mathbf{J} \in \mathbb{R}^{t\times(n+s)}$. Then, according to the assumptions on the expectation matrix and covariance matrix in (1), we obtain

$$\mathrm{cov}(\overrightarrow{\boldsymbol{\Phi}_1}, \overrightarrow{\boldsymbol{\Upsilon}}) = \boldsymbol{\sigma}^2(\mathbf{I}_m \otimes \mathbf{J})(\widehat{\boldsymbol{\Sigma}}_2 \otimes \widehat{\boldsymbol{\Sigma}}_1),\ \ \mathrm{cov}(\overrightarrow{\boldsymbol{\Phi}_1}, \overrightarrow{\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}}) = \boldsymbol{\sigma}^2(\mathbf{I}_m \otimes \mathbf{J})(\widehat{\boldsymbol{\Sigma}}_2 \otimes \widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}),$$
$$\mathrm{D}(\overrightarrow{\boldsymbol{\Phi}_1}) = \boldsymbol{\sigma}^2(\mathbf{I}_m \otimes \mathbf{J})(\widehat{\boldsymbol{\Sigma}}_2 \otimes \widehat{\boldsymbol{\Sigma}}_1)(\mathbf{I}_m \otimes \mathbf{J})'. \tag{7}$$

The best linear unbiased predictors (BLUPs) are defined according to the unbiasedness criteria of predictors and the minimum covariance matrix requirement in the LPO. In this study, we consider BLUPs as predictors. Under our considerations, we review the predictability/estimability requirement of $\boldsymbol{\Phi}_1$ in (6) and its special cases under $\mathcal{R}_1$ before giving the definition of the BLUP.

(a) $\boldsymbol{\Phi}_1$ is predictable by $\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ in $\mathcal{R}_1 \Longleftrightarrow \mathscr{C}(\mathbf{H}_1') \subseteq \mathscr{C}[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)'] \Longleftrightarrow \mathbf{H}_1\boldsymbol{\Theta}_1$ is estimable by $\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ in $\mathcal{R}_1$,
(b) $\widehat{\mathbf{X}}_1\boldsymbol{\Theta}_1$ is estimable by $\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ in $\mathcal{R}_1 \Longleftrightarrow \mathscr{C}(\widehat{\mathbf{X}}_1') \subseteq \mathscr{C}[(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)']$,
(c) $\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\boldsymbol{\Theta}_1$ is always estimable under $\mathcal{R}_1$ and $\widehat{\boldsymbol{\Psi}}$ is always predictable under $\mathcal{R}_1$,

see, e.g., [1]. Further, if $\boldsymbol{\Phi}_1$ is predictable under $\mathcal{R}_1$ then it is predictable under $\mathcal{R}$. Let $\boldsymbol{\Phi}_1$ be predictable under $\mathcal{R}_1$. If there exists $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ such that

$$\mathrm{D}(\overrightarrow{\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon} - \boldsymbol{\Phi}_1}) = \text{ min s.t. } \mathrm{E}(\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon} - \boldsymbol{\Phi}_1) = \mathbf{0} \tag{8}$$

holds in the LPO, the linear statistic $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ is defined to be the BLUP of $\boldsymbol{\Phi}_1$ under $\mathcal{R}_1$ and is denoted by $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon} = \mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1) = \mathrm{BLUP}_{\mathcal{R}_1}(\mathbf{H}_1\boldsymbol{\Theta}_1 + \mathbf{J}\widehat{\boldsymbol{\Psi}})$. If $\mathbf{J} = \mathbf{0}$ in $\boldsymbol{\Phi}_1$, $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ corresponds the best linear unbiased estimator (BLUE) of $\mathbf{H}_1\boldsymbol{\Theta}_1$, denoted by $\mathrm{BLUE}_{\mathcal{R}_1}(\mathbf{H}_1\boldsymbol{\Theta}_1)$, under $\mathcal{R}_1$; see, e.g., [2] and [8].

The results, in the present paper, are established by making use of some quadratic matrix optimization methods. The related subject can also be found in [3]-[6], [13] and [15]. We may refer to the studies [4], [5], [10] and [14], among others, in which both a constrained and unconstrained multivariate linear model on unknown parameters has been considered from different perspectives.

The following well-known result was given by [7].

Lemma 1.1. *The linear matrix equation* $\mathbf{AX} = \mathbf{B}$ *is consistent if and only if* $\boldsymbol{r}\left[\mathbf{A}, \quad \mathbf{B}\right] = \boldsymbol{r}(\mathbf{A})$*, or equivalently,* $\mathbf{AA}^+\mathbf{B} = \mathbf{B}$. *In this case, the general solution of* $\mathbf{AX} = \mathbf{B}$ *can be written in the following form* $\mathbf{X} = \mathbf{A}^+\mathbf{B} + (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{U}$*, where* $\mathbf{U}$ *is an arbitrary matrix.*

Lemma 1.2. *Let* $\mathbf{B} \in \mathbb{R}^{m\times p}$*,* $\mathbf{A} \in \mathbb{R}^{n\times p}$ *be given matrices, and let* $\mathbf{Q} \in \mathbb{R}^{n\times n}$ *psd matrix. Suppose that there exists* $\mathbf{X}_0 \in \mathbb{R}^{m\times n}$ *such that* $\mathbf{X}_0\mathbf{B} = \mathbf{A}$. *Then the maximal positive inertia of* $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{XQX}'$ *subject to all solutions of* $\mathbf{XB} = \mathbf{A}$ *is*

$$\max_{\mathbf{XB}=\mathbf{A}} \boldsymbol{i}_+(\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{XQX}') = \mathbf{r}\begin{bmatrix} \mathbf{X}_0\mathbf{Q} \\ \mathbf{B}' \end{bmatrix} - \mathbf{r}(\mathbf{B}) = \mathbf{r}(\mathbf{X}_0\mathbf{Q}\mathbf{B}^{\perp}).$$

Hence a solution $\mathbf{X}_0$ *of* $\mathbf{X}_0\mathbf{B} = \mathbf{A}$ *exists such that* $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' \preccurlyeq \mathbf{XQX}'$ *holds for all solutions of* $\mathbf{XB} = \mathbf{A} \Leftrightarrow$ *both the equations* $\mathbf{X}_0\mathbf{B} = \mathbf{A}$ *and* $\mathbf{X}_0\mathbf{Q}\mathbf{B}^{\perp} = \mathbf{0}$ *are satisfied by* $\mathbf{X}$.

2 Main results

The fundamental BLUP equation and some of the properties of the BLUPs under $\mathcal{R}_1$ are given as follows; see, e.g., [12].

Theorem 2.1. *Let $\boldsymbol{\Phi}_1$ be predictable under $\mathcal{R}_1$. For $\boldsymbol{\Phi}_1$ under $\mathcal{R}_1$, let $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ and $\mathbf{K}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ be unbiased linear predictors. Then the maximal positive inertia of $\mathrm{D}(\overrightarrow{\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1}) - \mathrm{D}(\overrightarrow{\mathbf{K}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1})$ subject to $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 = \mathbf{H}_1$ is*

$$\begin{aligned}\max_{\mathrm{E}(\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1)=\mathbf{0}} &\ \boldsymbol{i}_{+}\left(\mathrm{D}(\overrightarrow{\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1}) - \mathrm{D}(\overrightarrow{\mathbf{K}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1})\right) \\ &= \boldsymbol{r}\left(\begin{bmatrix}\mathbf{L}_1, & -\mathbf{I}_t\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+s}\\ \mathbf{J}\end{bmatrix}\mathrm{cov}(\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon})\begin{bmatrix}\mathbf{I}_{n+s}\\ \mathbf{J}\end{bmatrix}'\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\\ \mathbf{H}_1\end{bmatrix}^{\perp}\right). \qquad (9)\end{aligned}$$

Hence, $\mathrm{D}(\overrightarrow{\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1}) = \min$ *s.t.* $\mathrm{E}(\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1)=\mathbf{0} \Leftrightarrow \mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon} = \mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1)$

$$\Leftrightarrow \mathbf{L}_1\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix} = \begin{bmatrix}\mathbf{H}_1, & \mathbf{J}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}. \qquad (10)$$

The matrix equation in (10) *is consistent and the* $\mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1)$ *is given by using the general solution* $\mathbf{L}_1$ *of* (10),

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1) &= \mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon} = \left(\begin{bmatrix}\mathbf{H}_1, & \mathbf{J}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp} + \mathbf{U}_1\mathbf{W}_1^{\perp}\widehat{\mathbf{X}}_2^{\perp}\right)\boldsymbol{\Upsilon}, \\ \overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1)} &= (\mathbf{I}_m\otimes\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp})\overrightarrow{\boldsymbol{\Upsilon}} \\ &= \left(\mathbf{I}_m\otimes\left(\begin{bmatrix}\mathbf{H}_1, & \mathbf{J}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp} + \mathbf{U}_1\mathbf{W}_1^{\perp}\widehat{\mathbf{X}}_2^{\perp}\right)\right)\overrightarrow{\boldsymbol{\Upsilon}},\end{aligned} \qquad (11)$$

where $\mathbf{W}_1 = \begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}$ *and* $\mathbf{U}_1\in\mathbb{R}^{t\times(n+s)}$ *is an arbitrary matrix. In particular,*

$$\mathbf{L}_1 \textit{ is unique } \Longleftrightarrow \boldsymbol{r}(\mathbf{W}_1) = n+s,$$

$$\mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1) \textit{ is unique with probability 1 } \Longleftrightarrow \mathcal{R}_1 \textit{ is consistent.}$$

Further, the rank of $\mathbf{W}_1$ *satisfies the equalities*

$$\boldsymbol{r}(\mathbf{W}_1) = \boldsymbol{r}\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & (\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\end{bmatrix}. \qquad (12)$$

The covariance matrices of $\mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1)$ *and* $\boldsymbol{\Phi}_1 - \mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1)$ *are unique and satisfy the equalities*

$$\mathrm{D}[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1)}] = \boldsymbol{\sigma}^2\widehat{\boldsymbol{\Sigma}}_2\otimes\mathbf{M}_1\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\left(\mathbf{M}_1\mathbf{W}_1^{+}\right)', \qquad (13)$$

$$\mathrm{D}[\overrightarrow{\boldsymbol{\Phi}_1} - \overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\boldsymbol{\Phi}_1)}] = \boldsymbol{\sigma}^2\widehat{\boldsymbol{\Sigma}}_2\otimes\left(\mathbf{M}_1\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{J}\right)\widehat{\boldsymbol{\Sigma}}_1\left(\mathbf{M}_1\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp}-\mathbf{J}\right)', \qquad (14)$$

where $\mathbf{M}_1 = \begin{bmatrix}\mathbf{H}_1, & \mathbf{J}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}$ *and* $\mathbf{W}_1 = \begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, & \widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\end{bmatrix}$, *and* $\mathbf{U}_1\in\mathbb{R}^{t\times(n+s)}$ *is an arbitrary matrix.*

Proof of Theorem 2.1*:* Suppose that $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ and $\mathbf{K}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ are two unbiased linear predictors for $\boldsymbol{\Phi}_1$ in $\mathcal{R}_1$. Then, the expected value and covariance matrix of $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1$ are written as

$$\begin{aligned}\mathrm{E}(\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1)=\mathbf{0} &\Longleftrightarrow \mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 = \mathbf{H}_1 \\ &\Longleftrightarrow \begin{bmatrix}\mathbf{L}_1, & -\mathbf{I}_t\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1\\ \mathbf{H}_1\end{bmatrix} = \mathbf{0}\end{aligned} \qquad (15)$$

and

$$\begin{aligned}\mathrm{D}\left(\overrightarrow{\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}-\boldsymbol{\Phi}_1}\right) &= (\mathbf{I}_m\otimes(\mathbf{L}_1-\mathbf{J}))\,\mathrm{cov}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Psi}})\,(\mathbf{I}_m\otimes(\mathbf{L}_1-\mathbf{J}))' \\ &= \boldsymbol{\sigma}^2\,(\mathbf{I}_m\otimes(\mathbf{L}_1-\mathbf{J}))\,(\widehat{\boldsymbol{\Sigma}}_2\otimes\widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp})\,(\mathbf{I}_m\otimes(\mathbf{L}_1-\mathbf{J}))' \\ &= \boldsymbol{\sigma}^2\widehat{\boldsymbol{\Sigma}}_2\otimes(\mathbf{L}_1-\mathbf{J})\widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\mathbf{L}_1-\mathbf{J})' \\ &= \boldsymbol{\sigma}^2\widehat{\boldsymbol{\Sigma}}_2\otimes\begin{bmatrix}\mathbf{L}_1, & -\mathbf{I}_t\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+s}\\ \mathbf{J}\end{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\begin{bmatrix}\mathbf{I}_{n+s}\\ \mathbf{J}\end{bmatrix}'\begin{bmatrix}\mathbf{L}_1, & -\mathbf{I}_t\end{bmatrix}' := \widehat{\boldsymbol{\Sigma}}_2\otimes\boldsymbol{f}(\mathbf{L}_1),\end{aligned} \qquad (16)$$

where

$$\boldsymbol{f}(\mathbf{L}_1) = \begin{bmatrix}\mathbf{L}_1, & -\mathbf{I}_t\end{bmatrix}\begin{bmatrix}\mathbf{I}_{n+s}\\ \mathbf{J}\end{bmatrix}\widehat{\mathbf{X}}_2^{\perp}\widehat{\boldsymbol{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\begin{bmatrix}\mathbf{I}_{n+s}\\ \mathbf{J}\end{bmatrix}'\begin{bmatrix}\mathbf{L}_1, & -\mathbf{I}_t\end{bmatrix}'.$$

By using $\mathbf{K}_1$ in place of $\mathbf{L}_1$, the equivalent formulas as in (15) and (16) may also be given for the other unbiased linear predictor $\mathbf{K}_1\widehat{\mathbf{X}}_2^{\perp}\boldsymbol{\Upsilon}$ for $\boldsymbol{\Phi}_1$ under $\mathcal{R}_1$. In order to obtain solution $\mathbf{L}_1$ of the consistent linear matrix equation $\mathbf{L}_1\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 = \mathbf{H}_1$, the matrix minimization problem described in (8) for finding the BLUP of $\mathbf{\Phi}_1$ under $\mathcal{R}_1$ can be expressed such that

$$\mathbf{\Sigma}_2 \otimes \boldsymbol{f}(\mathbf{L}_1) \preccurlyeq \mathbf{\Sigma}_2 \otimes \boldsymbol{f}(\mathbf{K}_1) \text{ s.t. } \mathbf{K}_1\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 = \mathbf{H}_1 \tag{17}$$

or equivalently,

$$\boldsymbol{f}(\mathbf{L}_1) \preccurlyeq \boldsymbol{f}(\mathbf{K}_1) \text{ s.t. } \mathbf{K}_1\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1 = \mathbf{H}_1$$

because $\mathbf{\Sigma}_2$ is a non-null matrix. According to Lemma 1.2, (17) is a typical constrained quadratic matrix-valued function optimization problem in the LPO. Lemma 1.2 gives us the basic formula for the BLUP of $\mathbf{\Phi}_1$ in (10), and Lemma 1.1 gives us the expression for the BLUP of $\mathbf{\Phi}_1$ under $\mathcal{R}_1$ in (11). The expressions in (12) are well-known results; see also [11, Lemma 2.1(a)]. From (7), equalities are established in (13) and (14). □

Many consequences can be derived from Theorem 2.1 for different choices of the matrices $\mathbf{H}_1$ and $\mathbf{J}$. Some of these are given in the following.

Corollary 2.1. *Let* $\mathbf{H}_1\mathbf{\Theta}_1$ *be estimable and* $\widehat{\mathbf{\Psi}}$ *be predictable under* $\mathcal{R}_1$. *Then,*

$$\mathrm{BLUE}_{\mathcal{R}_1}(\mathbf{H}_1\mathbf{\Theta}_1) = \left(\left[\mathbf{H}_1, \quad \mathbf{0}\right]\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp} + \mathbf{U}_1\mathbf{W}_1^{\perp}\widehat{\mathbf{X}}_2^{\perp}\right)\mathbf{\Upsilon},$$
$$\overrightarrow{\mathrm{BLUE}_{\mathcal{R}_1}(\mathbf{H}_1\mathbf{\Theta}_1)} = \left(\mathbf{I}_m \otimes \left(\left[\mathbf{H}_1, \quad \mathbf{0}\right]\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp} + \mathbf{U}_1\mathbf{W}_1^{\perp}\widehat{\mathbf{X}}_2^{\perp}\right)\right)\overrightarrow{\mathbf{\Upsilon}},$$

and

$$\mathrm{BLUP}_{\mathcal{R}_1}(\widehat{\mathbf{\Psi}}) = \left(\left[\mathbf{0}, \quad \widehat{\mathbf{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp} + \mathbf{U}_1\mathbf{W}_1^{\perp}\widehat{\mathbf{X}}_2^{\perp}\right)\mathbf{\Upsilon},$$
$$\overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\widehat{\mathbf{\Psi}})} = \left(\mathbf{I}_m \otimes \left(\left[\mathbf{0}, \quad \widehat{\mathbf{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp} + \mathbf{U}_1\mathbf{W}_1^{\perp}\widehat{\mathbf{X}}_2^{\perp}\right)\right)\overrightarrow{\mathbf{\Upsilon}},$$

where $\mathbf{U}_1 \in \mathbb{R}^{t\times(n+s)}$ *is an arbitrary matrix and* $\mathbf{W}_1 = \left[\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1, \quad \widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]$. *The covariance matrices of* $\mathrm{BLUE}_{\mathcal{R}_1}(\mathbf{H}_1\mathbf{\Theta}_1)$ *and* $\mathrm{BLUP}_{\mathcal{R}_1}(\widehat{\mathbf{\Psi}})$ *are unique and satisfy the equalities*

$$\mathrm{D}[\overrightarrow{\mathrm{BLUE}_{\mathcal{R}_1}(\mathbf{H}_1\mathbf{\Theta}_1)}] = \boldsymbol{\sigma}^2\widehat{\mathbf{\Sigma}}_2 \otimes \left(\left[\mathbf{H}_1, \quad \mathbf{0}\right]\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\left(\left[\mathbf{H}_1, \quad \mathbf{0}\right]\mathbf{W}_1^{+}\right)\right)',$$

and

$$\mathrm{D}[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\widehat{\mathbf{\Psi}})}] = \boldsymbol{\sigma}^2\widehat{\mathbf{\Sigma}}_2 \otimes \left(\left[\mathbf{0}, \quad \widehat{\mathbf{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_1^{+}\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}\left(\left[\mathbf{0}, \quad \widehat{\mathbf{\Sigma}}_1\widehat{\mathbf{X}}_2^{\perp}(\widehat{\mathbf{X}}_2^{\perp}\widehat{\mathbf{X}}_1)^{\perp}\right]\mathbf{W}_1^{+}\right)\right)'.$$

3 Conclusion

In this study, we consider a CMLM and its corresponding reduced model. We obtain new models by combining two parts of the models, namely the model parts and the constraint parts. Thus, the explicit CMLMs are transformed into implicit CMLMs. This combination process is one of the approaches used when considering such models. We compute the BLUPs of all unknown parameter matrices under these models by taking this approach into account. We use some quadratic matrix optimization methods to derive analytical formulas for calculating the BLUPs. The obtained analytical results provide a broad perspective on the BLUPs under the considered models.

Another popular approach to such models is to reparameterize them subject to exact linear restrictions. Note that the linear restriction equations $\mathbf{C\Theta} = \mathbf{D}$ and $\mathbf{C}_1\mathbf{\Theta}_1 = \mathbf{D}$ in $\mathcal{M}$ and $\mathcal{M}_1$, respectively, are consistent. The general solutions of these matrix equations can be written as $\mathbf{\Theta} = \mathbf{C}^{+}\mathbf{D} + \mathbf{F_C}\mathbf{\Omega}$ and $\mathbf{\Theta}_1 = \mathbf{C}_1^{+}\mathbf{D} + \mathbf{F}_{\mathbf{C}_1}\mathbf{\Omega}_1$, respectively, where $\mathbf{\Omega} \in \mathbb{R}^{p\times m}$ and $\mathbf{\Omega}_1 \in \mathbb{R}^{p_1\times m}$ are reparameterized but arbitrary matrices. Substituting these solutions into the model equations in $\mathcal{M}$ and $\mathcal{M}_1$ yields unconstrained multivariate linear models. Both approaches can yield equivalent results for the BLUP of $\mathbf{\Phi}_1$.

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A New Transform Method and Its Application

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Abstract: Recently, various numerical or semi-analytical methods, such as Homotopy Perturbation Method, Adomian Decomposition Method, generalized Adams-Bashforth Moulton Method, Shooting Method, Differential Transformation Method, etc.have been developed for solution of linear and nonlinear differential equations due to the complexity of searching for exact solutions. One of these approximate methods is the differential transformation method (DTM, for short). This method was introduced by Zhou [1] in 1986, to solve boundary value problems appearing in modeling electrical circuits. The main goal of this work is to present a new generalization of the DTM to find numerical and semi-analytical solutions of various type differential equations. Our own method, which we call parameter dependent differential transform method (PdDTM, for short), depends on an auxiliary real parameter $p \quad (0 \leq p \leq 1)$. It is important to note that in the special cases $p = 0$ and $p = 1$ the presented PdDTM reduced to the classical DTM. We also solved an illustrative boundary value problem using PdDTM and drew a graph of the PdDTM-solution and exact solution to justify the presented PdDTM. The results obtained, showed that the proposed parameter-dependent DTM can be alternative way to solve various type boundary value problems. **Keywords:** Boundary value problems, differential transform method, numerical solution.

1 Introduction

Differential transformation method (DTM, for short) is one of the approximate methods, which enables to find an numerical solutions of various type linear and nonlinear differential equations. Zhou [1] first developed DTM to solve differential equations appearing in modeling electrical circuits. This method allows us to find not only numerical solutions, but also exact solutions in a closed form. We know that many numerical and analytical methods are not effective enough in solving various discontinuous and/or singular boundary value problems, since they may require complex algebraic calculations. However, the DTM can provide effective numerical solutions for most discontinuous and singular boundary value problems, since it is based on a Taylor's expansion. Chen and Ho [2] developed two dimensional DTM to solve linear and nonlinear boundary value problems for partial differential equations. Ayaz [3] applied differential transform technique to the system of differential equations. In [4], a differential transformation method is used to obtain the solution of momentum and heat transfer equations of non-Newtonian fluid flow in an axisymmetric channel with porous wall. In [5], the DTM is applied to linear and nonlinear system of ordinary differential equations. In [6], the DTM, is generalized to analyze the free vibration problem of pipes conveying fluid with several typical boundary conditions.

In this study we propose a new generalization of the classical DTM, which we call parameter dependent differential transform method (Pd DTM, for short) to solve initial and/or boundary value problems, as well as spectral problems.

2 The Main Properties of PdDTM

Let $\Omega = [a, b]$ be any finite interval and $f : \Omega \to \mathbb{R}$ is an analytic function, $p \quad (0 \leq p \leq 1)$ is a real parameter. Let $Y_k(f, x_0)$ be $k - th$ Taylor's coefficient of the function f, that is

$$Y_k(f, x_0) := \frac{1}{k!}\frac{d^k f}{dx^k}|_{x=x_0}, \quad k = 0, 1, 2, ... \tag{1}$$

Definition 1. *([7]) The sequence $Z_p(f)$, defined by*

$$Z_p(f) = (Z_p(f, 0), Z_p(f, 1), ..., Z_p(f, k), ...) \tag{2}$$

is called parameter dependent differential transformation (PdDT, for short) of the function f, where

$$Z_p(f, k) := pY_k(f; a) + (1 - p)Y_k(f; b). \tag{3}$$

Definition 2. *([7]) The parameter dependent inverse differential transformation (PdIDT) of the sequence $Z_p(f)$, is defined as*

$$Z_p^{-1}(Z_p(f)) := \sum_{k=0}^{\infty} Z_p(f,k)(x-x_p)^k, \tag{4}$$

where $x_p := pa + (1-p)b$. The function $f_p^*(x) := Z_p^{-1}(Z_p(f))$ is called the PdDTM- approximation of the function f.

Theorem 1. *([7]) For $p = 0$ and $p = 1$ the equality $f_p^*(x) = f(x)$ is hold.*

Remark 1. *([7]) In solving many problems by PdDTM instead of $f^*(x)$ it is convenient to introduced n-term parameter dependent approximation of the function $f(x)$ by*

$$\begin{aligned} f_{p,n}^*(x) &= Z_{p,n}^{-1}(Z_p(f)) \\ &= \sum_{k=0}^{n} Z_p(f,k)(x-x_p)^k \end{aligned} \tag{5}$$

Theorem 2. *([7]) If f(x) is constant function then $f_p^*(x) = f(x)$ and $f_{p,n}^*(x) = f(x)$ for each n.*

Theorem 3. *([7]) If $f(x) = cg(x)$, $c \in \mathbb{R}$, then $Z_p(f) = cZ_p(g)$ and $f_p^*(x) = cg_p^*(x)$.*

Theorem 4. *([7]) If $f(x) = g(x) \pm h(x)$ then $Z_p(f) = Z_p(g) \pm Z_p(h)$ and $f_p^*(x) = g_p^*(x) \pm h_p^*(x)$.*

Theorem 5. *([7]) Let $f(x) = \frac{d^m g}{dx^m}$ and $m \in \mathbb{N}$. Then*

$$Z_p(g,k) = \frac{(k+m)!}{k!} Z_p(f,k+m)$$

and

$$(f_p^m)^*(x) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} Z_p(f,k+m)(x-x_p)^k$$

where $x_p = pa + (1-p)b$.

Theorem 6. *([7]) Let $f(x) = x^m$, $m \in \mathbb{N}$. Then*

$$Z_p(f,k) = \begin{cases} \begin{pmatrix} m \\ k \end{pmatrix} (pa^{m-k} + (1-p)b^{m-k}) & for\ k < m \\ 1 & for\ \ k = m \\ 0 & for\ \ k > m \end{cases}$$

Theorem 7. *([7]) If $f(x) = g(x)h(x)$ then $Z_p(f,k) = \sum\limits_{m=0}^{k} [pZ_p(g,m)Z_p(h;k-m) + (1-p)Z_p(g,m)Z_p(h;k-m)]$*

3 Justification of the PdDTM

Let us consider the equation

$$y''(x) + xy'(x) + \left(\frac{1}{4}x^2 + \frac{1}{2}\right) y(x) = 0, \quad x \in [0,1] \tag{6}$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 1. \tag{7}$$

By using the fundamental operations of PdDTM, we have

$$(k+1)(k+2)Z_p(\overline{y},k+2) + \sum_{r=0}^{k}(k-r+1)Z_p(\overline{y},k-r+1)\delta(r-1) + \frac{1}{4}\sum_{r=0}^{k} Z_p(\overline{y},k-r)\delta(r-2) + \frac{1}{2}Z_p(\overline{y},k) = 0$$

$$Z_p(\overline{y},k+2) = \frac{-1}{(k+1)(k+2)}[\sum_{r=0}^{k}(k-r+1)Z_p(\overline{y},k-r+1)\delta(r-1) + \frac{1}{4}\sum_{r=0}^{k} Z_p(\overline{y},k-r)\delta(r-2) + \frac{1}{2}Z_p(\overline{y},k)] \tag{8}$$

Denoting $Z_p(\overline{y},0) = 0$ and $Z_p(\overline{y},1) = A$ then substituting in the recursive relation (8), we get $Z_p(\overline{y},2) = 0$, $Z_p(\overline{y},3) = \frac{-1}{4}A$, $Z_p(\overline{y},4) = 0$, $Z_p(\overline{y},5) = \frac{-1}{32}A$, $Z_p(\overline{y},6) = 0$, $Z_p(\overline{y},7) = \frac{-A}{384}A$, $Z_p(\overline{y},8) = 0, ...$

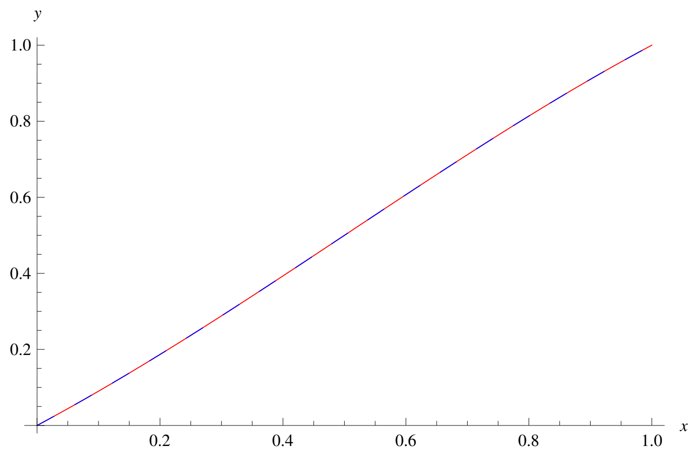

Fig. 1: Comparison of the exact solution (blue line) and PdTM solution for α=1/2 (red line).

Thus we obtain

$$\overline{y} = Ax + (\frac{-1}{4})Ax^3 + (\frac{1}{32})Ax^5 + (\frac{-1}{384})x^7 + ...$$

By using the fundamental operations of PdDTM for $x_0 = 1$, we have

$$Z_p(\overline{\overline{y}}, 0) = 1$$

$$Z_p(\overline{\overline{y}}, 1) = B$$

$$2Z_p(\overline{\overline{y}}, 2) + Z_p(\overline{\overline{y}}, 1) + (\frac{3}{4})Z_p(\overline{\overline{y}}, 0) = 0$$

$$6Z_p(\overline{\overline{y}}, 3) + 2Z_p(\overline{\overline{y}}, 2) + (\frac{7}{4})Z_p(\overline{\overline{y}}, 1) + (\frac{1}{2})Z_p(\overline{\overline{y}}, 0) = 0 \tag{9}$$

$$(k+1)(k+2)Z_p(\overline{\overline{y}}, k+2) + (k+1)Z_p(\overline{\overline{y}}, k+1) + (k+\frac{3}{4})Z_p(\overline{\overline{y}}, k) + \frac{1}{2}Z_p(\overline{\overline{y}}, k-1) + \frac{1}{4}Z_p(\overline{\overline{y}}, k-2) = 0, \quad k = 2, 3, ...$$

Then we can obtain $Z_p(\overline{\overline{y}}, 2) = \frac{1}{2}(\frac{-3}{4} - B)$, $Z_p(\overline{\overline{y}}, 3) = \frac{1-3B}{24}$, $Z_p(\overline{\overline{y}}, 4) = \frac{21+40B}{384}$, $Z_p(\overline{\overline{y}}, 5) = \frac{-18+5B}{1920}$, ... Consequently we can write

$$\begin{aligned} \overline{\overline{y}}(x) &= \sum_{k=0}^{n} Z_p(\overline{\overline{y}}, k)(x-1)^k \\ &= Z_p(\overline{\overline{y}}, 0) + Z_p(\overline{\overline{y}}, 1)(x-1) + Z_p(\overline{\overline{y}}, 2)(x-1)^2 + ... \\ &= 1 + B(x-1) + \frac{1}{2}(\frac{-3}{4} - B)(x-1)^2 + \frac{1-3B}{24}(x-1)^3 + \frac{21+40B}{384}(x-1)^4 + \frac{-18+5B}{1920}(x-1)^5 + ... \end{aligned}$$

Thus, we have

$$\begin{aligned} y_p(x) &= (1-p) + (pA + (1-p)B)(x-(1-p)) + (1-p)\frac{1}{2}(\frac{-3}{4} - B)(x-(1-p))^2 \\ &+ (p(\frac{-A}{4}) + (1-p)\frac{1-3B}{24})(x-(1-p))^3 + (1-p)\frac{21+40B}{384}(x-(1-p))^4 \\ &+ (p\frac{A}{32} + (1-p)\frac{-18+5B}{1920})(x-(1-p))^5 \end{aligned} \tag{10}$$

Now, by using boundary conditions we find $A = 2.9047$ and $B = -0.762363$.

4 Conclusion

In this work we presented a new generalization of differential transformation method, which we call parameter dependent differential transformation method (PdDTM). The presented method depends on an auxiliary parameter $p \quad (0 \leq p \leq 1)$. In the special cases $p = 0$ and $p = 1$ the PdDTM reduced to the well-know DTM. By applying the presented PdDTM we solved an illustrative differential equation. The results obtained showed that the present new method is quite reliable and can be applied to various type boundary value problems.

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Semi Global Domination Number in Product Fuzzy Graphs

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Abstract: In this paper, we introduced the concepts of semi-global domination numbers in product fuzzy graph, which is denoted by $\gamma_{sg}(G)$ and semi complimentary product fuzzy graph. We determine the semi-global domination number $\gamma_{sg}(G)$ for several classes of product fuzzy graph and obtain Nordhaus-Gaddum type results for this parameter. Further, some bounds of $\gamma_{sg}(G)$ are investigated. Also the relationship between $\gamma_{sg}(G)$ and other known parameters in Product fuzzy graphs are investigated.

Keywords: Product fuzzy graphs, semi complimentary product fuzzy graph, semi-global domination number in product fuzzy graph.
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1 Introduction

A graph is a mathematical representation of a network, and it describes the relationship between vertices and edges. Graph theory is used to represent real-life phenomena, but sometimes graphs are not able to properly represent many phenomena because the uncertainty of different attributes of the systems exists naturally. Many real-world phenomena provided motivation to define fuzzy graphs. Kauffman [4]. introduced fuzzy graphs using Zadeh's fuzzy relation [15]. The fuzzy-graph theory is growing rapidly, with numerous applications in many domains, including networking, communication, data mining, clustering, image capturing, image segmentation, planning, and scheduling.

The origin of graph theory dates back to Euler's solution to the puzzle of Koigsberg's bridges in 1736 [1]. Graph theory has numerous applications to problems in systems analysis, operations research, transportation, chemical structures, and economics [2, 3, 5, 7]. However, in many cases, some aspects of a graph theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it is natural to deal with the uncertainty using fuzzy set theory. The fuzzy graph theory as a generalization of Euler's graph theory was first introduced by Rosenfeld in 1975 [12]. Up to the present, fuzzy graphs have been studied by some researchers [8]-[9], [11] [13] [14]. For example, the concept of a semi-global dominating set in fuzzy graphs was introduced by A. Gani et al. [10]. The concept of global domination number in product fuzzy graphs was introduced by H. Ahmed and M. Shubatah (2020) [6]. This motivated us to introduce the concepts of semi-global dominating sets in product fuzzy graphs.

2 Definitions

In this section, we review briefly some definitions in Graphs, fuzzy graphs, product fuzzy graphs, semi domination number in a fuzzy graph and global domination number in a product fuzzy graph.

A crisp graph G is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex sets and the edges set of G are denoted by V(G) and E(G) respectively.

A fuzzy graph G = (μ,ρ) is a set V with two function μ :V $\rightarrow$ [0,1] and ρ : E $\rightarrow$ [0,1] such that $\rho(\{u, v\}) \leq \mu(u) \wedge \mu(v)$ for all $u, v \in V$. We write $\rho(\{u, v\})$ for $\rho(u, v)$.

The order p and size q of a fuzzy graph $G = (\mu, \rho)$ are defined to be $p = \sum_{u \in V} \mu(u)$ and $q = \sum_{(u,v) \in E} \rho(u, v)$.

A subset D of V is called a dominating set of G if for every $v \in V - D$ there exists $u \in D$ such that u dominates v.

A dominating set D of a fuzzy graph G is called a minimal dominating set if $D - \{v\}$ is not dominating set of G for all $v \in D$.

The minimum fuzzy cardinality has taken over all minimal dominating set in a fuzzy graph G is called domination number of G and denoted by $\gamma(G)$.

A subset D of V in a fuzzy graph G is said to be global dominating set if D is a dominating set in bath G and complement of G.

The global dominating set D of a fuzzy graph G is said to be minimal global dominating set if D-{v} is not global dominating set of G for all $v \in D$.

The minimum fuzzy cardinality taken over all minimal global dominating sets in a fuzzy graph is called the global domination number and is denoted by γ_g(G).

A global dominating set D of fuzzy cardinality $|D|$={$\sum \mu(u)$ for all $u \in D\} = \gamma_g(G)$ is denoted by $\gamma_g - set$.

Let D be a $\gamma_g - set$ then D is connected if the fuzzy subgraph $< D >$ induced by D is connected.
The connected global domination number of a fuzzy graph G is the minimum cardinality taken over all connected global dominating set of G and is denoted by $\gamma_{cg}(G)$.

A global dominating set D of a fuzzy graph G is called an independent global dominating set if it is also independent.

The minimum fuzzy cardinality has taken over all an independent global dominating sets is called independent global domination number and are denoted by γ_{ig}(G).

An arc uv of a fuzzy graph is called a strong arc if $COND_{-uv}(u,v) \geq \rho^{\propto}(u,v)$, where the $COND_{-uv}(u,v)$ is the strength of connectedness between u and v.

Let G be a graph whose vertex set is V, μ be a fuzzy subset of V and ρ be a fuzzy subset of V× V, we call (μ, ρ) a product partial fuzzy subgraph of G(in short, a product fuzzy graph) $if \rho(u,v) \leq \mu$(u)× μ(v) for all $u, v \in V$.

A product fuzzy graph $G = (\mu, \rho)$ is called complete product fuzzy graph if $\rho(u,v) = \mu(u) \times \mu(v)$ for all $u, v \in V$.

A product fuzzy graph G is said to be a bipartite product fuzzy graph if the vertex set V can be partitioned into two nonempty sets $V_1 and V_2$ such that $\rho(u,v) = 0$ if $u, v \in V_1$ or $u, v \in V_2$.
We say that a bipartite product fuzzy graph is a complete bipartite product fuzzy graph if $\rho(\{u,v\}) = \mu$ (u)×$\mu(v)$ for all $u \in V_1, v \in V_2$.

The complement of a product fuzzy graph $G = (V, \mu, \rho)$ is denoted by $\overline{G} = (V, \overline{\mu}, \overline{\rho})$ where $\mu = \overline{\mu}$ and $\overline{\rho}\ (u,v) = \mu(u) \times \mu(v) - \rho(u,v)$.

Let $G = (V, \mu, \rho)$ be a product fuzzy graph and $u, v \in V(G)$ then we say u dominates v if $\rho(u,v) = \mu(u) \times \mu(v)$ for all $u, v \in V$.

Let $G = (V, \mu, \rho)$ be a product fuzzy graph then a vertex subset D of V(G) is said to be dominating set of G if for every vertex $v \in (V - D)$ their exists a vertex $u \in D$ such that $\rho(u,v) = \mu(u) \times \mu(v)$.
The dominating set D of a product fuzzy graph is called a minimal product dominating set if $D - \{v\}$ is not dominating set of G, for all vertices in D.

The minimum fuzzy cardinality that has taken over all minimal dominating sets in a product fuzzy graph G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of a product fuzzy graph G is called an independent dominating set if D is independent.

The maximum fuzzy cardinality taken over all independent sets of a product fuzzy graph is called an independence number and is denoted by $\beta_0(G)$. If $e = (u,v)$ is an edge in a product fuzzy graph G. Then we say that u and v cover the edge e. A subset D of V is called a covering set of a product fuzzy graph G if all edge in G their is a vertex v in D such that v cover e.

The minimum fuzzy cardinality taken over all covering sets of a product fuzzy graph is called a vertex covering number and is denoted by $\alpha_0(G)$.

A dominating set D of a product fuzzy graph $G = (V, \mu, \rho)$ is called connected dominating set of G if the fuzzy subgraph $< D >$ induced by D is connected.

The connected domination number of a product fuzzy graph G is the minimum cardinality taken over all connected dominating sets in G and is denoted by $\gamma_c(G)$.

A dominating set D of a product fuzzy graph G is called an independent dominating set if D is an independent.

The independence domination number of fuzzy graph G is the minimum fuzzy cardinality taken over all independent dominating sets in G and is denoted by $\gamma_i(G)$.

The independence domination number of a product fuzzy graph G is the minimum fuzzy cardinality taken over all independent dominating sets in G and is denoted by $\gamma_i(G)$. Let $G = (V, \mu, \rho)$ be any fuzzy product graph where a vertex subset D of $V(G)$ is called global dominating set of G if D is also a dominating set of the complement of G.

A global dominating set D of a product fuzzy graph G is called a minimal global dominating set if $D - \{v\}$ is not global dominating set of G for all $v \in D$.

The minimum fuzzy cardinality taken over all minimal global dominating sets in a product fuzzy graph G is called the global domination number and is denoted by $\gamma_g(G)$.

Let $G = (V, \mu, \rho)$ be any fuzzy graph. Then a semi-complementary fuzzy graph which is denoted by $G^{sc} = (V, \mu^{sc}, \rho^{sc})$ defined as where
$(i) \quad \mu^{sc}(v) = \mu(v)$ and
$(ii) \quad \rho^{sc} = \{\ xy \in \rho^* and\ \exists\ u\ such\ that\ xu\ and\ uy\ in\ E. Then\ \rho^{sc}(x,y) = \mu(x) \wedge \mu(y)$.

Let $G = (V, \mu, \rho)$ be any fuzzy graph with strong arcs. A vertex subset D of $V(G)$ is called semi-global dominating set of G if D is also a dominating set of G^{sc}.

A semi-global dominating set D of a fuzzy graph G is called minimal semi global dominating set if $D - \{v\}$ is not semi global dominating set of G for all $v \in D$.

The minimum fuzzy cardinality taken over all minimal semi global dominating sets in a fuzzy graph G is called the semi global domination number and is denoted by $\gamma_{sg}(G)$.

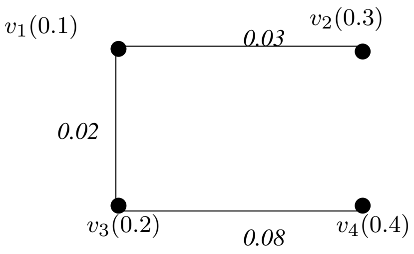
The maximum fuzzy cardinality taken over all semi global dominating sets in a fuzzy graph G is called the upper semi global domination number and is denoted by $\Gamma_{sg}(G)$.

3 Semi Complementary Product Fuzzy Graph and Semi Complete Product Fuzzy Graph

The aim of this section is to introduce and study the concepts of semi complementary product fuzzy graph and semi complete product fuzzy graph.

Definition 1. *Let $G = (V, \mu, \rho)$ be any product fuzzy graph. Then a semi complementary product fuzzy graph which is denoted by $G^{sc} = (V, \mu^{sc}, \rho^{sc})$ defined as*
$(i) \quad \mu^{sc}(v) = \mu(v)$ *and*
$(ii) \quad \rho^{sc} = \{\ xy \notin \rho^* and\ \exists\ u\ such\ that\ xu\ and\ uy\ in\ E. Then\ \rho^{sc}(x,y) = \mu(x) \times \mu(y)\}$.

Example 1. *Consider a product fuzzy graph and semi- complementary product fuzzy graph shown in Figure (1).*



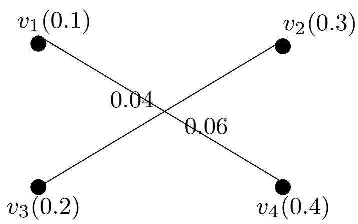


Figure 1- Product fuzzy graph (G) G^{sc}*-Complementary product fuzzy graph*

Theorem 1. (i) *If* $G = (\mu, \rho)$ *be a connected product fuzzy graph, but* G^{sc} *is not connected product fuzzy graph;*
(ii) $(G^c)^c) = G$*, but* $(G^{sc})^c \neq G^{sc}$*;*
(iii) $(G^{sc})^c)$ *is spanning subgraph of* G *and* $|E(G^c)| \geq |(E(G^{sc})|$*;*
(iv) *every edge* $\rho(u, v)$ *in* (G^{sc}) *is not neighbor in* G*;*
(v) *If* G *be complete product fuzzy graph. Then* $(G^{sc}) = G^c$ *= null graph;*
(vi) *in* (G^{sc})*, all the edges are effective edges.*

Her, we proved (i)

Proof: Let $G = (V, \mu, \rho)$ be connected product fuzzy graph. Then $\forall\ (u, v) \in\ \rho*.\ \ \rho(u, v) = \mu(u) \times \mu(v)$ and

$$\rho^{sc} = \{\ \ xy \notin \rho^* and \; \exists \; u \; such \; that \; xu \; and \; uy \; in \; E. \; Then \; \rho^{sc}(x, y) = \mu(x) \times \mu(y)$$

. Then $\rho^{\bar{s}c}(u, v) = 0$ for all $(u, v) \in \rho^{sc*}$. Hence, the proof. □

Definition 2. *Let* $G = (V, \mu, \rho)$ *be any strong product fuzzy graph. We say that* G *is a semi complete product fuzzy graph, if every pair of vertices have a common neighbor in* G*.*

Example 2. *Consider a semi complete product fuzzy graph shown in Figure(2)*

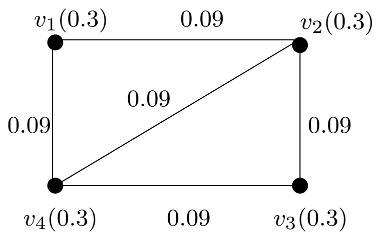


Fig 2 *Semi complete product fuzzy graph*

.

Remark 1. *Every complete product fuzzy graph is semi complete product fuzzy graph but the converse is not true.*

Proof: Let $G = (V, \mu, \rho)$ be a complete product fuzzy graph. Then every pair of vertices have a common neighbor in G. Thus G is a semi complete product fuzzy graph.
To show that the converse of the above theorem is not true we give the following example.

Example 3. *Consider a semi complete product fuzzy graph shown in Figure 3*

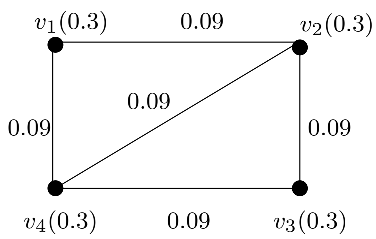


Fig 3

.
From (Figure 3) we see that (G) *is a semi-complete product fuzzy graph but is not a complete product fuzzy graph.*

□

4 Semi Global domination number in product fuzzy graph

Definition 3. *Let $G = (V, \mu, \rho)$ be any strong product fuzzy graph. A vertex subset D of $V(G)$ is called semi global dominating set of G if D is also a dominating set of G^{sc}.*

Definition 4. *A semi global dominating set D of a product fuzzy graph G is called minimal semi global dominating set if $D - \{v\}$ is not semi global dominating set of G for all $v \in D$.*

Definition 5. *The minimum fuzzy cardinality taken over all minimal semi global dominating sets in a product fuzzy graph G is called the semi global domination number and is denoted by $\gamma_{sg}(G)$.*

Definition 6. *The maximum fuzzy cardinality taken over all semi global dominating sets in a product fuzzy graph G is called the upper semi global domination number and is denoted by $\Gamma_{sg}(G)$.*

Example 4. *Consider a product fuzzy graph and semi- complementary product fuzzy graph as shown in Figure (4).*

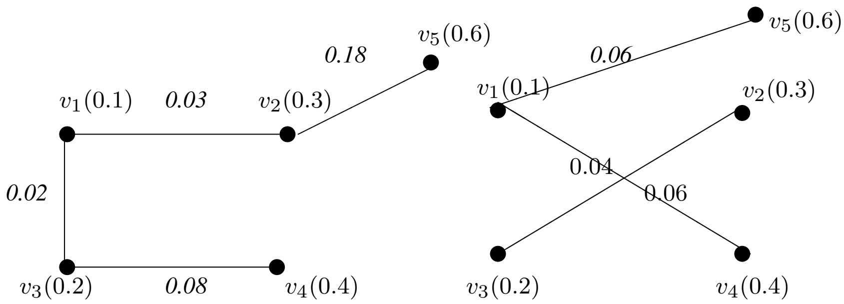


Figure 4- Product fuzzy graph G and G^{sc}

By (Figure 4) we see that $\gamma_{sg}(G)$. = 0.6. and $\Gamma_{sg}(G)$ = 0.8.

Corollary 1. *The semi global dominating set in product fuzzy graph is not singleton*

Proof: Let G be a semi product fuzzy graph and D a semi global dominating set. Since a semi global dominating set D is dominating set for both G and G^{sc}. Then D contains at least two vertices. Thus (s. g. d. set) containing more than two vertices. Hence, the result. □

In the following results, we give γ_{sg} for some standard product fuzzy graphs we begin with the complete product fuzzy graph K_μ.

Theorem 2. *If $G = (\mu, \rho)$ is a complete product fuzzy graph. Then*

$$\gamma_{sg}(G) = p.$$

Proof: Let $G = K_\mu$ be a complete product fuzzy graph. Then every vertex of G has $(n-1)$ neighbors. Since the complement of G is the null graph. Then V is only the semi global dominating set of G and G^{sc}. Hence, $\gamma_{sg}(G) \leq |v| = p$. □

The following theorem gives γ_{sg} of the complete bipartite product fuzzy graphs $K_{n,m}$.

Theorem 3. *If $G = K_{n,m}$ is complete bipartite product fuzzy graph, where $n = |V_1|$ and $m = |V_2|$. Then*

$$\gamma_{sg}(G) = p.$$

Proof: Let G be a complete bipartite product fuzzy graph and let D is a minimal semi-global dominating set of G. Then D is a dominating set of G and G^{sc}. Since V_1 and V_2 are independent. Then G^{sc}. is a null graph. Hence,

$$\gamma_{sg}(G) = p.$$

□

Theorem 4. *For any product fuzzy graph G,*

$$(i)\ \gamma_g(G) \leq \gamma_{sg}(G);$$

$$(ii)\ \gamma_g(\overline{G}) \leq \gamma_s g(G).$$

Proof: Let G be any product fuzzy graph and D be a minimal semi-global dominating set of G. Therefore, D is a global dominating set of G. Hence, $\gamma_g(G) \leq |D| = \gamma_{sg}(G)$. Similarly, $\gamma_g(\overline{G}) \leq \gamma_s g(\overline{G})$. □

The following corollary follows directly from theorem (4).

Corollary 2. *: For any product fuzzy graph G,*

$$\frac{\gamma_g + \overline{\gamma_g}}{2} \leq \gamma_s g(G) \leq \gamma_g + \overline{\gamma_g}.$$

For the semi-global domination number $\gamma_{sg}(G)$ the following theorem gives a Nordhaus-Gaddum-type result

Theorem 5. *For any product fuzzy graph G,*

$$\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) \leq 2p.$$

Further, equality holds if $\rho(u, v) < \mu(u) \times \mu(v)$ for all $u, v \in V$.

Proof: Let G be a product fuzzy graph. Since V itself is a semi global dominating set of G. Then $\gamma_{sg}(G) \leq |V| = p$ and $\gamma_{sg}(\bar{G}) \leq |V| = p$. Therefore, $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) \leq 2p$. If $\rho(u, v) < \mu(u) \times \mu(v)$ for all $u, v \in V$. Then $\gamma_{sg}(G) = \gamma_{sg}(\bar{G}) = p$. Hence, $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2p$. □

Proposition 1. *Let D be a $\gamma - set$ of a product fuzzy graph G, If there exists a vertex v in $V - D$ adjacent to only vertices in D. Then*

$$\gamma_{sg}(G) \leq \gamma + \mu(v).$$

Proof: This follows, since $D \cup \{v\}$ is a semi-global dominating set. □

Theorem 6. *For any product fuzzy graph G of order p without isolates.*

$$(i)\ \ \gamma_i(G) + \gamma_{sg}(G) \leq p + t;$$

$$(ii)\ \ \gamma(G) + \gamma_{sg}(G) \leq p + t.$$

Proof: Let D be a $\gamma i - set$ of a product fuzzy graph G and let $v \in D$ such that, $\mu(u)$ = max $\{\mu(u)$ for all $u \in V(G)\}$. Then $V - D \cup \{v\}$ is a semi-global domination set. Therefore, $\gamma_{sg}(G) \leq |V - D| + \mu(v) = p - \gamma_i(G) + t$. Hence, $\gamma_i(G) + \gamma_{sg}(G) \leq p + t$. holds Since every independent dominating set of G is a semi dominating set of G. Hence, (ii) holds. □

5 Conclusion

In this paper, we introduced the concept of semi-complementary and semi-global domination number in product fuzzy graphs. We obtained the bounds and some properties for semi-global domination number of product fuzzy graphs. Relationships between semi-global domination number on product fuzzy graphs and some other parameters were established.

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Degree-Based Topological Descriptors of Triphenylene Benzenoid System

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Abstract:
Topological descriptors are used to model various chemical and physical properties of molecules, such as boiling points, solubility, reactivity, biological activity, etc. By analyzing the relationship between the values of these indices and the properties of interest, QSPR and QSAR models can be developed to predict the properties of new molecules. Triphenylene is an aromatic hydrocarbon with the molecular formula $C_{18}H_{12}$. It is a planar molecule consisting of three fused benzene rings. The benzenoid series of triphenylene consists of compounds derived from triphenylene by replacing one or more of its benzene rings with other aromatic rings. In this study, we computed some topological descriptors of the benzenoid triphenylene series S_r, Moreover, polynomial formulae for these topological descriptors were derived in terms of the number of the Triphenylene benzenoid rings (S_r) in the chain graph.

Keywords: Benzenoid triphenylene series, Molecular Graph, Topological Descriptors.
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1 Introduction

Topological indices and coindices are numerical quantities that are derived from the chemical graph of a molecule, which is a graph representation of the molecular structure. These indices are important tools in the fields of quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) research. QSAR and QSPR studies aim to establish quantitative relationships between the physicochemical properties or biological activities of molecules and their structural characteristics[1–3].

Some examples of commonly used topological indices include the Wiener index, the Zagreb index, the Randic index, and the Balaban index [4–7]. In (1972) Gutman and Trinajstić [8, 9] presented the first degree-based structure descriptors (first and 2^{nd} Zagreb indices) (1972). Došlić (2008) introduced Zagreb coindices while computing weighted Wiener polynomials of certain composite graphs [33, 34]. They are defined as:

$$\overline{M}_1(\Gamma) = \sum_{\mu\nu \notin E(\Gamma)} [\delta(\mu) + \delta(\nu)]$$

$$\overline{M}_2(\Gamma) = \sum_{\mu\nu \notin E(\Gamma)} [\delta(\mu)\delta(\nu)]$$

Furtula et al. in (2015) introduced the forgotten index (F-index) [10, 11].

In 2016, N. De et al. [35, 36] introduce the F-coindex which is defined as follows.

$$\overline{F}(\Gamma) = \sum_{\mu\nu \notin E(\Gamma)} [\delta^2(\mu) + \delta^2(\nu)]$$

Alameri et al. [12] (2020) defined a new degree-based structure descriptor denoted by (Y-index). On the other hand, in the same year [37] authors defined new degree-based descriptors, denoted by the $(Y - coindex)$, and defined as:

$$\overline{Y}(\Gamma) = \sum_{\mu\nu \notin E(\Gamma)} [\delta^3(\mu) + \delta^3(\nu)]$$

In (2005) Li and Zheng [13] introduced the first general Zagreb index. Then, Shirdel and Sayadi [14] (2013) computed the Hyper-Zagreb index of some graph operations. Wei et al. [15] (2016) studied the First and 2^{nd} Zagreb and First and 2^{nd} Hyper-Zagreb Indices of Carbon Nanocones $CNC_k[n]$. Also, the general Rendić index is defined by Li. and Gutman [16]. In (2013), Ranjini et al. Re-defined the Zagreb indices

Topological indices	*Formulae of indices*
1^{st} Zagreb index ($M_1 - index$) [8]	$M_1(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta(\mu) + \delta(\nu)]$
Forgotten index ($F - index$) [10]	$F(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta^2(\mu) + \delta^2(\nu)]$
Yemen index ($Y - index$) [12]	$Y(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta^3(\mu) + \delta^3(\nu)]$
1^{st} general Zagreb index (M_1^{α}) [13]	$M_1^{\alpha}(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta^{\alpha}(\mu) + \delta^{\alpha}(\nu)]$
2^{nd} Zagreb index ($M_2 - index$) [8]	$M_2(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta(\mu) \cdot \delta(\nu)]$
Hyper-Zagreb index ($HM - index$) [14]	$HM(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta(\mu) + \delta(\nu)]^2$
2^{nd} Hyper-Zagreb index ($HM_2 - index$) [15]	$HM_2(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta(\mu) \cdot \delta(\nu)]^2$
General Rendić index ($R^{\alpha} - index$) [16]	$R^{\alpha}(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta(\mu) \cdot \delta(\nu)]^{\alpha}$
Redefined 1^{st} Zagreb index $ReZG_1$ [17]	$ReZG_1(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} \frac{\delta(\mu) + \delta(\nu)}{\delta(\mu)\delta(\nu)}$
Redefined 2^{nd} Zagreb index $ReZG_2$ [18]	$ReZG_2(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} \frac{\delta(\mu) \cdot \delta(\nu)}{\delta(\mu) + \delta(\nu)}$
Redefined 3^{rd} Zagreb index $ReZG_3$ [18]	$ReZG_3(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} [\delta(\mu) \cdot \delta(\nu)][\delta(\mu) + \delta(\nu)]$
Sombor index ($SO - index$) [19]	$SO(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} \sqrt{\delta^2(\mu) + \delta^2(\nu)}$
General sum-connectivity index χ^{α} [20]	$\chi^{\alpha}(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} \big[\delta(\mu) + \delta(\nu)\big]^{\alpha}$
Geometric arithmetic index (GA) [22]	$GA(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} \frac{2\sqrt{\delta(\mu)\delta(\nu)}}{\delta(\mu) + \delta(\nu)}$
Atom-bond connectivity index (ABC) [23]	$ABC(\Gamma) = \sum_{\mu\nu \in E(\Gamma)} \sqrt{\frac{\delta(\mu) + \delta(\nu) - 2}{\delta(\mu)\delta(\nu)}}$

Table 1 Some well-known topological indices.

[17, 18], the Sombor index (SO) was introduced by Gutman [19], General Sum Connectivity index defined in [20, 21]. Geometric Arithmetic index introduced by Ghorbani and Azimi [22], Atom-bond Connectivity (ABC) index defined by Estrada et al. [23].

In general, topological indices and coindices are powerful tools for understanding the structure-property relationships of molecules and are widely used in QSAR and QSPR research.

Benzenoid systems such as phenylene, biphenylene, and triphenylene are important classes of polycyclic aromatic hydrocarbons (PAHs). PAHs are a class of organic compounds composed of multiple aromatic rings. The benzenoid hydrocarbons are a subset of the alternant PAHs, but are considered to include unstable or hypothetical compounds like triangulene or heptacene. More than 300 benzenoid hydrocarbons have been isolated and characterized. These compounds are fully-conjugated hydrocarbons whose molecules are essentially planar with all rings six-membered. The benzenoid hydrocarbons are largely a subset of the alternant PAHs. Benzenoid systems are important in drug design and modeling studies and have been used to develop physicochemical descriptors of molecules that convey aromaticity-related character. The benzenoid triphenylene series consists of polycyclic aromatic hydrocarbons (PAHs) composed of three fused benzene rings arranged in a planar, disc-like structure. The general formula for these compounds is $C_{18}H_{12}$, and they have a molecular weight of 228.29 g/mol [24–30, 38].

The molecular structure of triphenylene can be represented in (Fig. 1):

2 Main results

In this section, the formulae for the 1^{st} Zagreb, Forgotten, Yemen, general Zagreb, 2^{nd} Zagreb, Hyper-Zagreb, 2^{nd} Hyper-Zagreb, General Rendić, Redefined 1^{st} Zagreb, Redefined 2^{nd} Zagreb, Redefined 3^{rd} Zagreb, Sombor, General Sum Connectivity, Geometric Arithmetic, Atom Bond Connectivity, Yemen-Sombor, the 2^{nd}, 3^{rd}, and Generalized General sum-connectivity of the chain molecular graph of Triphenylene S_r have been investigated. Moreover, polynomial formulae for all the above-mentioned topological descriptors have been introduced.

Theorem 1. *Let S_r be the r^{th} level in the chain of the Benzenoid System (See Figure 1). Then*

1. $M_1(S_r) = 78r + 24.$

2. $F(S_r) = 210r + 48.$

3. $Y(S_r) = 582r + 96.$

4. $M_1^{\alpha}(S_r) = 18n \cdot 2^{\alpha+1} + [20n + 8] \cdot 3^{\alpha+1}.$

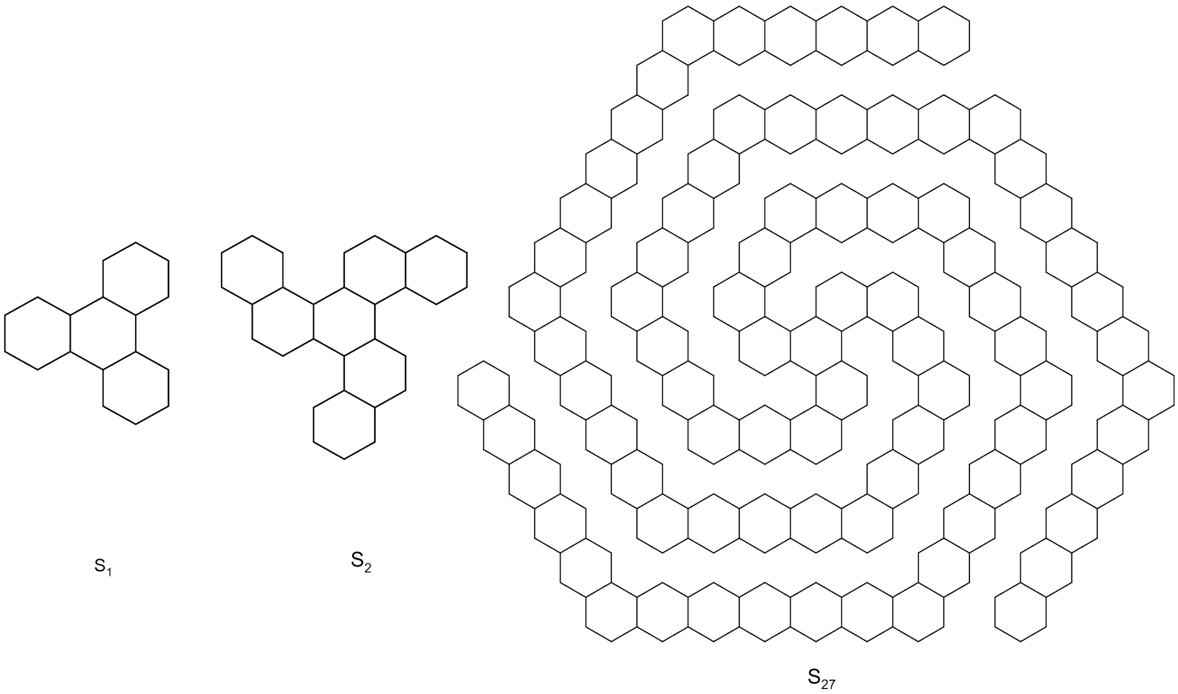

Fig. 1: Molecular structure of the Benzenoid triphenylene Series

5. $M_2(S_r) = 276n + 60.$

6. $HM(S_r) = 1128n + 228.$

7. $HM_2(S_r) = 1932n + 636.$

8. $R^{\alpha}(S_r) = 6(n+1)4^{\alpha} + 12(2n-1)6^{\alpha} + 12(n+1)9^{\alpha}.$

9. $ReZG_1(S_r) = 34n + 4.$

10. $ReZG_2(S_r) = 52.8n + 9.6.$

11. $ReZG_3(S_r) = 1464n + 384.$

12. $SO(S_r) = 154.42n + 24.62.$

13. $\chi^{\alpha}(S_r) = 4^{\alpha}[6(n+1)] + 5^{\alpha}[12(2n-1)] + 6^{\alpha}[12(n+1)].$

14. $GA(S_r) = 65.52n + 30.24.$

15. $ABC(S_r) = 29.21n + 3.76.$

16. $YS(S_r) = 12(r+2) + 6\sqrt{35}r + 18\sqrt{6}r.$

17. $\chi_2^{\alpha}(S_r) = (3r+6)\cdot 8^{\alpha} + 6r\cdot 13^{\alpha} + 6r\cdot 18^{\alpha}.$

18. $\chi_3^{\alpha}(S_r) = (3r+6)\cdot 16^{\alpha} + 6r\cdot 35^{\alpha} + 6r\cdot 54^{\alpha}.$

19. $\chi_{\alpha}^{\alpha}(S_r) = (3r+6)\cdot 2^{\alpha^2+\alpha} + 6r[2^{\alpha}+3^{\alpha}]^{\alpha} + 6r\cdot 2^{\alpha}\cdot 3^{\alpha^2}.$

Proof: We now consider the molecular graph S_r =Chain of Triphenylene Benzenoid System in r^{th} level,(Fig. 1). It is easy to obtain the $|V(S_r)| = 6(6n+1)$ and $|E(S_r)| = 6(7n+1)$, and the edge set of S_r can be divided into three edge sets as follows:

$E_{2,2}(S_r) = \{st \in E(S_r) : \delta_{S_r}(s) = 2, \delta_{S_r}(t) = 2, \delta_{S_r}(s) + \delta_{S_r}(t) = 4, \delta_{S_r}(s)\delta_{S_r}(t) = 4\},$
$E_{2,3}(S_r) = \{st \in E(S_r) : \delta_{S_r}(s) = 2, \delta_{S_r}(t) = 3, \delta_{S_r}(s) + \delta_{S_r}(t) = 5, \delta_{S_r}(s)\delta_{S_r}(t) = 6\},$
$E_{3,3}(S_r) = \{st \in E(S_r) : \delta_{S_r}(s) = 3, \delta_{S_r}(t) = 3, \delta_{S_r}(s) + \delta_{S_r}(t) = 6, \delta_{S_r}(s)\delta_{S_r}(t) = 9\},$

The Cardinality of all types of edges is shown in (Table. 2),

Table 2 The edge partition of S_r.

Edge partition	$E_{2,2}$	$E_{2,3}$	$E_{3,3}$
$Cardinality$	$3r+6$	$6r$	$6r$

Now, by the concepts of the First-Zagreb index ($M_1 - index$), Forgotten-Index ($F - index$), Yemen-Index ($Y - index$), First General-Zagreb index (M_1^{α}), 2^{nd} Zagreb index ($M_2 - index$), Hyper-Zagreb index ($HM - index$), 2^{nd} Hyper-Zagreb index ($HM_2 - index$), General-Rendić index ($R^{\alpha} - index$), Redefined First-Zagreb index ($ReZG_1$), Redefined 2^{nd} Zagreb index ($ReZG_2$), Redefined 3^{rd} Zagreb index ($ReZG_3$), Sombor index(SO), General Sum Connectivity index, Geometric-Arithmetic index, Atom Bond Connectivity index (ABC), Yemen-Sombor, the 2^{nd}, 3^{rd}, and Generalized General sum-connectivity indices respectively, we have

$$
\begin{aligned}
1.\quad M_1(S_r) &= \sum_{st\in E(S_r)}[\delta_{S_r}(s)+\delta_{S_r}(t)] = \sum_{st\in E_{2,2}}[\delta_{S_r}(s)+\delta_{S_r}(t)] \\
&+ \sum_{st\in E_{2,3}}[\delta_{S_r}(s)+\delta_{S_r}(t)] + \sum_{st\in E_{3,3}}[\delta_{S_r}(s)+\delta_{S_r}(t)] \\
&= (2+2)|E_{2,2}(S_r)| + (2+3)|E_{2,3}(S_r)| + (3+3)|E_{3,3}(S_r)| \\
&= (2+2)[3r+6] + (2+3)[6r] + (3+3)[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
2.\quad F(S_r) &= \sum_{st\in E(S_r)}[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)] = \sum_{st\in E_{2,2}}[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)] \\
&+ \sum_{st\in E_{2,3}}[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)] + \sum_{st\in E_{3,3}}[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)] \\
&= (2^2+2^2)|E_{2,2}(S_r)| + (2^2+3^2)|E_{2,3}(S_r)| + (3^2+3^2)|E_{3,3}(S_r)| \\
&= (2^2+2^2)[3r+6] + (2^2+3^2)[6r] + (3^2+3^2)[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
3.\quad Y(S_r) &= \sum_{st\in E(S_r)}[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)] = \sum_{st\in E_{2,2}}[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)] \\
&+ \sum_{st\in E_{2,3}}[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)] + \sum_{st\in E_{3,3}}[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)] \\
&= (2^3+2^3)|E_{2,2}(S_r)| + (2^3+3^3)|E_{2,3}(S_r)| + (3^3+3^3)|E_{3,3}(S_r)| \\
&= (2^3+2^3)[3r+6] + (2^3+3^3)[6r] + (3^3+3^3)[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
4.\quad M_1^{\alpha}(S_r) &= \sum_{st\in E(S_r)}[\delta^{\alpha}_{S_r}(s)+\delta^{\alpha}_{S_r}(t)] = \sum_{st\in E_{2,2}}[\delta^{\alpha}_{S_r}(s)+\delta^{\alpha}_{S_r}(t)] \\
&+ \sum_{st\in E_{2,3}}[\delta^{\alpha}_{S_r}(s)+\delta^{\alpha}_{S_r}(t)] + \sum_{st\in E_{3,3}}[\delta^{\alpha}_{S_r}(s)+\delta^{\alpha}_{S_r}(t)] \\
&= (2^{\alpha}+2^{\alpha})|E_{2,2}(S_r)| + (2^{\alpha}+3^{\alpha})|E_{2,3}(S_r)| + (3^{\alpha}+3^{\alpha})|E_{3,3}(S_r)| \\
&= (2^{\alpha}+2^{\alpha})[3r+6] + (2^{\alpha}+3^{\alpha})[6r] + (3^{\alpha}+3^{\alpha})[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
5.\quad M_2(S_r) &= \sum_{st\in E(S_r)}[\delta_{S_r}(s)\delta_{S_r}(t)] = \sum_{st\in E_{2,2}}[\delta_{S_r}(s)\delta_{S_r}(t)] + \sum_{st\in E_{2,3}}[\delta_{S_r}(s)\delta_{S_r}(t)] \\
&+ \sum_{st\in E_{3,3}}[\delta_{S_r}(s)\delta_{S_r}(t)] = (2\cdot 2)|E_{2,2}(S_r)| + (2\cdot 3)|E_{2,3}(S_r)| + (3\cdot 3)|E_{3,3}(S_r)| \\
&= (2\cdot 2)[3r+6] + (2\cdot 3)[6r] + (3\cdot 3)[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
6.\quad HM(S_r) &= \sum_{st\in E(S_r)}[\delta_{S_r}(s)+\delta_{S_r}(t)]^2 = \sum_{st\in E_{2,2}}[\delta_{S_r}(s)+\delta_{S_r}(t)]^2 \\
&+ \sum_{st\in E_{2,3}}[\delta_{S_r}(s)+\delta_{S_r}(t)]^2 + \sum_{st\in E_{3,3}}[\delta_{S_r}(s)+\delta_{S_r}(t)]^2 \\
&= (2+2)^2|E_{2,2}(S_r)| + (2+3)^2|E_{2,3}(S_r)| + (3+3)^2|E_{3,3}(S_r)| \\
&= (2+2)^2[3r+6] + (2+3)^2[6r] + (3+3)^2[6r] \\
&= 1128n+228. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
7.\quad HM_2(S_r) &= \sum_{st\in E(S_r)}[\delta_{S_r}(s)\delta_{S_r}(t)]^2 = \sum_{st\in E_{2,2}}[\delta_{S_r}(s)\delta_{S_r}(t)]^2 + \sum_{st\in E_{2,3}}[\delta_{S_r}(s)\delta_{S_r}(t)]^2 \\
&+ \sum_{st\in E_{3,3}}[\delta_{S_r}(s)\delta_{S_r}(t)]^2 \\
&= (2\cdot 2)^2|E_{2,2}(S_r)| + (2\cdot 3)^2|E_{2,3}(S_r)| + (3\cdot 3)^2|E_{3,3}(S_r)| \\
&= (2\cdot 2)^2[3r+6] + (2\cdot 3)^2[6r] + (3\cdot 3)^2[6r] \\
&= 1932n+636. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
8.\quad R^{\alpha}(S_r) &= \sum_{st\in E(S_r)}[\delta_{S_r}(s)\delta_{S_r}(t)]^{\alpha} = \sum_{st\in E_{2,2}}[\delta_{S_r}(s)\delta_{S_r}(t)]^{\alpha} + \sum_{st\in E_{2,3}}[\delta_{S_r}(s)\delta_{S_r}(t)]^{\alpha} \\
&+ \sum_{st\in E_{3,3}}[\delta_{S_r}(s)\delta_{S_r}(t)]^{\alpha} \\
&= (2\cdot 2)^{\alpha}|E_{2,2}(S_r)| + (2\cdot 3)^{\alpha}|E_{2,3}(S_r)| + (3\cdot 3)^{\alpha}|E_{3,3}(S_r)| \\
&= (2\cdot 2)^{\alpha}[6(n+1)] + (2\cdot 3)^{\alpha}[12(2n-1)] + (3\cdot 3)^{\alpha}[12(n+1)] \\
&= (3r+6)4^{\alpha} + 6r6^{\alpha} + 6r9^{\alpha}. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
9.\quad ReZG_1(S_r) &= \sum_{st\in E(S_r)}\frac{\delta_{S_r}(s)+\delta_{S_r}(t)}{\delta_{S_r}(s)\delta_{S_r}(t)} = \sum_{st\in E_{2,2}}\frac{\delta_{S_r}(s)+\delta_{S_r}(t)}{\delta_{S_r}(s)\delta_{S_r}(t)} + \sum_{st\in E_{2,3}}\frac{\delta_{S_r}(s)+\delta_{S_r}(t)}{\delta_{S_r}(s)\delta_{S_r}(t)} \\
&+ \sum_{st\in E_{3,3}}\frac{\delta_{S_r}(s)+\delta_{S_r}(t)}{\delta_{S_r}(s)\delta_{S_r}(t)} \\
&= \frac{2+2}{2\cdot 2}|E_{2,2}(S_r)| + \frac{2+3}{2\cdot 3}|E_{2,3}(S_r)| + \frac{3+3}{3\cdot 3}|E_{3,3}(S_r)| \\
&= \frac{2+2}{2\cdot 2}[3r+6] + \frac{2+3}{2\cdot 3}[6r] + \frac{3+3}{3\cdot 3}[6r] \\
&= 34n+4. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
10.\quad ReZG_2(S_r) &= \sum_{st\in E(S_r)}\frac{\delta_{S_r}(s)\delta_{S_r}(t)}{\delta_{S_r}(s)+\delta_{S_r}(t)} = \sum_{st\in E_{2,2}}\frac{\delta_{S_r}(s)\delta_{S_r}(t)}{\delta_{S_r}(s)+\delta_{S_r}(t)} + \sum_{st\in E_{2,3}}\frac{\delta_{S_r}(s)\delta_{S_r}(t)}{\delta_{S_r}(s)+\delta_{S_r}(t)} \\
&+ \sum_{st\in E_{3,3}}\frac{\delta_{S_r}(s)\delta_{S_r}(t)}{\delta_{S_r}(s)+\delta_{S_r}(t)} \\
&= \frac{2\cdot 2}{2+2}|E_{2,2}(S_r)| + \frac{2\cdot 3}{2+3}|E_{2,3}(S_r)| + \frac{3\cdot 3}{3+3}|E_{3,3}(S_r)| \\
&= \frac{2\cdot 2}{2+2}[3r+6] + \frac{2\cdot 3}{2+3}[6r] + \frac{3\cdot 3}{3+3}[6r] \\
&= 52.8n+9.6. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
11. \quad ReZG_3(S_r) &= \sum_{st\in E(S_r)} [\delta_{S_r}(s)\delta_{S_r}(t)][\delta_{S_r}(s)+\delta_{S_r}(t)] = \sum_{st\in E_{2,2}} [\delta_{S_r}(s)\delta_{S_r}(t)][\delta_{S_r}(s)+\delta_{S_r}(t)] \\
&+ \sum_{st\in E_{2,3}} [\delta_{S_r}(s)\delta_{S_r}(t)][\delta_{S_r}(s)+\delta_{S_r}(t)] + \sum_{st\in E_{3,3}} [\delta_{S_r}(s)\delta_{S_r}(t)][\delta_{S_r}(s)+\delta_{S_r}(t)] \\
&= [2\cdot 2][2+2]|E_{2,2}(S_r)| + [2\cdot 3][2+3]|E_{2,3}(S_r)| + [3\cdot 3][3+3]|E_{3,3}(S_r)| \\
&= [2\cdot 2][2+2][3r+6] + [2\cdot 3][2+3][6r] + [3\cdot 3][3+3][6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
12. \quad SO(S_r) &= \sum_{st\in E(S_r)} \sqrt{\delta^2_{S_r}(s)+\delta^2_{S_r}(t)} = \sum_{st\in E_{2,2}} \sqrt{\delta^2_{S_r}(s)+\delta^2_{S_r}(t)} \\
&+ \sum_{st\in E_{2,3}} \sqrt{\delta^2_{S_r}(s)+\delta^2_{S_r}(t)} + \sum_{st\in E_{3,3}} \sqrt{\delta^2_{S_r}(s)+\delta^2_{S_r}(t)} \\
&= \sqrt{2^2+2^2}|E_{2,2}(S_r)| + \sqrt{2^2+3^2}|E_{2,3}(S_r)| + \sqrt{3^2+3^2}|E_{3,3}(S_r)| \\
&= \sqrt{2^2+2^2}[3r+6] + \sqrt{2^2+3^2}[6r] + \sqrt{3^2+3^2}[6r] \\
&= 154.42n + 24.62. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
13. \quad \chi^\alpha(S_r) &= \sum_{st\in E(S_r)} [\delta_{S_r}(s)+\delta_{S_r}(t)]^\alpha = \sum_{st\in E_{2,2}} [\delta_{S_r}(s)+\delta_{S_r}(t)]^\alpha \\
&+ \sum_{st\in E_{2,3}} [\delta_{S_r}(s)+\delta_{S_r}(t)]^\alpha + \sum_{st\in E_{3,3}} [\delta_{S_r}(s)+\delta_{S_r}(t)]^\alpha \\
&= (2+2)^\alpha|E_{2,2}(S_r)| + (2+3)^\alpha|E_{2,3}(S_r)| + (3+3)^\alpha|E_{3,3}(S_r)| \\
&= (2+2)^\alpha[6(n+1)] + (2+3)^\alpha[12(2n-1)] + (3+3)^\alpha[12(n+1)] \\
&= 4^\alpha[3r+6] + 5^\alpha[6r] + 6^\alpha[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
14. \quad GA(S_r) &= \sum_{st\in E(S_r)} \frac{2\sqrt{\delta_{S_r}(s)\delta_{S_r}(t)}}{\delta_{S_r}(s)+\delta_{S_r}(t)} = \sum_{st\in E_{2,2}} \frac{2\sqrt{\delta_{S_r}(s)\delta_{S_r}(t)}}{\delta_{S_r}(s)+\delta_{S_r}(t)} + \sum_{st\in E_{2,3}} \frac{2\sqrt{\delta_{S_r}(s)\delta_{S_r}(t)}}{\delta_{S_r}(s)+\delta_{S_r}(t)} \\
&+ \sum_{st\in E_{3,3}} \frac{2\sqrt{\delta_{S_r}(s)\delta_{S_r}(t)}}{\delta_{S_r}(s)+\delta_{S_r}(t)} \\
&= \frac{2\sqrt{2\cdot 2}}{2+2}|E_{2,2}(S_r)| + \frac{2\sqrt{2\cdot 3}}{2+3}|E_{2,3}(S_r)| + \frac{2\sqrt{3\cdot 3}}{3+3}|E_{3,3}(S_r)| \\
&= \frac{2\sqrt{2\cdot 2}}{2+2}[3r+6] + \frac{2\sqrt{2\cdot 3}}{2+3}[6r] + \frac{2\sqrt{3\cdot 3}}{3+3}[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
15. \quad ABC(S_r) &= \sum_{st\in E(S_r)} \sqrt{\frac{\delta_{S_r}(s)+\delta_{S_r}(t)-2}{\delta_{S_r}(s)\delta_{S_r}(t)}} = \sum_{st\in E_{2,2}} \sqrt{\frac{\delta_{S_r}(s)+\delta_{S_r}(t)-2}{\delta_{S_r}(s)\delta_{S_r}(t)}} \\
&+ \sum_{st\in E_{2,3}} \sqrt{\frac{\delta_{S_r}(s)+\delta_{S_r}(t)-2}{\delta_{S_r}(s)\delta_{S_r}(t)}} + \sum_{st\in E_{3,3}} \sqrt{\frac{\delta_{S_r}(s)+\delta_{S_r}(t)-2}{\delta_{S_r}(s)\delta_{S_r}(t)}} \\
&= \sqrt{\frac{2+2-2}{2\cdot 2}}|E_{2,2}(S_r)| + \sqrt{\frac{2+3-2}{2\cdot 3}}|E_{2,3}(S_r)| + \sqrt{\frac{3+3-2}{3\cdot 3}}|E_{3,3}(S_r)| \\
&= \sqrt{\frac{2+2-2}{2\cdot 2}}[3r+6] + \sqrt{\frac{2+3-2}{2\cdot 3}}[6r] + \sqrt{\frac{3+3-2}{3\cdot 3}}[6r]. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
16.\quad YS(S_r) &= \sum_{st\in E(S_r)} \sqrt{\delta^3_{S_r}(s)+\delta^3_{S_r}(t)} = \sum_{st\in E_{2,2}(S_r)} \sqrt{\delta^3_{S_r}(s)+\delta^3_{S_r}(t)} \\
&+ \sum_{st\in E_{2,3}(S_r)} \sqrt{\delta^3_{S_r}(s)+\delta^3_{S_r}(t)} + \sum_{st\in E_{3,3}(S_r)} \sqrt{\delta^3_{S_r}(s)+\delta^3_{S_r}(t)} \\
&= \sqrt{2^3+2^3}|E_{2,2}(S_r)| + \sqrt{2^3+3^3}|E_{2,3}(S_r)| + \sqrt{3^3+3^3}|E_{3,3}(S_r)| \\
&= 12(r+2)+6\sqrt{35}r+18\sqrt{6}r. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
17.\quad \chi^\alpha_2(S_r) &= \sum_{st\in E(S_r)} \left[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)\right]^\alpha = \sum_{st\in E_{2,2}(S_r)} \left[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)\right]^\alpha \\
&+ \sum_{st\in E_{2,3}(S_r)} \left[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)\right]^\alpha + \sum_{st\in E_{3,3}(S_r)} \left[\delta^2_{S_r}(s)+\delta^2_{S_r}(t)\right]^\alpha \\
&= [2^2+2^2]^\alpha|E_{2,2}(S_r)| + [2^2+3^2]^\alpha|E_{2,3}(S_r)| + [3^2+3^2]^\alpha|E_{3,3}(S_r)| \\
&= (3r+6)\cdot 8^\alpha + 6r\cdot 13^\alpha + 6r\cdot 18^\alpha. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
18.\quad \chi^\alpha_3(S_r) &= \sum_{st\in E(S_r)} \left[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)\right]^\alpha = \sum_{st\in E_{2,2}(S_r)} \left[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)\right]^\alpha \\
&+ \sum_{st\in E_{2,3}(S_r)} \left[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)\right]^\alpha + \sum_{st\in E_{3,3}(S_r)} \left[\delta^3_{S_r}(s)+\delta^3_{S_r}(t)\right]^\alpha \\
&= [2^3+2^3]^\alpha|E_{2,2}(S_r)| + [2^3+3^3]^\alpha|E_{2,3}(S_r)| + [3^3+3^3]^\alpha|E_{3,3}(S_r)| \\
&= (3r+6)\cdot 16^\alpha + 6r\cdot 35^\alpha + 6r\cdot 54^\alpha. \quad \square
\end{aligned}
$$

$$
\begin{aligned}
19.\quad \chi^\alpha_\alpha(S_r) &= \sum_{st\in E(S_r)} \left[\delta^\alpha_{S_r}(s)+\delta^\alpha_{S_r}(t)\right]^\alpha = \sum_{st\in E_{2,2}(S_r)} \left[\delta^\alpha_{S_r}(s)+\delta^\alpha_{S_r}(t)\right]^\alpha \\
&+ \sum_{st\in E_{2,3}(S_r)} \left[\delta^\alpha_{S_r}(s)+\delta^\alpha_{S_r}(t)\right]^\alpha + \sum_{st\in E_{3,3}(S_r)} \left[\delta^\alpha_{S_r}(s)+\delta^\alpha_{S_r}(t)\right]^\alpha \\
&= [2^\alpha+2^\alpha]^\alpha|E_{2,2}(S_r)| + [2^\alpha+3^\alpha]^\alpha|E_{2,3}(S_r)| + [3^\alpha+3^\alpha]^\alpha|E_{3,3}(S_r)| \\
&= (3r+6)\cdot 2^{\alpha^2+\alpha} + 6r[2^\alpha+3^\alpha]^\alpha + 6r\cdot 2^\alpha\cdot 3^{\alpha^2}. \quad \square
\end{aligned}
$$

$\square$

Corollary 1. *Let S_r be the r^{th} level in the Triphenylene chain. Then*

1. $M_1(S_r,x) = (3r+6)x^4 + 6rx^5 + 6rx^6$.
2. $F(S_r,x) = (3r+6)x^8 + 6rx^{13} + 6rx^{18}$.
3. $Y(S_r,x) = (3r+6)x^{16} + 6rx^{35} + 6rx^{54}$.
4. $M_1^\alpha(S_r,x) = (3r+6)x^{2\cdot 2^\alpha} + (6r)x^{2^\alpha+3^\alpha} + 6rx^{2\cdot 3^\alpha}$.
5. $M_2(S_r,x) = (3r+6)x^4 + (6r)x^6 + 6rx^9$.
6. $HM(S_r,x) = (3r+6)x^{16} + (6r)x^{25} + 6rx^{36}$.
7. $HM_2(S_r,x) = (3r+6)x^{16} + (6r)x^{36} + 6rx^{81}$.
8. $R^\alpha(S_r,x) = (3r+6)x^{4^\alpha} + (6r)x^{6^\alpha} + 6rx^{9^\alpha}$.
9. $ReZG_1(S_r,x) = (3r+6)x + (6r)x^{\frac{5}{6}} + 6rx^{\frac{2}{3}}$.
10. $ReZG_2(S_r,x) = (3r+6)x + (6r)x^{\frac{6}{5}} + 6rx^{\frac{3}{2}}$.

11. $ReZG_3(S_r,x) = (3r+6)x^{16} + (6r)x^{30} + 6rx^{54}$.

12. $SO(S_r,x) = (3r+6)x^{2\sqrt{2}} + 6rx^{\sqrt{13}} + 6rx^{3\sqrt{2}}$.

13. $\chi^{\alpha}(S_r,x) = (3r+6)x^{4^{\alpha}} + 6rx^{5^{\alpha}} + 6rx^{6^{\alpha}}$.

14. $GA(S_r,x) = (9r+6)x + 6rx^{\frac{2\sqrt{6}}{5}}$.

15. $ABC(S_r,x) = (9r+6)x^{\frac{1}{\sqrt{2}}} + 6rx^{\frac{2}{3}}$.

16. $YS(S_r,x) = (3r+6)x^{4} + 6rx^{\sqrt{35}} + 6rx^{3\sqrt{6}}$.

17. $\chi_2^{\alpha}(S_r,x) = (3r+6)x^{8^{\alpha}} + 6rx^{13^{\alpha}} + 6rx^{18^{\alpha}}$.

18. $\chi_3^{\alpha}(S_r,x) = (3r+6)x^{16^{\alpha}} + 6rx^{35^{\alpha}} + 6rx^{54^{\alpha}}$.

19. $\chi_{\alpha}^{\alpha}(S_r,x) = (3r+6)x^{2^{\alpha^2+\alpha}} + 6rx^{[2^{\alpha}+3^{\alpha}]^{\alpha}} + 6rx^{2^{\alpha}\cdot 3^{\alpha^2}}$.

3 Conclusion

This is interesting research on computing various topological descriptors and deriving their polynomial formulae for the Triphenylene benzenoid chain molecular graphs. Some highlights:

• Topological descriptors are graph-based numerical parameters that correlate well with molecular properties and activities. They can be used for structure-property relationship studies and molecular design.

• The study has computed a range of topological descriptors for the Triphenylene Benzenoid chain molecular graphs, including Zagreb indices, Hyper-Zagreb indices, Rendic indices, sum connectivity indices, and others.

• Polynomial formulae for these topological descriptors were derived in terms of the number of the Triphenylene benzenoid rings (S_r) on the chain graph. This allows the topological descriptors to be easily calculated for any Triphenylene benzenoid chain of a given size.

• The results provide useful information on the topological characteristics of the Triphenylene Benzenoid chain molecular graphs as a function of their size. This can help in understanding how properties may change with the size of the Triphenylene benzenoid chains.

• The derived formulae could be applied in predictive models for the structure-property relationships and molecular design of the Triphenylene benzenoid chain compounds.

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Childhood Cancer Risk Analysis with Integrated Decision-Making Method

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Abstract: The top cause of death for children worldwide, particularly in low- and middle-income nations, is cancer in childhood. The nation in which they reside has a significant impact on their chances of survival. In high-income nations, the likelihood of treating childhood cancer is greater than 80%, compared to just 45% in low- and middle-income nations. Effective, evidence-based therapy combined with compassionate care is the most effective strategy to lessen the impacts of childhood cancer. The current risk assessment procedure will be addressed, and a new risk evaluation method will be presented. In order to cope with ambiguity in the risk assessment process for pediatric cancer, the suggested strategy makes use of MCDM, which has a hybrid structure made up of the Neutrosophic AHP and Fermatean Fuzzy AHP methods.

Keywords: Childhood cancer, cosine similarity, FF-AHP, NF-AHP, risk analysis

1 Introduction

Childhood cancer is the leading cause of mortality for children globally, particularly in low- and middle-income countries. The nation in which they reside has a significant impact on their chances of survival. In high-income nations, the likelihood of treating childhood cancer is greater than 80%, compared to just 45% in low- and middle-income nations. This discrepancy in cure rates is caused by a variety of variables, including late diagnosis and cancer detected in its advanced stages due to a lack of resources, the expense of therapy (which is higher in the later stages of the disease), inaccurate diagnosis, and unsuitable treatment. If low- and middle-income nations gain access to vital drugs and technology, the survival rate could rise. In general, effective, evidence-based therapy combined with compassionate care is the most effective strategy to lessen the impacts of childhood cancer.

If it is detected early and the proper therapy is given right away, the chances of curing children's cancer and the cost of treatment with less suffering can be improved. A proper diagnosis is necessary for effective treatment of children's cancer with the appropriate measures, which may include operations, radiation, and chemotherapy. The following three factors should be considered for early diagnosis:

- It is important for parents to be aware of children's cancer so they can spot the signs and seek medical attention.
- To offer the best care, the medical expert must be qualified to look at the situation right away.
- The patient is given the appropriate care at the appropriate time.

Even with the least degree of physical and financial pain, if cancer is discovered early enough, there is a greater possibility of recovery and survival. With the aid of qualified doctors, low- and middle-income nations should launch parental education initiatives to help parents react quickly if their children exhibit signs. Both non-governmental groups and civil society must work together to complete this mission. The World Health Organization started a global program on childhood cancer in 2018. They offered the government professional advice and support as part of this campaign in order to keep up high-quality programs to combat childhood cancer. By 2030, they hope to enhance childhood cancer survival rates, which should be at least 60%.

Any decision-making process must account for imprecision. To deal with the ambiguous environment of collective decision-making, many tools and strategies have been proposed. Fermatean fuzzy sets (FFS) [14] are one of the newest techniques for coping with uncertainty. Compared to the intuitionistic fuzzy sets(IFS) [3] and Pythagorean fuzzy sets(PFS) [18], [19], which are extensions of Zadeh's fuzzy set [20], these sets offer a larger range of applications. Recently, FFs have inspired many studies.([2], [8], [9], [10], [11], [12], [15], [16]).

Smarandache [17] extended the idea of IFSs to create neutrosophic sets(NS), which offer a fresh perspective on ambiguity, imprecision, consistency, and vagueness. Smarandache [3] defined a NS by its three ingredient: truth membership, indeterminacy membership, and falsity membership. Smarandache also added the degree of indeterminacy or neutrality as a new and independent ingredient of FSs. The intuitionistic neutrosophic soft set has been defined and examined some properties by Broumi and Smarandache [6].

The AHP improved by Saaty [13] is among the most widely used MCDM techniques. Researchers can methodically specify the weight of the criteria and alternatives. The traditional AHP approach has been expanded into a number of fuzzy variations due to inadequate information and ambiguity. Since 2013, NSs have been widely utilized in decision-making procedures. To the best of our knowledge, Abdel-Basset et al. [1] and

Radwan et al. [4] both published works on NF-AHP. Both interval-valued NF-AHP alone and interval-valued NF-AHP combined with a cosine similarity measure are presented by Boltürk and Kahraman [5]. Based on Bhattacharya's distance, Broumi and Smarandache [7] presented a new cosine similarity between two interval-valued NSs. The FF-AHP was first introduced by Alkan et al. [2].

A measure of similarity between two vectors of n dimensions using the cosine of the angle between them is known as cosine similarity. It compares simply the direction of two vectors to determine how similar they are. Each user is viewed as a vector of prior judgments in this similarity. The cosine value of the angle between the two vectors in this instance expresses how similar the two vectors are to one another. By dividing the inner product of the two vectors by the product of their lengths, the cosine similarity formula determines the angle between the two vectors. The closeness between two users in cosine similarity ranges from 0 to 1. The likelihood of users increases as the result gets closer to 1. A closeness in cosine similarity between two users does not mean that they rate things similarly; rather, it means that there is a consistent correlation between their evaluations. The following formula is used to express the cosine similarity between two vectors, d_i and d_j: $sim_{cos}(d_i, d_j) = \frac{\overrightarrow{d_i}.\overrightarrow{d_j}}{\|\overrightarrow{d_i}\|.\|\overrightarrow{d_j}\|}$.

The originality: Risks according to childhood cancer are prioritized using an MCDM approach. The fuzzy approach employed in this work captures the erroneous information that distinguishes decision-makers assessments. In conclusion, this study provides further insight into the specific risk landscape for childhood cancer in the future. The strategy put forward in this study offers a sophisticated and enhanced manner of managing uncertainty in risk prioritization. For MCDM, a hybrid technique based on FF-AHP, NF-AHP, and Cosine Similarity procedures has been suggested in order to give physicians more dependable options. The suggested approach is a helpful tool that may be used to solve various complicated choice issues with many competing criteria because of its adaptable structure.

2 Method

U, the initial universe set, will be used as a symbol throughout the article.

The FFS $\mathcal{F}$ is shown by $\mathcal{F} = \{(u, \zeta_{\mathcal{F}}(u), \eta_{\mathcal{F}}(u)) : u \in U\}$, where $\zeta_{\mathcal{F}} : U \to [0,1]$ and $\eta_{\mathcal{F}} : U \to [0,1]$ and the inequality $0 \le \zeta^3_{\mathcal{F}}(u) + \eta^3_{\mathcal{F}}(u) \le 1$ [14] is valid. It is defined as $\theta_{\mathcal{F}}(u) = \sqrt[3]{1 - (\zeta^3_{\mathcal{F}}(u) + \eta^3_{\mathcal{F}}(u))}$ degree of indeterminacy of u to $\mathcal{F}$.

Take three FFSs $\mathcal{F} = \{\zeta_{\mathcal{F}}, \eta_{\mathcal{F}}\}$, $\mathcal{F}_1 = \{\zeta_{\mathcal{F}_1}, \eta_{\mathcal{F}_1}\}$ and $\mathcal{F}_2 = \{\zeta_{\mathcal{F}_2}, \eta_{\mathcal{F}_2}\}$. Then, some operations as follows [14]:

i. $\mathcal{F}_1 \cap \mathcal{F}_2 = (\min\{\zeta_{\mathcal{F}_1}, \zeta_{\mathcal{F}_2}\}, \max\{\eta_{\mathcal{F}_1}, \eta_{\mathcal{F}_2}\})$;
ii. $\mathcal{F}_1 \cup \mathcal{F}_2 = (\max(\zeta_{\mathcal{F}_1}, \zeta_{\mathcal{F}_2}), \min(\eta_{\mathcal{F}_1}, \eta_{\mathcal{F}_2}))$;
iii. $\mathcal{F}^t = (\eta_{\mathcal{F}}, \zeta_{\mathcal{F}})$;
iv. $\mathcal{F}_1 \boxplus \mathcal{F}_2 = \left(\sqrt[3]{\zeta^3_{\mathcal{F}_1} + \zeta^3_{\mathcal{F}_2} - \zeta^3_{\mathcal{F}_1}\zeta^3_{\mathcal{F}_2}}, \eta_{\mathcal{F}_1}\eta_{\mathcal{F}_2}\right)$;
v. $\mathcal{F}_1 \boxtimes \mathcal{F}_2 = \left(\zeta^3_{\mathcal{F}_1}\zeta^3_{\mathcal{F}_2}, \sqrt[3]{\eta^3_{\mathcal{F}_1} + \eta^3_{\mathcal{F}_2} - \eta^3_{\mathcal{F}_1}\eta^3_{\mathcal{F}_2}}\right)$;
vi. $\mu\mathcal{F} = \left(\sqrt[3]{1 - (1 - \zeta^3_{\mathcal{F}})^\mu}, \eta^\mu_{\mathcal{F}}\right), \quad \mu > 0$;
vii. $\mathcal{F}^\mu = \left(\zeta^3_{\mathcal{F}_1}, \sqrt[3]{1 - (1 - \eta^3_{\mathcal{F}})^\mu}\right), \quad \mu > 0$.

Definition 1. *For two interval numbers $t = [t^-, t^+]$ and $s = [s^-, s^+]$, the operations*

$$t + s = [t^- + s^-, t^+ + s^+], \quad t - s = [t^- - s^+, t^+ - s^-],$$
$$t^n = [(t^-)^n, (t^+)^n], \textit{here} \quad t^- \ge 0, \quad n \in \mathbb{N}.$$

are called interval arithmetic.

The set $A = \{u, T_A(u), I_A(u), F_A(u) : u \in U\}$ is called a NS, where $T_A(u), I_A(u), F_A(u) : U \to [0,1]$ and $0 \le T_A(u)^3 + I_A(u)^3 + F_A(u)^3 \le 3^+$.

Table 1 Scale for FF-AHP [2]

Linguistic Terms	ζ_L	ζ_U	η_L	η_U
Certainly low importance(CLI)	0	0	0.9	1
Very low importance(VLI)	0.1	0.2	0.8	0.9
Low importance(LI)	0.2	0.35	0.65	0.8
Below average importance(BAI)	0.35	0.45	0.55	0.65
Average importance(AI)	0.45	0.55	0.45	0.55
Above average importance(AAI)	0.55	0.65	0.35	0.45
High importance(HI)	0.65	0.8	0.35	0.2
Very high importance(VHI)	0.8	0.9	0.1	0.2
Certainly high importance(CHI)	0.9	1	0	0
Exactly equal(EE)	0.1965	0.1965	0.1965	0.1965

For the FF-AHP:

Step 1: Create the pairwise comparison matrix $\mathcal{R} = (r_{ik})_{m \times m}$ using Table 1 to reflect the experts' assessments.

Table 2 Scale for NF-AHP

Linguistic Terms	Neutrosophic values
Equal Importance	([0.5, 0.5], [0.5, 0.5], [0.5, 0.5])
Weakly More Importance	([0.5, 0.6], [0.35, 0.45], [0.4, 0.5])
Moderate Importance	([0.55, 0.65], [0.3, 0.4], [0.35, 0.45])
Moderately More Importance	([0.6, 0.7], [0.25, 0.35], [0.3, 0.4])
Strong Importance	([0.65, 0.75], [0.2, 0.3], [0.25, 0.35])
Strongly More Importance	([0.7, 0.8], [0.15, 0.25], [0.2, 0.3])
Very Strong Importance	([0.75, 0.85], [0.1, 0.2], [0.15, 0.25])
Very Strongly More Importance	([0.8, 0.9], [0.05, 0.1], [0.1, 0.2])
Extreme Importance	([0.9, 0.95], [0.0, 0.05], [0.05, 0.15])
Extremely High Importance	([0.95, 1.0], [0.0, 0.0], [0.0, 0.1])
Absolutely More Importance	([1.0, 1.0], [0.0, 0.0], [0.0, 0.0])

Step 2: Using the bottom and upper values of the membership and non-membership functions, calculate the differences matrix $\mathcal{D} = (\delta_{ik})_{m\times m}$ using:

$$\delta_{ikL} = \mu_{ikL}^3 - \nu_{ikU}^3 \quad (1)$$
$$\delta_{ikU} = \mu_{ikU}^3 - \nu_{ikL}^3. \quad (2)$$

Step 3: Obtain the interval multiplicative matrix $\mathcal{S} = (\sigma_{ik})_{m\times m}$ by using:

$$\sigma_{ikL} = \sqrt{1000^{\delta_{ikL}}} \quad (3)$$
$$\sigma_{ikU} = \sqrt{1000^{\delta_{ikU}}}. \quad (4)$$

Step 4: Compute the determinacy value $T = (\rho_{ik})_{m\times m}$ of the r_{ik} by employing:

$$\rho_{ik} = 1 - (\mu_{ikL}^3 - \nu_{ikU}^3) - (\mu_{ikU}^3 - \nu_{ikL}^3). \quad (5)$$

Step 5: To get the weights matrix before $\mathcal{Z} = (\zeta_{ik})_{m\times m}$ normalization, multiply the determinacy values by the $\mathcal{S} = (\sigma_{ik})_{m\times m}$ matrix using:

$$\zeta_{ik} = \left[\frac{\sigma_{ikL} + \sigma_{ikU}}{2}\right]\rho_{ik}. \quad (6)$$

Step 6: Compute the normalized priority weights ω_i by adpting Equation 7:

$$\omega_i = \frac{\sum_{k=1}^{m} t_{ik}}{\sum_{i=1}^{m}\sum_{k=1}^{m} t_{ik}}. \quad (7)$$

For the NF-AHP:

Step 1: Find the neutrosophic rating scale using interval values.

Step 2: Divide the issue into a hierarchy of objectives, criteria, sub-criteria, and alternatives.

Step 3: Use interval-valued neutrosophic sets to build the pairwise comparison matrices ($\tilde{P}$). The deneutrosophication equation has been used to assess the consistency of the pairwise comparison matrices:

$$D(x) = \left(\frac{T_x^L + T_x^U}{2} + \left(1 - \frac{I_x^L + I_x^U}{2}\right) * I_x^U - \left(\frac{F_x^L + F_x^U}{2}\right) * (1 - F_x^U)\right) \quad (8)$$

for a collection of IV-Neutrosophic number $\tilde{x}_j = ([T_x^L, T_x^U], [I_x^L, I_x^U], [F_x^L, F_x^U])$, $(j = 1, 2, \cdots, n)$. It follows that if the neutrosophic pairwise comparison matrix is consistent, so is the deneutrosophicated pairwise comparison matrix. There will be provided pairwise comparison matrices for the criteria and options in relation to the aim.

Step 4: Utilize the proposed interval-valued neutrosophic evaluation scale to determine the normalized weights of the various criteria.

Step 4.1: Sum the values in each column as in

$$\tilde{S_{ij}} = \left(\left[\sum_{k=1}^{m} T_{k_j}^{L}, \sum_{k=1}^{m} T_{k_j}^{U}\right], \left[\sum_{k=1}^{m} I_{k_j}^{L}, \sum_{k=1}^{m} I_{k_j}^{U}\right], \left[\sum_{k=1}^{m} F_{k_j}^{L}, \sum_{k=1}^{m} F_{k_j}^{U}\right]\right). \tag{9}$$

Step 4.2: In Equation 9, choose the highest limit for each parameter. Then, divide each term by the appropriate element to get $\tilde{N_{ij}}$:

$$\tilde{N_{kj}} = \left(\left[\frac{T_{k_j}^{L}}{\sum_{k=1}^{m} T_{k_j}^{U}}, \frac{T_{k_j}^{U}}{\sum_{k=1}^{m} T_{k_j}^{U}}\right], \left[\frac{I_{k_j}^{L}}{\sum_{k=1}^{m} I_{k_j}^{U}}, \frac{I_{k_j}^{U}}{\sum_{k=1}^{m} I_{k_j}^{U}}\right], \left[\frac{F_{k_j}^{L}}{\sum_{k=1}^{m} F_{k_j}^{U}}, \frac{F_{k_j}^{U}}{\sum_{k=1}^{m} F_{k_j}^{U}}\right]\right). \tag{10}$$

Step 4.3: To obtain the neutrosophic priority vector of the choices as in Equation 11, compute the average of each row.

$$\tilde{\omega_A} = \left(\frac{\left[\sum_{k=1}^{m} \frac{T_{k_j}^{L}}{\sum_{k=1}^{m} T_{k_j}^{U}}, \sum_{k=1}^{m} \frac{T_{k_j}^{U}}{\sum_{k=1}^{m} T_{k_j}^{U}}\right]}{m}, \frac{\left[\sum_{k=1}^{m} \frac{I_{k_j}^{L}}{\sum_{k=1}^{m} I_{k_j}^{U}}, \sum_{k=1}^{m} \frac{I_{k_j}^{U}}{\sum_{k=1}^{m} I_{k_j}^{U}}\right]}{m} \frac{\left[\sum_{k=1}^{m} \frac{F_{k_j}^{L}}{\sum_{k=1}^{m} F_{k_j}^{U}}, \sum_{k=1}^{m} \frac{F_{k_j}^{U}}{\sum_{k=1}^{m} F_{k_j}^{U}}\right]}{m}\right) \tag{11}$$

Step 4.4: To produce neutrosophic weights vectors for each choice, repeat the previous stages with regard to each criterion. To get the priority weights of the criterion, the same procedure is done.

Step 5: Build the $\tilde{\Psi}$ matrx whose rows are the weights of the alternatives $(\tilde{\omega_{A_i}})$ and the columns are the weights of the criteria $(\tilde{\omega_{C_j}})$ to get the final combined priority weights.

Step 6: Equation 12 can be used to determine the final combined interval-valued neutrosophic weights of the alternatives.

$$\begin{aligned}\tilde{\Psi_A} &= \left(\left[T_{\omega_{C1}^L}, T_{\omega_{C1}^U}\right], \left[I_{\omega_{C1}^L}, I_{\omega_{C1}^U}\right], \left[F_{\omega_{C1}^L}, F_{\omega_{C1}^U}\right]\right)\left(\left[T_{\omega_{A1}^L}, T_{\omega_{A1}^U}\right], \left[I_{\omega_{A1}^L}, I_{\omega_{A1}^U}\right], \left[F_{\omega_{A1}^L}, F_{\omega_{A1}^U}\right]\right) \\ &+ \left(\left[T_{\omega_{C2}^L}, T_{\omega_{C2}^U}\right], \left[I_{\omega_{C2}^L}, I_{\omega_{C2}^U}\right], \left[F_{\omega_{C2}^L}, F_{\omega_{C2}^U}\right]\right)\left(\left[T_{\omega_{A2}^L}, T_{\omega_{A2}^U}\right], \left[I_{\omega_{A2}^L}, I_{\omega_{A2}^U}\right], \left[F_{\omega_{A2}^L}, F_{\omega_{A2}^U}\right]\right) \\ &+ \cdots + \left(\left[T_{\omega_{Cn}^L}, T_{\omega_{Cn}^U}\right], \left[I_{\omega_{Cn}^L}, I_{\omega_{Cn}^U}\right], \left[F_{\omega_{Cn}^L}, F_{\omega_{Cn}^U}\right]\right)\left(\left[T_{\omega_{An}^L}, T_{\omega_{An}^U}\right], \left[I_{\omega_{An}^L}, I_{\omega_{An}^U}\right], \left[F_{\omega_{An}^L}, F_{\omega_{An}^U}\right]\right).\end{aligned} \tag{12}$$

Step 7: To get the crisp weights of alternatives, use Equation 8's deneutrosophication formula.

Step 8: Normalize the crisp weights of alternatives.

Step 9: Choose the option with the most weight after ranking the alternatives.

3 Childhood Cancer Risk Assessment

3.1 Major Factors of Childhood Cancer

Many studies have tried to identify the causes of childhood cancer. Some factors are related to the environment, such as radiation exposure and chemical exposure. Some are lifestyle-related, such as drugs, alcohol, cell phone use, and smoking. Some kids receive DNA alterations from a parent that raise their risk of developing a particular kind of cancer. Here we list possible risk factors for childhood cancer with a small description of each factor.

Gender (S1): Gender can be male or female. Age (S2): The age of a child is considered between 0 and 19 years.Height (S3): The height of a child. BMI (S4): The body mass index (BMI) is a measure of body fat according to height and weight. Drugs (S5): A medication is a drug used to diagnose, cure, treat, or prevent disease. Alcohal (S6): It is a substance that contains the recreational drug ethanol, alcohol is made by fermentation of fruits, grains, or any source of sugar. Cell Phone Usage (S7): The use of cell phones on a daily basis. Pagets Disease (S8): This bone condition prevents the body's regular recycling process, which sees new bone tissue progressively replace old bone tissue. Compromised bones may grow weak and deformed over time as a result of the disease. Genetic Disposition (S9): There is an increased chance of acquiring a specific disease based on a person's ancestral genes. Smoking (S10): The habit of inhaling and exhaling tobacco or drug smoke. Blood Disorder (S11): These are conditions that affect the blood's ability to function. Birth Defects (S12): It is a disease that, despite its cause, is present at birth. Birth defects can appear as disabilities that can be physical, mental, or developmental in nature. Immunity (S13): Immunity is the capability ofmulti-cellular organisms to resist harmful microorganisms. Auto Immune Diseases (S14): It is a disease in which your immune system unintentionally attacks your body. Certain Syndromes(S15): Any syndrome already present in children such as Down syndrome, Li-Fraumeni syndrome, etc. Race (S16): Identification of a group of people. Certain Radiation Exposure (S17): Exposed to certain electromagnetic radiation, or living in the vicinity of a source of electromagnetic radiation. Certain Chemical Exposure (S18): Exposure to certain chemicals or polluted groundwater used for drinking. Socioeconomic Status (S19): A family's financial status in society.

3.2 Types of Childhood Cancers

Children and teenagers tend to get different types of childhood cancers. The most common childhood cancers are discussed below:
Leukemia (D1): It is bone marrow and blood cancer. Twenty-eight percent of childhood cancer cases fall into this category. Brain and spinal cord tumors (D2): The second most common cancer in children is the brain and spinal cord cancer. In this type of cancer, abnormal growth in tissues of the brain and spinal cord is seen causing headache, nausea, vomiting, blurred vision, and difficulty in walking and holding objects.

About 26 children develop this type of cancer every year. Neuroblastoma (D3): Neuroblastoma begins in the early forms of nerve cells seen in a developing egg or fetus. About 6 percent of cancers in adolescents are neuroblastomas. This type of cancer occurs in newborns and adolescents. It is uncommon in children over 10 years of age. Neuroblastomas mostly occur in and around the adrenal glands. However, neuroblastomas can develop in other areas of the stomach and ribs, neck, and near the spine where there are clusters of nerve cells. Wilms Tumor (D4): Wilms' tumor begins in one or, rarely, both kidneys. It is usually found in children around 3 to 4 years of age and is rare in more mature children and adults. Wilms' tumor accounts for around 5 percent of childhood cancers. Its symptoms are fever, pain, nausea, or loss of appetite. Lymphomas (D5): It is a disease that attacks infection-fighting cells in the immune system. These cells are called lymphocytes. These cells are found in the lymph nodes, spleen, thymus gland, bone marrow, and other parts of the body. In this disease, abnormal growth of lymphocytes has been observed. Symptoms include weight loss, fever, sweats, fatigue, and lumps under the skin in the neck, armpits, or groin area. Retinoblastoma (D6): This type of cancer is related to the eyes. It is a rare type of cancer in which a child could not distinguish the colors of light, also had impaired vision and sensitive eyes. The pupil of the eyes becomes large. Rhabdomyosarcoma (D7): It is an intrusive and very dangerous cancer that originates from skeletal muscle cells. It is widely believed to be a childhood disease as the vast majority of cases found are under the age of 18. It is about 3 percent of childhood cancers. Bone Cancer (D8): This type of cancer usually occurs in older children. This type of cancer causes severe bone pain all the time. The bones become weak and can also be broken. In some cases, weight loss is also observed.

3.3 New Method

The new method involves the FF-AHP and NF-AHP methods. First, the FF-AHP steps are given:

Step 1: Establish the criteria and options before building the hierarchical structure.
Step 2: Consult the partners to organize these risks into a hierarchy, then convert the issue into a hierarchy of objectives and standards.
Step 3: Create binary comparison matrices for the criterion based on Table 1's range-valued sets.
Step 4: Applying the suggested interval valuation scale, determine the normalized criteria weights:
Step 4.1: The matrix's values are gathered for each of its columns.
Step 4.2: Each parameter is divided by the highest value chosen after choosing the maximum values for each parameter.
Step 4.3: In order to determine the priority vectors, compute the average of each sequence.
Step 4.4: Each criterion is subjected to the same procedures as before, and weight vectors are produced for each. To get the priority weights for each criterion, these steps were repeated.
Step 5: Utilizing the prioritized weights that were collected, compute the cosine similarity between each alternative pair.
Step 6: The appropriate AHP score is determined using the linear regression algorithm.
Step 7: Based on the conventional AHP procedures, alternative weights were obtained.
Step 8: The probability and severity criteria of the alternatives are graded in accordance with alternative weights to enable the use of the L matrix approach.
Step 9: Utilize the grades you have earned to apply the L matrix approach.

Second, the NF-AHP steps are given:

Step 1: Define the neutrosophic numbers, which are utilized in the proposed neutrosophic fuzzy AHP approach to compare various criteria and correlate to the 1–9 Saaty scale.
Step 2: Determine the decision-making problem's criteria, sub-criteria, and options. Next, create a hierarchy for the problem under consideration.
Step 3: By comparing each criterion and sub-criterion pairwise, you may determine the neutrosophic preference. Compare the options under each criterion or sub-criterion that are provided in accordance with the assessments of the experts based on Table 2.
Step 4: Construct the neutrosophic preference connection and ensure that each paired comparison is consistent. Utilizing the prioritized weights that were collected, compute the cosine similarity between each alternative pair.
Step 5: Give the results of the calculation for each preference relation's neutrosophic relative weight.
Step 6: Rank the overall weights.

3.4 Evaluations

The primary criteria, sub-criteria, and alternatives will be included in the hierarchical structure that will be established about the risks of children cancer. Three doctors will assess these options and criteria for this study by building paired comparison matrices using linguistic concepts. A consistency ratio will be supplied along with pairwise comparison matrices that include linguistic phrases for the major criteria, sub-criteria, and alternatives. The steps of the created method will be presented but not shown due to space restrictions. According to the results obtained, the rating of the disease was found as $D_1 = 1$, $D_2 = 6$, $D_3 = 5$, $D_4 = 7$, $D_5 = 2$, $D_6 = 8$, $D_7 = 4$, $D_8 = 3$.

4 Conclusion

According to the World Health Organization (WHO), around 400,000 children are diagnosed with cancer each year and the rate of cure in low and middle-income countries is only 45 percent, which is highly unsatisfactory. To improve this percentage, WHO has launched a global initiative and provided appropriate professional guidance and resources. Their goal is to increase the survival rate up to sixty percent by the end of 2030. To help achieve this goal, we have proposed a novel model that allows doctors to diagnose the type of childhood cancer early so that appropriate treatment can be given at the right time. This ultimately reduces the physical and financial suffering of the patient and their parents. Our model takes nineteen symptoms as inputs and determines the type of cancer. We have used Fermatean fuzzy and Neutrosophic fuzzy decision-making techniques for diagnostic purposes.

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Risk Assessment of Autonomous Vehicles in terms of Public Health

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Abstract: To minimize the negative repercussions of Autonomous Vehicle installation and optimize its benefits, risk assessment is essential. Risk factors associated with transportation that affect public health will be presented first in this study, which will focus on the favorable and negative effects of Autonomous Vehicles on public health. When uncertainty in hazard and risk assessment for transport-related exposures and risk factors that may affect public health is encountered, a new approach will be developed to overcome these issues. Examining the health consequences of Autonomous Vehicles, identifying hazards, and attempting to decrease these risks will aid in the development of strategies that benefit overall public health.

Keywords: AHP, autonomous vehicles, MABAC, public health, risk assessment, TOPSIS.

1 Introduction

In 2000, roughly 47 percent of the population was predicted to reside in cities, with that percentage expected to rise to 60 percent by 2030. As the global urbanization process proceeds, urban health is becoming a growing public concern. The aim of public health is the promotion and control of a population's general health within a region. Modern urban planning is becoming increasingly intertwined with public health problems. In the early twentieth century, urban planners separated land use and development laws (e.g., public gardens and pollution) to help improve public health. However, this segregation led in sprawling cities, which harmed population health [10]. The goal of public health is to promote and safeguard the health of all individuals in all communities. This science-based, evidence-based field attempts to provide a safe environment for everyone to live, learn, work, and play. The health-care business focuses on treating people who are sick, whereas public health focuses on preventing people from becoming sick or wounded in the first place. Public health is also concerned with entire populations, whereas health care is concerned with individual patients.

AVs powered by artificial intelligence were long thought to be a pipe dream for mainstream deployment. Recent advancements in AI technology and early testing have made widespread adoption much more possible. The growing pervasiveness of AVs heralds a new era for transportation networks. As earlier technological improvements in transportation have shown, they may suppress alternative types of movement, particularly cycling. As a result, there would be fewer transportation options. A reduction in transportation options opposes social goals such as addressing climate change, public health, and traffic congestion through cycling [3].

Since the introduction of AVs, several advantages and disadvantages have been projected. large potential benefits include reduced traffic-related death and injury, improved pedestrian and bicycle safety, large reductions in emissions, increased mobility for the elderly and disabled, and the liberation of parking spaces for other land uses. Among the potential drawbacks are increased traffic congestion (due to increased travel overall due to improved availability and lower costs, as well as empty cars traveling the roads when collecting their owners or returning from trips), privacy, security, insurance, and liability concerns, and job losses. The amount to which these positive and negative outcomes occur will be heavily influenced by the most widely utilized AV applications [11]. Transportation-related technological improvements, for example, have the potential to dramatically improve people's health and well-being. Vehicle automation is fast advancing, with extensive experiments involving both personal and commercial vehicles taking place all over the world. Despite the rapid advancement of AV technology, most jurisdictions' road rules have yet to be modified to allow the use of fully autonomous vehicles, and governments have yet to develop comprehensive policy approaches to AVs in order to capitalize on the health benefits offered by this technology while also applying appropriate safety standards. AVs are expected to significantly improve a range of health-related areas. Crash prevention, pollution reduction, improved mobility (and hence quality of life) for people who cannot drive, stress reduction, and increased bicycle safety are all examples. Despite the fact that these are serious health issues that contribute to disease burden and demand greater preventative efforts, there is little recognition among public health professionals of the role that AVs can play in their amelioration. Recent initiatives to draw attention to the need for the health sector to realize the potential for AVs to make major contributions to public health and to build effective methods to manage the adoption process to optimize these results are instances of exceptions. The public health benefits of AVs will be realized when all vehicles are automated, leading to calls for governments to develop strategies to promote rapid adoption. Because public support is an important factor in governments' decisions to implement new policies, the relative newness of AVs and a general lack of understanding among the general public about the extent of their health-enhancing potential may pose significant barriers to governments and businesses proactively developing policies and programs to accelerate adoption [12].

Many decision-makers and professionals are curious about how AVs will affect future travel and, as a result, demand for public transportation, parking lots, and roadways. They also want to know what public policies might be put in place to mitigate these issues while maximizing the benefits of this new technology. Optimists predict that by 2030, AVs will have replaced the majority of driving, resulting in significant cost and benefit savings. However, there are compelling reasons to remain cautious. The development of AVs, their benefits and drawbacks, the effects they will have on travel, and customer demand are all fraught with uncertainty.

Any decision-making(DM) process must account for imprecision. To deal with the ambiguous environment of collective DM, many tools and strategies have been proposed. Fermatean fuzzy sets (FFS) [13] are one of the newest techniques for coping with uncertainty. Compared to the intuitionistic fuzzy sets(IFS) [2] and Pythagorean fuzzy sets(PFS) [17], [18], which are extensions of Zadeh's fuzzy set [19], these sets offer a larger range of applications. Recently, FFs have inspired many studies.([1], [4], [5], [7], [8], [9], [14], [15]).

The goal of this research is to compile a list of the public health risks posed by AVs. Prioritizing risks is a multi-criteria DM (MCDM) challenge that necessitates taking into account a variety of potential solutions as well as competing tangible and intangible elements. An integrated MCDM technique over FFSs is proposed to address this MCDM difficulty. This proposed method addresses the prioritization of AV-related public health problems in a Fermatean turbid environment by introducing unique integrated MCDM methodologies based on AHP, TOPSIS, and MABAC.

2 Preliminaries

Let E be a universal set. The set $F = \{(e, m_F(e), n_F(e)) : e \in E\}$ is called the FFS with $0 \le m_F^3 + n_F^3 \le 1$ and $m_F, n_F \in [0,1]$. The hesitation degree has been shown with $\theta_F = (1 - m_F^3 + n_F^3)^{1/3}$ [13].

Let $Int[0,1]$ show the set of all closed subintervals of $[0,1]$. The set $F = \{(e, m_F(e), n_F(e)) : e \in E\}$ is called an IVFFS on a set $E \neq \emptyset$, where $m_F(e), n_F(e) \in Int[0,1]$ with the condition $0 < \sup_e (m_F(e))^3 + \sup_e (n_F(e))^3 \le 1$ [5].

Furthermore, F can be written as: $F = \{(e, [m_{F_L}(e), m_{F_U}(e)], [n_{F_L}(e), n_{F_U}(e)]) : e \in E\}$ with $0 \le (m_{F_U}(e))^3 + (n_{F_U}(e))^3 \le 1$.

Choose the three IVFFSs $F = ([m_{F_L}(e), m_{F_U}(e)], [n_{F_L}(e), n_{F_U}(e)])$, $F_1 = ([m_{F_{1L}}(e), m_{F_{1U}}(e)], [n_{F_{1L}}(e), n_{F_{1U}}(e)])$, $F_2 = ([m_{F_{2L}}(e), m_{F_{2U}}(e)], [n_{F_{2L}}(e), n_{F_{2U}}(e)])$. Then [5],

- $F_1 \cup F_2 = ([\max(m_{F_{1L}}, m_{F_{2L}}), \max(m_{F_{1U}}, m_{F_{2U}})], [\min(m_{F_{1L}}, m_{F_{2L}}), \min(m_{F_{1U}}, m_{F_{2U}})])$
- $F_1 \cap F_2 = ([\min(m_{F_{1L}}, m_{F_{2L}}), \min(m_{F_{1U}}, m_{F_{2U}})], [\max(m_{F_{1L}}, m_{F_{2L}}), \max(m_{F_{1U}}, m_{F_{2U}})])$
- $F^c = ([n_{F_L}, n_{F_U}], [m_{F_L}, m_{F_U}])$
- $F_1 \oplus F_2 = \left(\left[\sqrt[3]{(m_{F_{1L}}(e))^3 + (m_{F_{2L}}(e))^3 - (m_{F_{1L}}(e))^3.(m_{F_{2L}}(e))^3}, \sqrt[3]{(m_{F_{1U}}(e))^3 + (m_{F_{2U}}(e))^3 - (m_{F_{1U}}(e))^3.(m_{F_{2U}}(e))^3} \right], [n_{F_{1L}} n_{F_{2L}}, n_{F_{1U}} n_{F_{2U}}] \right)$
- $F_1 \otimes F_2 = \left([m_{F_{1L}} m_{F_{2L}}, m_{F_{1U}} m_{F_{2U}}], \left[\sqrt[3]{(n_{F_{1L}}(e))^3 + (n_{F_{2L}}(e))^3 - (n_{F_{1L}}(e))^3.(n_{F_{2L}}(e))^3}, \sqrt[3]{(n_{F_{1U}}(e))^3 + (n_{F_{2U}}(e))^3 - (n_{F_{1U}}(e))^3.(n_{F_{2U}}(e))^3} \right] \right)$
- $\lambda F = \left(\left[\sqrt[3]{1 - \left(1 - m_{F_L}^3\right)^{\lambda}}, \sqrt[3]{1 - \left(1 - m_{F_U}^3\right)^{\lambda}} \right], \left[n_{F_L}^{\lambda}, n_{F_U}^{\lambda} \right] \right)$
- $F^{\lambda} = \left(\left[m_{F_L}^{\lambda}, m_{F_U}^{\lambda} \right], \left[\sqrt[3]{1 - \left(1 - n_{F_L}^3\right)^{\lambda}}, \sqrt[3]{1 - \left(1 - n_{F_U}^3\right)^{\lambda}} \right] \right)$

Let $F = ([m_{F_L}(e), m_{F_U}(e)], [n_{F_L}(e), n_{F_U}(e)])$, $F_1 = ([m_{F_{1L}}(e), m_{F_{1U}}(e)], [n_{F_{1L}}(e), n_{F_{1U}}(e)])$, and $F_2 = ([m_{F_{2L}}(e), m_{F_{2U}}(e)], [n_{F_{2L}}(e), n_{F_{2U}}(e)])$ be three IVFFSs. Then, for $\lambda, \lambda_1, \lambda_2 > 0$ [5],

- $F_1 \oplus F_2 = F_2 \oplus F_1$
- $F_1 \otimes F_2 = F_2 \otimes F_1$
- $\lambda(F_1 \oplus F_2) = \lambda F_1 \oplus \lambda F_2$
- $(\lambda_1 + \lambda_2)F = \lambda_1 F + \lambda_2 F$
- $(F_1 \otimes F_2)^{\lambda} = F_1^{\lambda} \otimes F_2^{\lambda}$
- $F^{\lambda_1} \otimes F^{\lambda_2} = F^{\lambda_1 + \lambda_2}$

3 Method

The equation $CRT = \frac{CIX}{RIX}$ is called the consistency ratio, where $CIX = \frac{\lambda_{max}}{n-1}$, RIX is the consistency index and λ_{max} is the random index, and principal eigenvalue for CRT, respectively.

The IVFFWG aggregation operation is used to combine the pairwise comparison matrix that each professional represents. Let $U_k = \{U_1, U_2, \cdots, U_k\}$, $(k = 1, 2, \cdots, K)$, show the set of professional having influence weights ω_k for every E_k; $\sum_{k=1}^{K} \omega_k = 1$.

Table 1 Scale Values according to IVFF

Linguistic Terms	ζ_L	ζ_U	η_L	η_U
Certainly High Importance($\mathfrak{CH}$)	0.95	1	0	0
Very High Importance($\mathfrak{VH}$)	0.8	0.9	0.1	0.2
High Importance($\mathfrak{H}$)	0.7	0.8	0.2	0.3
Slightly More Importance($\mathfrak{SM}$)	0.6	0.65	0.35	0.4
Equally Importance($\mathfrak{E}$)	0.5	0.5	0.5	0.5
Slightly Less Importance($\mathfrak{SL}$)	0.35	0.4	0.6	0.65
Low Importance($\mathfrak{L}$)	0.2	0.3	0.7	0.8
Very Low Importance($\mathfrak{VL}$)	0.1	0.2	0.8	0.9
Certainly Low Importance($\mathfrak{CL}$)	0	0	0.95	1

$$IVFFWG(c_1, c_2, \cdots, c_k) = \left(\left[\prod_{k=1}^{K} \left(\zeta_k^L\right)^{\omega_k}, \prod_{k=1}^{K} \left(\zeta_k^U\right)^{\omega_k} \right], \left[\sqrt[3]{\left(1 - \prod_{k=1}^{K} (1 - (\zeta_k^L)^3)^{\omega_k}\right)}, \sqrt[3]{\left(1 - \prod_{k=1}^{K} (1 - (\zeta_k^U)^3)^{\omega_k}\right)} \right] \right) \quad (1)$$

The difference matrix $F = (f_{ij})_{m\times m}$ between the upper and lower points of the MF and NF:

$$f_{ik_L} = \zeta^3_{ik_L} - \eta^3_{ik_U} \quad (2)$$
$$f_{ik_U} = \zeta^3_{ik_U} - \eta^3_{ik_L}. \quad (3)$$

The interval multiplicative matrix $M = (m_{ij})_{m\times m}$:

$$m_{ik_L} = \sqrt[3]{1000^{f_L}} \quad (4)$$
$$m_{ik_U} = \sqrt[3]{1000^{f_U}}. \quad (5)$$

The indeterminacy value $T = (t_{ij})_{m\times m}$ of the c_{ij}:

$$t_{ij} = 1 - \left(\zeta^3_{ij_U} - \zeta^3_{ij_L}\right) - \left(\eta^3_{ij_U} - \eta^3_{ij_L}\right). \quad (6)$$

Multiply the indeterminacy degrees with $S = (s_{ij})_{m\times m}$ matrix to obtain the matrix of unnormalized weights $R = (r_{ij})_{m\times m}$:

$$r_{ij} = \left(\frac{m_{ik_L} + m_{ik_U}}{2}\right) t_{ij}. \quad (7)$$

The normalized priority weights ω_i:

$$\omega_i = \frac{\sum_{j=1}^{m} r_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{m} r_{ij}} \quad (8)$$

PIS and NIS are determined as

$$P_1 = \left(\left[\max\left(\zeta'_{L11}, \cdots, \zeta'_{Lm1}\right), \max\left(\zeta'_{U11}, \cdots, \zeta'_{Um1}\right)\right], \left[\min\left(\eta'_{L11}, \cdots, \eta'_{Lm1}\right), \min\left(\eta'_{U11}, \cdots, \eta'_{Um1}\right)\right] \right) \quad (9)$$
$$\vdots$$
$$P_n = \left(\left[\max\left(\zeta'_{L1n}, \cdots, \zeta'_{Lmn}\right), \max\left(\zeta'_{U1n}, \cdots, \zeta'_{Umn}\right)\right], \left[\min\left(\eta'_{L1n}, \cdots, \eta'_{Lmn}\right), \min\left(\eta'_{U1n}, \cdots, \eta'_{Umn}\right)\right] \right)$$

$$N_1 = \left(\left[\min\left(\zeta'_{L11}, \cdots, \zeta'_{Lm1}\right), \min\left(\zeta'_{U11}, \cdots, \zeta'_{Um1}\right)\right], \left[\max\left(\eta'_{L11}, \cdots, \eta'_{Lm1}\right), \max\left(\eta'_{U11}, \cdots, \eta'_{Um1}\right)\right] \right) \quad (10)$$
$$\vdots$$
$$N_n = \left(\left[\min\left(\zeta'_{L1n}, \cdots, \zeta'_{Lmn}\right), \min\left(\zeta'_{U1n}, \cdots, \zeta'_{Umn}\right)\right], \left[\max\left(\eta'_{L1n}, \cdots, \eta'_{Lmn}\right), \max\left(\eta'_{U1n}, \cdots, \eta'_{Umn}\right)\right] \right)$$

The Euclidean distance for IVFFSs and the distances between alternative F_i and PIS, NIS are defined as

$$D_{IVFF}(F_1,F_2) = \left\{ \frac{\left(\zeta^3_{F_{1L}} - \zeta^3_{F_{2L}}\right)^2 + \left(\zeta^3_{F_{1U}} - \zeta^3_{F_{2U}}\right)^2 + \left(\eta^3_{F_{1L}} - \eta^3_{F_{2L}}\right)^2 + \left(\eta^3_{F_{1U}} - \eta^3_{F_{2U}}\right)^2)}{6} \right. \tag{11}$$
$$+ \left. \frac{\left(+(1-\zeta^3_{F_{1U}} - \eta^3_{F_{1U}} - (1-\zeta^3_{F_{2U}} - \eta^3_{F_{2U}})\right)^2 + \left((1-\zeta^3_{F_{1L}} - \eta^3_{F_{1L}}) - (1-\zeta^3_{F_{2L}} - \eta^3_{F_{2L}})\right)^2}{6} \right\}^{1/2}$$

$$d_i^+(F_i,P_n) = \sum_{j=1}^{n} \left(D_{IVFF}(c'_{ij},P_j) \right), \tag{12}$$
$$d_i^-(F_i,N_n) = \sum_{j=1}^{n} \left(D_{IVFF}(c'_{ij},N_j) \right) \tag{13}$$

Relative closeness is specified as

$$\xi(F_i) = \frac{d_i^-(F_i,N_n)}{d_i^+(F_i,P_n) + d_i^-(F_i,N_n)}. \tag{14}$$

For the pairwise comparison matrix $C=(c_{ij})_{m\times n}$, the normalized matrix of the decision matrix is constructed as:

$$n_{ij} = \begin{cases} \frac{c_{ij}-c_i^-}{c_i^+-c_i^-} & , \quad \textit{for beneficial criteria} \\ \frac{c_{ij}-c_i^+}{c_i^--c_i^+} & , \quad \textit{for non-beneficial criteria} \end{cases} \tag{15}$$

where $c_i^+ = \max(c_{i1}, c_{i2}, \cdots, c_{im})$ and $c_i^+ = \min(c_{i1}, c_{i2}, \cdots, c_{im})$.

The weighted normalized of the decision matrix $V=[v_{ij}]_{m\times n}$ is computed as:

$$v_{ij} = \omega_j(n_{ij}+1) \tag{16}$$

The BAA matrix is determined as:

$$G=[g_j]_{1\times n} = [g_1,g_2,\cdots,g_n] \tag{17}$$
$$g_j = \left(\prod_{i=1}^{m} v_{ij}\right)^{1/m} \tag{18}$$

Calculate the distance from the BAA as:

$$D = V - G = [d_{ij}]_{m\times n} \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mm} \end{bmatrix} = \begin{bmatrix} v_{11}-g_1 & v_{12}-g_2 & \cdots & v_{1n}-g_n \\ v_{21}-g_1 & v_{22}-g_2 & \cdots & v_{2n}-g_n \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}-g_1 & v_{m2}-g_2 & \cdots & v_{mn}-g_n \end{bmatrix} \tag{19}$$

The total distance from the BAA is determined as:

$$S_i = \sum_{j=1}^{n} d_{ij}. \tag{20}$$

3.1 Algorithms

IVFF-AHP Algorithm:
1. Determine the criteria and options before constructing the hierarchical structure.
2. Using Table 1, create the pairwise comparison matrix.
3. Examine each pairwise comparison matrix for consistency.
4. Add together professional opinions.
5. Calculate the difference matrix using the Equations (2), (3).

6. Compute the multiplicative matrix using the Equations (4), (5).
7. Obtain the indeterminacy value of the c_{ij} using the Equation (6).
8. Obtain the matrix of un-normalized weights using the Equation (7).
9. Determine the normalized priority weights using the Equation (8).

Output: Normalized priority weights.

IVFF-TOPSIS Algorithm:

1. Build the decision matrix using the Table 1.
2. To generate the aggregated IVFF decision matrix, add the decision matrices together.
3. Normalize the decision matrix that has been aggregated.
4. Determine the weighted decision matrix $C^{'} = (c_{ij}^{'})$ with $c_{ij}^{'} = c_{ij}\omega_j$.
5. Compute the PIS and NIS using the Equations 9, 10
6. Compute the distances between an alternative with PIS and NIS using the Euclidean distance equation 11and the Equations 12, 13.
7. Calculate the relative closeness using the Equation 14.

Output: Relative closeness of an alternative.

IVFF-MABAC Algorithm:

1. Build the decision matrix according to Table 1.
2. Determine the normalized matrix of the decision-matrix using the Equation 15.
3. Compute the weighted normalized of the decision matrix using the Equation 16.
4. Calculate the BAA matrix using the Equations 17, 18.
5. Calculate the distance from the BAA using the Equation 19.
6. Identify the total distance from the BAA using 20.

Output: Comprehensive evaluation result.

3.2 Problem and Computations

Risk analysis purposes to characterize the risks that impact the public health of AVs. Criteria and sub-criteria for this study were selected from the Detrimental to Health section of the Transportation and Health Conceptual Model in the study of Khreis et al. [6] and from Table 1 in the study of Sohrabi et al. [16].

A_1 - Losing transportation-related job; A_{11} - Social Exclusion
A_2 - Transportation Equity; A_{21} - Community severance, A_{22} - Contamination, A_{23} - Greenhouse gases
A_3 - Land Use and Built Environment; A_{31} - Heat, A_{32} - Noise, A_{33} - Air Pollution
A_4 - Traffic Flow; A_{41} - Stress
A_5 - Trip, mode, and route choice; A_{51} - Physical Inactivate, A_{52} - Electromagnetic Field
A_6 - Traffic safety; A_{61} - Motor vehicle crashes

In the first stage, the weights of the criterion must be computed using the IVFF-AHP approach while taking into account fuzzy linguistic variables and pairwise comparisons. The IVFFS can tolerate severe fuzziness, ambiguity, and imprecision throughout the DM process. In addition, an FFS is selected to assess the risks associated with AVs using TOPSIS and MABAC. The primary objective of using an FFS is to decrease computation complexity and calculation execution time while improving how AV dangers are ranked in hybrid MCDM approaches.

Table 2 contains the rating scales, which three experts are asked to use to evaluate their pairwise judgments of the dangers. The pairwise comparison matrices consisting of linguistic terms for the main criteria, sub-criteria, and alternatives are computed. The pairwise comparison matrix's expert ratings are evaluated using the consistency check to see if they are fair. The CRTs of each matrix are calculated. Due to space constraints, sub-criteria tables are not provided. The IVFFSs for the primary criteria that correlate to the linguistic words in Table 1 are denoted in Table 3. Using the Equations 2 and 3, the matrix F is obtained 4. Equations 2 and 3 are then employed to compute the difference matrix D of the primary criterion between the higher and lower values of the MF and NF, which is denoted in Table 4. Equations 4 and 5 are employed to build the interval multiplicative matrix in Table 5. The weights before normalization are displayed in Table 6 and were calculated using Equation 7. The results of all these computations were also implemented to the sub-criteria, and Table 7 provides the final priority weights for both the primary and secondary criteria. The findings show that, with a weight of 0.338, the information security requirements are the most crucial. Nonetheless, with a weight of 0.04, the criteria for social development are the least significant.

Table 2 Pairwise comparison matrix of main criteria

	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$\mathfrak{E}$	$\mathfrak{SM}$	$\mathfrak{SM}$	$\mathfrak{VH}$	$\mathfrak{H}$	$\mathfrak{H}$
A_2	$\mathfrak{SL}$	$\mathfrak{E}$	$\mathfrak{SM}$	$\mathfrak{VH}$	$\mathfrak{SM}$	$\mathfrak{H}$
A_3	$\mathfrak{SL}$	$\mathfrak{SL}$	$\mathfrak{E}$	$\mathfrak{H}$	$\mathfrak{SM}$	$\mathfrak{SM}$
A_4	$\mathfrak{VL}$	$\mathfrak{VL}$	$\mathfrak{L}$	$\mathfrak{E}$	$\mathfrak{L}$	$\mathfrak{SM}$
A_5	$\mathfrak{L}$	$\mathfrak{SL}$	$\mathfrak{SM}$	$\mathfrak{H}$	$\mathfrak{E}$	$\mathfrak{H}$
A_6	$\mathfrak{L}$	$\mathfrak{L}$	$\mathfrak{SL}$	$\mathfrak{SL}$	$\mathfrak{L}$	$\mathfrak{E}$

Table 3 IVFF values for main criteria

	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$([0.5, 0.5], [0.5, 0.5])$	$([0.5, 0.65], [0.35, 0.40])$	$([0.5, 0.65], [0.35, 0.4])$	$([0.8, 0.9], [0.1, 0.2])$	$([0.65, 0.8], [0.2, 0.35])$	$([0.65, 0.8], [0.2, 0.35])$
A_2	$([0.35, 0.4], [0.5, 0.65])$	$([0.5, 0.5], [0.5, 0.5])$	$([0.5, 0.65], [0.35, 0.4])$	$([0.8, 0.9], [0.1, 0.2])$	$([0.5, 0.65], [0.35, 0.4])$	$([0.65, 0.8], [0.2, 0.35])$
A_3	$([0.35, 0.4], [0.5, 0.65])$	$([0.35, 0.4], [0.5, 0.65])$	$([0.5, 0.5], [0.5, 0.5])$	$([0.65, 0.8], [0.2, 0.35])$	$([0.4, 0.5], [0.4, 0.5])$	$([0.5, 0.65], [0.35, 0.4])$
A_4	$([0.1, 0.2], [0.8, 0.9])$	$([0.1, 0.2], [0.8, 0.9])$	$([0.2, 0.35], [0.65, 0.8])$	$([0.5, 0.5], [0.5, 0.5])$	$([0.2, 0.35], [0.65, 0.8])$	$([0.5, 0.65], [0.35, 0.4])$
A_5	$([0.2, 0.35], [0.65, 0.8])$	$([0.35, 0.4], [0.5, 0.65])$	$([0.4, 0.5], [0.4, 0.5])$	$([0.65, 0.8], [0.2, 0.35])$	$([0.5, 0.5], [0.5, 0.5])$	$([0.65, 0.8], [0.2, 0.35])$
A_6	$([0.2, 0.35], [0.65, 0.8])$	$([0.2, 0.35], [0.65, 0.8])$	$([0.35, 0.4], [0.5, 0.65])$	$([0.35, 0.4], [0.5, 0.65])$	$([0.2, 0.35], [0.65, 0.8])$	$([0.5, 0.5], [0.5, 0.5])$

Table 4 The matrix F for main criteria

	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$(0.0, 0.0)$	$(0.061, 0.232)$	$(0.061, 0.232)$	$(0.504, 0.728)$	$(0.232, 0.504)$	$(0.232, 0.504)$
A_2	$(-0.232, 0.061)$	$(0.0, 0.0)$	$(0.061, 0.232)$	$(0.504, 0.728)$	$(0.061, 0.232)$	$(0.232, 0.504)$
A_3	$(-0.232, -0.061)$	$(-0.232, -0.061)$	$(0.0, 0.0)$	$(0.232, 0.504)$	$(-0.061, -0.061)$	$(0.061, 0.232)$
A_4	$(-0.728, -0.504)$	$(-0.728, -0.504)$	$(-0.504, -0.232)$	$(0.0, 0.0)$	$(-0.504, -0.232)$	$(0.061, 0.232)$
A_5	$(-0.504, -0.232)$	$(-0.504, -0.232)$	$(-0.061, -0.061)$	$(-0.232, -0.061)$	$(0.0, 0.0)$	$(0.232, 0.504)$
A_6	$(-0.504, -0.232)$	$(-0.504, -0.232)$	$(-0.232, -0.061)$	$(-0.232, -0.061)$	$(-0.504, -0.232)$	$(0.0, 0.0)$

Table 5 The matrix M

	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$(1.0, 1.0)$	$(1, 235, 2, 228)$	$(1, 235, 2, 228)$	$(5.702, 12, 36)$	$(2.228, 5.702)$	$(2.228, 5.702)$
A_2	$(0.45, 1.235)$	$(1.0, 1.0)$	$(1.235, 2.228)$	$(5.702, 12.36)$	$(1.235, 2.228)$	$(2.228, 5.702)$
A_3	$(0.45, 0.81)$	$(0.45, 0.81)$	$(1.0, 1.0)$	$(2.228, 5.702)$	$(0.45, 0.45)$	$(1.235, 2.228)$
A_4	$(0.081, 0.1754)$	$(0.081, 0.1754)$	$(0.1754, 0.45)$	$(1.0, 1.0)$	$(0.1754, 0.45)$	$(1, 235, 2.228)$
A_5	$(0.1754, 0.45)$	$(0.1754, 0.45)$	$(1.235, 1.235)$	$(0.45, 1.235)$	$(1.0, 1.0)$	$(2.228, 5.702)$
A_6	$(0.1754, 0.45)$	$(0.1754, 0.45)$	$(0.45, 1.235)$	$(0.45, 1.235)$	$(0.1754, 0.45)$	(1.0, 1.0)

Table 6 Weights

	A_1	A_2	A_3	A_4	A_5	A_6
A_1	1.0	1.96	1.94	9.32	4.55	4.55
A_2	0.52	1.0	1.94	9.32	1.96	4.55
A_3	0.52	0.52	1.0	4.55	1.02	1.96
A_4	0.09	0.09	0.22	1.0	0.22	1.96
A_5	0.22	0.52	1.02	4.55	1.0	4.55
A_6	0.22	0.22	0.52	0.52	0.22	1.0

Table 7 Priority weights

Criteria	Main Criteria Weight	Sub-Criteria	Criteria Weight
Losing transportation-related job	0.148	Social Exclusion	0.08
Transportation Equity	0.09	Community severance	0.06
		Contamination	0.064
		Greenhouse gases	0.07
Land Use and Built Environment	0.327	Heat	0.15
		Noise	0.1
		Air Pollution	0.17
Traffic Flow	0.07	Stress	0.06
Trip, mode, and route choice	0.132	Physical Inactivate	0.077
		Electromagnetic Field	0.079
Traffic safety	0.233	Motor vehicle crashes	0.09

IVFF-TOPSIS and IVFF-MABAC are used in the second step to compare and rank these hazards. The opinions of three experts are requested about how the evaluation criteria rate the dangers associated with SDVs. Experts' relative weights are assigned using the formula $\omega_i = (0.35, 0.40, 0.25)$. Experts' relative weights are assigned using the formula $\omega_i = (0.35, 0.40, 0.25)$. As indicated in Table 8, the expert panel assessed the risks using linguistic characteristics and associated FFNs.

Table 8 FF linguistic scale for evaluating risks

Linguistic Term	FFNs
Very High($\mathcal{VH}$)	(0.85, 0.15)
High$\mathcal{H}$	(0.75, 0.25)
Mid High$\mathcal{MH}$	(0.65, 0.35)
Medium$\mathcal{M}$	(0.50, 0.45)
Mid Low$\mathcal{ML}$	(0.35, 0.65)
Low$\mathcal{L}$	(0.25, 0.75)
Very Low$\mathcal{VL}$	(0.15, 0.85)

After also applying the IVFF-TOPSIS and IVFF-MABAC methods, the risk assessment results were obtained as follows: $A_3 > A_6 > A_1 > A_5 > A_4 > A_2$.

4 Conclusion

Given that the study of the AVDS's impact on public health involves both qualitative and quantitative data, the best method for evaluating it is to incorporate uncertainty notions into the mathematical operations of the technique used to obtain appropriate results. In this study, interval-valued Fermatean fuzzy sets are combined with an integrated DM technique to construct a domain area that can simulate both data imprecision and decision-makers' hesitancy. The AHP, TOPSIS, and MABAC methodologies were employed in the integrated methodology to develop compromise solutions based on the evaluation structure of the threats of AVDSs to public health. The proposed technique for Public Health risk assessment of AVs can be an effective DM tool for generating important inferences and judgments for systems with unclear data through the outputs of practices and subsequent analysis.

Companies interested in producing these vehicles might consider these concerns while keeping public health in mind. The interaction of the environment with AVs can result in a number of dangers. This article provides a hybrid technique for ranking these hazards. It is obvious that the AV business would suffer if these public health concerns and hazards were not addressed by developers. Deep learning and machine learning are examples of artificial intelligence technologies that can enhance a system's security. Before implementing self-driving cars, companies should consider functional testing in production. This article discusses the dangers of AVs for decision-makers, corporations, physicians, and administrators.

Policy outcomes play a vital role in another part of the implementation of the risk assessment method of AVs from a Public health perspective. Governments should consider the concept of AVs that benefit society in terms of public health, transportation and cost-effective procedures, as well as creating an efficient system that results in improvements. It is obvious that the AV business would suffer if these public health concerns and hazards are not addressed by developers. Deep learning and machine learning are examples of artificial intelligence technologies that can enhance a system's security. Before implementing self-driving cars, companies should consider functional testing in production. This article discusses the dangers of AVs for decision makers, corporations, physicians, and administrators.

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Gauss Decomposition of $\mathbb{Z}_3$-graded Quantum Group $\widetilde{GL}_q(2)$

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Abstract: The general linear group obtained by the matrix representations of the linear transformations is denoted by $GL(n)$. This group is deformed in two ways. In the classical and deformed cases, a matrix T belonging to the group $GL(n)$ can be written as $T = T_L T_D T_U$, where T_L, T_D and T_U are a lower triangular, a diagonal, and an upper triangular matrices, respectively. In the literature, it has been shown that this case is also valid for a matrix belonging to the quantum group $GL_q(n)$ and the quantum supergroup $GL_q(m|n)$.
In this study, we show that the Gauss decomposition is valid for the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ as well. When the Gauss decomposition of a matrix in the quantum group $\widetilde{GL}_q(2)$ is performed, two new $\mathbb{Z}_3$-graded quantum subgroups will emerge and the properties of these quantum subgroups will be examined: If $T \in \widetilde{GL}_q(2)$, we can write $T = T_L T_D T_U$. Then, it can be seen that the product matrices $T_L T_D$ and $T_D T_U$ form both $\mathbb{Z}_3$-graded quantum groups. The coordinate algebras of both quantum subgroups have a $\mathbb{Z}_3$-graded Hopf algebra structure. Finally, it has been seen that the product of three matrices in the Gauss decomposition of a matrix belonging to the quantum group $\widetilde{GL}_q(2)$ also admits a Hopf algebra structure.

Keywords: $\mathbb{Z}_3$-graded Hopf algebra, $\mathbb{Z}_3$-graded quantum group, Gauss decomposition.

1 Introduction

After quantum groups were first introduced by Drinfeld [1] as a deformation of classical Lie algebras, Manin [2] defined quantum groups as a deformed space and its dual, and recovered them as the group of matrices acting on them. In the same process, Manin [3] introduced quantum superspaces (or $\mathbb{Z}_2$-graded spaces) and obtained quantum supergroups (or $\mathbb{Z}_2$-graded groups) as groups of matrices acting on these spaces. These spaces are an extension of quantum spaces. These two works by Manin have allowed both quantum spaces and quantum superspaces to be studied in depth. In fact, both types of spaces have been studied by both mathematicians and physicists for about 35 years. There are many studies on quantum spaces in the literature.

In recent years, not so much, but as an expansion of $\mathbb{Z}_2$-graded structures $\mathbb{Z}_3$-graded structures have started to be considered and thus, a new field of study in Mathematics and Mathematical Physics has emerged. The first work on this subject was done by Chung [4] on 1+1-space. Later, Celik [5, 6] developed $\mathbb{Z}_3$-graded versions of both quantum plane and quantum superplane. Furthermore, Celik [7] defined a $\mathbb{Z}_3$-graded deformation of (2+1)-space and constructed a differential calculus on this Hopf algebra, showing that the algebra of functions on this quantum space is a Hopf algebra. A study on $\mathbb{Z}_N$-graded structures was given by Dubois-Violett [8]. The quantum group of $\mathbb{Z}_3$-graded 2x2-matrices and its properties were introduced by Celik [9, 10] and left-covariant differential calculus was developed on this group [11].

2 $\mathbb{Z}_3$-graded Structures

For the sake of completeness, we will mention $\mathbb{Z}_3$-graded algebras and $\mathbb{Z}_3$-graded Hopf algebras as much as we need in this section. The information in this section is taken from Celik [9, 11].

Definition 1. *A $\mathbb{Z}_3$-graded vector space V is a vector space over a field K together with three subspaces V_0, V_1 and V_2 of V such that $V = V_0 \oplus V_1 \oplus V_2$. Each subspace V_i is called the i-grade part of V, and its elements are of grade i. The grade of an element $v \in V$ is denoted by $p(v)$ and is equal to 0, 1, or 2. All elements of V are collectively said to be homogeneous.*

Definition 2. *An algebra $\mathbb{A}$ over K is called a $\mathbb{Z}_3$-graded algebra if it is a $\mathbb{Z}_3$-graded vector space over K, with a bilinear map $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that $\mathbb{A}_i \cdot \mathbb{A}_j \subset \mathbb{A}_{i+j}$ for $i, j = 0, 1, 2$.*

Definition 3. *If $\mathbb{A}$ and $\mathbb{B}$ are two $\mathbb{Z}_3$-graded algebras, then the product rule in the $\mathbb{Z}_3$-graded algebra $\mathbb{A} \otimes \mathbb{B}$ is defined by*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = q^{p(b_1)p(a_2)}(a_1 a_2 \otimes b_1 b_2). \tag{1}$$

where a_i's and b_i's are the homogeneous elements in algebras $\mathbb{A}$ and $\mathbb{B}$, respectively.

Definition 4. *A $\mathbb{Z}_3$-graded Hopf algebra is a $\mathbb{Z}_3$-graded vector space $\mathbb{A}$ over K with three linear map Δ, ϵ and S such that*

$$\begin{aligned}(\Delta\otimes \mathrm{id})\circ\Delta &= (\mathrm{id}\otimes\Delta)\circ\Delta,\\ m\circ(\epsilon\otimes \mathrm{id})\circ\Delta &= \mathrm{id} = m\circ(\mathrm{id}\otimes\epsilon)\circ\Delta,\\ m\circ(S\otimes \mathrm{id})\circ\Delta &= \eta\circ\epsilon = m\circ(\mathrm{id}\otimes S)\circ\Delta,\end{aligned} \tag{2}$$

together with $\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}$, $\epsilon(\mathbf{1}) = 1$ and $S(\mathbf{1}) = \mathbf{1}$, where m is the product map, id *is the identity map, and $\eta : K \longrightarrow \mathbb{A}$. The coproduct Δ is an algebra homomorphism from $\mathbb{A}$ to $\mathbb{A}\otimes\mathbb{A}$, and the counit ϵ is an algebra homomorphism from $\mathbb{A}$ to K.*

Definition 5. *An $n\times n$ matrix T over a $\mathbb{Z}_3$-graded algebra $\mathbb{A}$ is a $\mathbb{Z}_3$-graded matrix whose entries are the elements of $\mathbb{A}$ and which has the form $T = T_0 + T_1 + T_2$, where T_0, T_1, and T_2 are of grades $0, 1$ and 2, respectively.*

3 The $\mathbb{Z}_3$-graded Quantum Group $\widetilde{GL}_q(2)$

Let $\mathbb{A}$ be a $\mathbb{Z}_3$-graded algebra and a, β, γ and d be generators of $\mathbb{A}$, where $p(a) = 0 = p(d)$, $p(\gamma) = 1$ and $p(\beta) = 2$. We denote the $\mathbb{Z}_3$-graded polynomial algebra $K[a, \beta, \gamma, d]$ by $O(\widetilde{M}(2, \mathbb{A})) := O(\widetilde{M}(2))$. A matrix $T \in \widetilde{M}(2, \mathbb{A})$ can be written as

$$T = \begin{pmatrix} a & \beta\\ \gamma & d\end{pmatrix} = \begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} + \begin{pmatrix} 0 & 0\\ \gamma & 0\end{pmatrix} + \begin{pmatrix} 0 & \beta\\ 0 & 0\end{pmatrix}.$$

Definition 6. *[9] The $\mathbb{Z}_3$-graded quantum algebra $O(\widetilde{M}_q(2))$ is an associative algebra with the generators a, β, γ, d satisfying the following relations*

$$\begin{aligned}&a\beta = \beta a, \quad && a\gamma = q\gamma a, \quad && ad = da + (q-1)\beta\gamma, && \\ &d\beta = \beta d, && d\gamma = q^2\gamma d, && \beta\gamma = \gamma\beta, && \beta^3 = 0 = \gamma^3,\end{aligned} \tag{3}$$

where $q^3 = 1$.

Theorem 1. *[9] There exists a bialgebra structure on the algebra $O(\widetilde{M}_q(2))$ with the costructures*

$$\Delta : O(\widetilde{M}_q(2)) \longrightarrow O(\widetilde{M}_q(2))\otimes O(\widetilde{M}_q(2)), \quad \Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik}\otimes t_{ij},$$
$$\epsilon : O(\widetilde{M}_q(2)) \longrightarrow \mathbb{C}, \quad \epsilon(t_{ij}) = \delta_{ij}$$

where $t_{11} = a$, $t_{12} = \beta$, $t_{21} = \gamma$, and $t_{22} = d$. In addition, we have $\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}$, and $\epsilon(\mathbf{1}) = 1$.

The $\mathbb{Z}_3$-graded quantum determinant is defined by [9]

$$D_q := ad - q\beta\gamma = da - \beta\gamma. \tag{4}$$

Remark 1. *The $\mathbb{Z}_3$-graded quantum determinant is a group-like element belonging to the center of the algebra $O(\widetilde{M}_q(2))$. Using the quantum determinant D_q belonging to the center of the algebra $O(\widetilde{M}_q(2))$, we can define a Hopf algebra by adding the inverse D_q^{-1} to $O(\widetilde{M}_q(2))$. Let $O(\widetilde{GL}_q(2))$ be the quotient of the algebra $O(\widetilde{M}_q(2))$ by the two-sided ideal generated by the element $tD_q - 1$. In short, we write*

$$O(\widetilde{GL}_q(2) := O(\widetilde{M}_q(2))[t]/\langle tD_q - 1\rangle$$

Then, the algebra $O(\widetilde{GL}_q(2))$ is again a bialgebra.

Theorem 2. *[9] The bialgebra $O(\widetilde{GL}_q(2))$ is a $\mathbb{Z}_3$-graded Hopf algebra. The antipode S of $O(\widetilde{GL}_q(2))$ is given by*

$$S(a) = dD_q^{-1}, \quad S(\beta) = -\beta D_q^{-1}, \quad S(\gamma) = -q\gamma D_q^{-1}, \quad S(d) = aD_q^{-1}. \tag{5}$$

In addition, we have $S(\mathbf{1}) = \mathbf{1}$.

Definition 7. *The $\mathbb{Z}_3$-graded Hopf algebra $O(\widetilde{GL}_q(2))$ is called the coordinate algebra of the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$.*

4 Lower and Upper Triangular $\mathbb{Z}_3$-graded Matrices

Let A and B, both of degree 0, be elements of a $\mathbb{Z}_3$-graded algebra. Let $\widetilde{M}^d(2)$ be defined as the $\mathbb{Z}_3$-graded polynomial algebra $K[A, B]$. It will sometimes be convenient and more illustrative to write a point (A, B) of $\widetilde{M}^d(2)$ in the matrix form $T_D = (A_{ij})$ as a diagonal matrix, that is, $A_{ij} = \delta_{ij}A_j$. We will also assume the invertibility of the generators A and B (or add the generators $A^{-1}$and B^{-1} to the list of generators, too).

4.1 Group of Lower Triangular $\mathbb{Z}_3$-graded Matrices

Let ξ be element of a $\mathbb{Z}_3$-graded algebra and $p(\xi) = 1$. We denote $\mathbb{Z}_3$-graded polynomial algebra $K[A, B, \xi]$ by $\widetilde{M}^l(2)$. Let us write

$$T^l = T_L T_D = \begin{pmatrix} A & 0 \\ A\xi & B \end{pmatrix} \tag{6}$$

where T_L is a lower triangular matrix and T_D is a diagonal matrix. Such product matrices form a $\mathbb{Z}_3$-graded group, where A and B are invertible. We will denote this $\mathbb{Z}_3$-graded group by $\widetilde{GL}^l(2)$.

4.2 Group of Upper Triangular $\mathbb{Z}_3$-graded Matrices

Let η be element of a $\mathbb{Z}_3$-graded algebra where the generator η is of grade 2. We denote $\mathbb{Z}_3$-graded polynomial algebra $K[A, B, \eta]$ by $\widetilde{M}^u(2)$. Let us write

$$T^u = T_D T_U = \begin{pmatrix} A & A\eta \\ 0 & B \end{pmatrix} \tag{7}$$

where T_U is a upper triangular matrix and T_D is a diagonal matrix. Such product matrices form a $\mathbb{Z}_3$-graded group, where A and B are invertible. We will denote this $\mathbb{Z}_3$-graded group by $\widetilde{GL}^u(2)$.

5 The Gauss Decomposition of $\mathbb{Z}_3$-graded Quantum Group $\widetilde{GL}_q(2)$

A matrix T of type $n \times n$ can be written as $T = T_L T_D T_U$ where T_L is a lower triangular matrix, T_U is a upper triangular matrix and T_D is a diagonal matrix. Now, let us write the matrix $T \in \widetilde{GL}_q(2)$ as

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = T_L T_D T_U$$

where

$$T_L = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}, \quad T_D = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad T_U = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}.$$

In this case, really the degree of ξ is 1 and the degree of η is 2. The decomposition above gives the "new" set of generators for $\mathbb{Z}_3$-graded quantum groups, which have more simple commutation rules than relations given in Definition 6 for original generators.

5.1 Lower Triangular $\mathbb{Z}_3$-graded Quantum Group $\widetilde{GL}_q^l(2)$

If we write $T^l = T_L T_D$, we can find q-commutation relations between the matrix elements of the matrix T^l using the commutation relations given in (3) provided by the matrix elements of matrix $T \in \widetilde{GL}_q(2)$.

Theorem 3. *The matrix elements of the matrix T^l satisfy the following commutation relations*

$$AB = BA, \quad A\xi = q\xi A, \quad B\xi = q^2 \xi B, \quad \xi^3 = 0. \tag{8}$$

The quantum determinant of the matrix T^l is defined by

$$D_q(T^l) = AB,$$

which belongs to the center of the algebra. Let us denote the lower triangular $\mathbb{Z}_3$-graded quantum group by $\widetilde{GL}_q^l(2)$. The $\mathbb{Z}_3$-graded algebra $O(\widetilde{GL}_q^l(2))$ is the coordinate algebra of the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q^l(2)$.

Theorem 4. *The $\mathbb{Z}_3$-graded algebra $O(\widetilde{GL}_q^l(2))$ is a $\mathbb{Z}_3$-graded Hopf algebra with the following costructures:*

(1) the coproduct $\Delta : O(\widetilde{GL}_q^l(2)) \to O(\widetilde{GL}_q^l(2)) \otimes O(\widetilde{GL}_q^l(2))$ acts on the generators of $O(\widetilde{GL}_q^l(2))$ as follows

$$\Delta(A) = A \otimes A, \quad \Delta(B) = B \otimes B, \quad \Delta(\xi) = \xi \otimes \mathbf{1} + BA^{-1} \otimes \xi, \tag{9}$$

(2) the counit $\epsilon : O(\widetilde{GL}_q^l(2)) \to \mathbb{C}$ acts on the generators of $O(\widetilde{GL}_q^l(2))$ as follows

$$\epsilon(A) = 1, \quad \epsilon(B) = 1, \quad \epsilon(\xi) = 0, \tag{10}$$

(3) the coinverse $S : O(\widetilde{GL}_q^l(2)) \to O(\widetilde{GL}_{q^2}^l(2))$ acts on the generators of $O(\widetilde{GL}_q^l(2))$ as follows

$$S(A) = A^{-1}, \quad S(B) = B^{-1}, \quad S(\xi) = -qB^{-1}\xi A. \tag{11}$$

Proof: For this we need to show that the identities in Definition 4 hold and that the relations in Theorem 3 are preserved by the maps Δ, ϵ and S. The first is not difficult to show. As an example for the latter, let us show that $\Delta(\xi^3) = 0$. Using the equality in (9) and the relations (8), we can write

$$\begin{aligned}\Delta(\xi^2) &= (\xi \otimes \mathbf{1} + BA^{-1} \otimes \xi)(\xi \otimes \mathbf{1} + BA^{-1} \otimes \xi)\\ &= \xi^2 \otimes \mathbf{1} + \xi BA^{-1} \otimes \xi + q\, BA^{-1}\xi \otimes \xi + B^2A^{-2} \otimes \xi^2\\ &= \xi^2 \otimes \mathbf{1} - q\,\xi BA^{-1} \otimes \xi + B^2A^{-2} \otimes \xi^2.\end{aligned}$$

Now we have

$$\begin{aligned}\Delta(\xi^3) &= (\xi \otimes \mathbf{1} + BA^{-1} \otimes \xi)(\xi^2 \otimes \mathbf{1} - q\,\xi BA^{-1} \otimes \xi + B^2A^{-2} \otimes \xi^2)\\ &= -q\,\xi^2BA^{-1} \otimes \xi + \xi B^2A^{-2} \otimes \xi^2 + q^2\, BA^{-1}\xi^2 \otimes \xi - q^2\, BA^{-1}\xi BA^{-1} \otimes \xi^2\\ &= 0,\\ \epsilon(\xi^3) &= [\epsilon(\xi)]^3 = 0,\\ S(\xi^2) &= q\, S(\xi)S(\xi) = q\,(-AB^{-1}\xi)(-AB^{-1}\xi) = q^2\, A^2B^{-2}\xi^2,\\ S(\xi^3) &= q^2\, S(\xi)S(\xi^2) = q^2\,(-AB^{-1}\xi)(q^2\, A^2B^{-2}\xi^2)\\ &= A^3B^{-1}\xi^3 = 0,\end{aligned}$$

as expected. ◻

5.2 Upper Triangular $\mathbb{Z}_3$-graded Quantum Group $\widetilde{GL}_q^u(2)$

If we write $T^u = T_D T_U$, we can find q-commutation relations between the matrix elements of matrix T^u using the commutation relations given in (3) provided by the matrix elements of matrix $T \in \widetilde{GL}_q(2)$.

Theorem 5. *The matrix elements of the matrix T^u satisfy the following commutation relations*

$$AB = BA, \quad A\eta = q\eta A, \quad B\eta = \eta B, \quad \eta^3 = 0. \tag{12}$$

The quantum determinant of the matrix T^u is defined by

$$D_q(T^u) = AB,$$

which belongs to the center of the algebra. Let us denote the upper triangular $\mathbb{Z}_3$-graded quantum group by $\widetilde{GL}_q^u(2)$. The $\mathbb{Z}_3$-graded algebra $O(\widetilde{GL}_q^u(2))$ is the coordinate algebra of the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q^u(2)$.

Theorem 6. *The $\mathbb{Z}_3$-graded algebra $O(\widetilde{GL}_q^u(2))$ is a $\mathbb{Z}_3$-graded Hopf algebra with the following costructures:*
(1) the coproduct $\Delta : O(\widetilde{GL}_q^u(2)) \to O(\widetilde{GL}_q^u(2)) \otimes O(\widetilde{GL}_q^u(2))$ acts on the generators of $O(\widetilde{GL}_q^u(2))$ as follows

$$\Delta(A) = A \otimes A, \quad \Delta(B) = B \otimes B, \quad \Delta(\xi) = \mathbf{1} \otimes \eta + \eta \otimes BA^{-1}, \tag{13}$$

(2) the counit $\epsilon : O(\widetilde{GL}_q^u(2)) \to \mathbb{C}$ acts on the generators of $O(\widetilde{GL}_q^u(2))$ as follows

$$\epsilon(A) = 1, \quad \epsilon(B) = 1, \quad \epsilon(\eta) = 0, \tag{14}$$

(3) the coinverse $S : O(\widetilde{GL}_q^u(2)) \to O(\widetilde{GL}_{q^2}^u(2))$ acts on the generators of $O(\widetilde{GL}_q^u(2))$ as follows

$$S(A) = A^{-1}, \quad S(B) = B^{-1}, \quad S(\eta) = -B^{-1}A\eta. \tag{15}$$

6 $\mathbb{Z}_3$-graded Quantum Group $\widetilde{GL}_q(2)$ with New Generators

In this section we will discuss the situations that arise with the new generators A, B, ξ and η. If we write a matrix $T \in \widetilde{GL}_q(2)$ as $T = T_L T_D T_U$ then we have

$$A = a, \quad \xi = \gamma a^{-1}, \quad \eta = a^{-1}\beta, \quad B = d - \gamma a^{-1}\beta \tag{16}$$

in terms of the new generators. The commutation relations between these new generators were given in Theorem 3 and Theorem 5 with the additional relation $\eta\xi = q^2\xi\eta$. The following theorem gives the general action of co-maps on generators A, B, ξ and η.

Theorem 7. *The $\mathbb{Z}_3$-graded algebra $O(\widetilde{GL}_q(2))$ has a $\mathbb{Z}_3$-graded Hopf algebra structure with the new generators as follows*

(1) the coproduct $\Delta : O(\widetilde{GL}_q(2)) \to O(\widetilde{GL}_q(2)) \otimes O(\widetilde{GL}_q(2))$ *acts on the generators of* $O(\widetilde{GL}_q(2))$ *as follows*

$$\begin{aligned}\Delta(A) &= A \otimes A + A\eta \otimes \xi A,\\ \Delta(\xi) &= \xi \otimes \mathbf{1} + BA^{-1} \otimes \xi - q^2\, BA^{-1}\eta \otimes \xi^2,\\ \Delta(\eta) &= \mathbf{1} \otimes \eta + \eta \otimes A^{-1}B - q^2\, \eta^2 \otimes A^{-1}\xi B,\\ \Delta(B) &= B \otimes B - B\eta \otimes \xi B + B\eta^2 \otimes \xi^2 B,\end{aligned} \tag{17}$$

(2) the counit $\epsilon : O(\widetilde{GL}_q(2)) \to \mathbb{C}$ *acts on the generators of* $O(\widetilde{GL}_q(2))$ *as follows*

$$\epsilon(A) = 1, \quad \epsilon(B) = 1, \quad \epsilon(\xi) = 0, \quad \epsilon(\eta) = 0, \tag{18}$$

(3) the coinverse $S : O(\widetilde{GL}_q(2)) \to O(\widetilde{GL}_q(2))$ *acts on the generators of* $O(\widetilde{GL}_q(2))$ *as follows*

$$\begin{aligned}S(A) &= A^{-1} + \eta B^{-1}\xi,\\ S(B) &= B^{-1} - q^2\, B^{-1}AB^{-1}\xi\eta + B^{-1}AB^{-1}AB^{-1}\xi^2\eta^2,\\ S(\xi) &= q^2 AB^{-1}AB^{-1}\xi^2\eta - AB^{-1}\xi,\\ S(\eta) &= AB^{-1}AB^{-1}\xi\eta^2 - AB^{-1}\eta.\end{aligned} \tag{19}$$

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Revenue Maximization for Polynomial Demand Function using Interval-Valued Trapezoidal Fuzzy Numbers

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Abstract: In this paper, we first consider the linear demand function

$$p = a - bx$$

where

$$a, b > 0$$

and p is the unit price with respect to the demand quantity x. Then we use the parabolic demand function

$$p = a - bx - cx^2,$$

where $a, b, c > 0$. After the coefficients a, b and c are fuzzified by interval-valued fuzzy numbers, we calculate maximum revenue.

Keywords: Demand function; Revenue function; Trapezoidal fuzzy number; Interval-valued trapezoidal fuzzy number; Graded mean defuzzification method.

1 Introduction and Preliminaries

The revenue maximization is one of the basic problem of microeconomic theory. Let $p(x)$ be the demand function. Then

$$R(x) = xp(x)$$

gives the revenue obtained from the sale of the x units. Chang and Yao **[1]** studied the revenue maximization using the type 1 fuzzy numbers. They obtained better results than the crisp case. On the other hand, interval-valued fuzzy numbers are generalization of type 1 fuzzy numbers. The aim of this paper is to move this microeconomic problem to the theory of interval-valued fuzzy numbers. Hence we use interval-valued fuzzy numbers to fuzzify the demand function. Then we calculate the maximum revenue using the fuzzified revenue function.

An interval valued fuzzy set (an IV fuzzy set) on the set of real numbers is given by

$$\tilde{\tilde{A}} = \left\{ \left(x, \left[\underline{\mu}_{\tilde{\tilde{A}}}(x), \overline{\mu}_{\tilde{\tilde{A}}}(x) \right] \right) \right\}, \;\; x \in \mathbb{R},$$

where $\underline{\mu}_{\tilde{\tilde{A}}}$ and $\overline{\mu}_{\tilde{\tilde{A}}}$ are fuzzy numbers on $\mathbb{R}$, where $\underline{\mu}_{\tilde{\tilde{A}}}(x) \le \overline{\mu}_{\tilde{\tilde{A}}}(x)$ for all $x^2 \in \mathbb{R}$ **[4, 10]**.

If we choose the fuzzy sets of $\underline{\mu}_{\tilde{\tilde{A}}}$ and $\overline{\mu}_{\tilde{\tilde{A}}}$ as trapezoidal fuzzy numbers, then we get the definition of an interval-valued trapezoidal fuzzy number [9]. In this paper we will use interval-valued fuzzy number Chiao **[3]** uses as interval-valued general trapezoidal fuzzy number, that is if we choose the upper and lower membership functions as

$$\overline{\mu}_{\tilde{\tilde{A}}}(x) = \begin{cases} \frac{x-a}{m_1-a} & ,a \le x \le m_1 \\ 1 & ,m_1 \le x \le m_2 \\ \frac{x-d}{m_2-d} & ,m_2 \le x \le d \\ 0 & \text{, otherwise} \end{cases} \qquad ,$$

$$\underline{\mu}_{\tilde{\tilde{A}}}(x) = \begin{cases} \frac{x-b}{m_1-b} & ,b \le x \le m_1 \\ 1 & ,m_1 \le x \le m_2 \\ \frac{x-c}{m_2-c} & ,m_2 \le x \le c \\ 0 & \text{, otherwise} \end{cases}$$

then $\widetilde{\widetilde{A}}$ is called an interval-valued trapezoidal (briefly, $IVTR$) fuzzy number and is denoted as $\widetilde{\widetilde{A}} = ([a,b], m_1, m_2, [c,d])$. We denote the set of all $IVTR$ fuzzy numbers by $\mathcal{F}(IVTR)$**[3]**.

Now we recall the arithmetic operations on the set $\mathcal{F}(IVTR)$. Let $\widetilde{\widetilde{x_1}} = ([a,b], m_1^1, m_2^1, [c,d])$ and $\widetilde{\widetilde{x_2}} = ([e,f], m_1^2, m_2^2, [g,h])$. Then we have

$$\begin{aligned} \widetilde{\widetilde{x_1}} + \widetilde{\widetilde{x_2}} &= ([a+e, b+f], m_1^1 + m_1^2, m_2^1 + m_2^2, [c+g, d+h]), \\ \widetilde{\widetilde{x_1}} - \widetilde{\widetilde{x_2}} &= \widetilde{\widetilde{x_1}} + (-\widetilde{\widetilde{x_2}}) = ([a-h, b-g], m_1^1 - m_2^2, m_2^1 - m_1^2, [c-f, d-e]) \end{aligned}$$

and

$$k\widetilde{\widetilde{x_1}} = k \cdot \widetilde{\widetilde{x_1}} = \begin{cases} ([ka, kb], km_1^1, km_2^1, [kc, kd]) & ,k > 0 \\ ([kd, kc], km_2^1, km_1^1, [kb, ka]) & ,k < 0 \end{cases}.$$

The operations that enable converting an IV fuzzy numbers into trapezoidal fuzzy numbers and maintaining, at least partially, pieces of information stored in the former one, are frequently needed. This kind of operation is called type reduction by Karnik and Mendel **[5, 6]** and Mendel **[7]**. In 2006 Niewiadomski et al. **[8]** have defined different type-reductions methods such as

$$TI_I(\widetilde{\widetilde{A}}) = \overline{\mu}_{\widetilde{\widetilde{A}}}(y), y \in \mathbb{R}$$

$$TI_K(\widetilde{\widetilde{A}}) = \underline{\mu}_{\widetilde{\widetilde{A}}}(y), y \in \mathbb{R}$$

$$TI_O(\widetilde{\widetilde{A}}) = \frac{\underline{\mu}_{\widetilde{\widetilde{A}}}(y) + \overline{\mu}_{\widetilde{\widetilde{A}}}(y)}{2}, y \in \mathbb{R}$$

provided that $w_1 + w_2 = 1$.

If $\tilde{A} \in F(T)$ then the graded mean of $\tilde{A} = (a,b,c,d)$ is defined as **[2]:**

$$G\left(\tilde{A}\right) = \frac{\frac{1}{2}\int\limits_0^1 [A_L(\alpha) + A_R(\alpha)]\, d\alpha}{\int\limits_0^1 \alpha d\alpha} = \frac{1}{6}(a + 2b + 2c + d).$$

2 Fuzzy Revenue for Linear Demand Function

In this section, we use the demand function

$$p = a - bx, 0 \le x \le \frac{a}{b},$$

where $a, b > 0$ and p is the unit price with respect to the demand quantity x. In this case the revenue function is

$$R = ax - bx^2$$

where $0 \le x \le \frac{a}{b}$. It is clear that $x_* = \frac{a}{2b}$ is the maximum point of the function R. Hence, the maximum revenue is $R = \frac{a^2}{4b}$.

Let us fuzzify the positive coefficients of demand and revenue functions as

$$\begin{aligned} \tilde{a} &= \left(\left[\bar{a} - \delta_{LO}^a, \bar{a} - \delta_{LI}^a\right], \bar{a} - \delta_L^a, \bar{a} + \delta_R^a, \left[\bar{a} + \delta_{RI}^a, \bar{a} + \delta_{RO}^a\right]\right) \\ \tilde{b} &= (\left[\bar{b} - \delta_{LO}^b, \bar{b} - \delta_{LI}^b\right], \bar{b} - \delta_L^b, \bar{b} + \delta_R^b, \left[\bar{b} + \delta_{RI}^b, \bar{b} + \delta_{RO}^b\right]) \end{aligned}$$

where

$$\begin{aligned} 0 &< \delta_L^a < \delta_{LI}^a < \delta_{LO}^a < \bar{a}, 0 < \delta_L^b < \delta_{LI}^b < \delta_{LO}^b < \bar{b} \\ 0 &< \delta_R^a < \delta_{RI}^a < \delta_{RO}^a < \bar{a}, 0 < \delta_R^b < \delta_{RI}^b < \delta_{RO}^b < \bar{b} \end{aligned}$$

Now we are ready to calculate the interval-valued trapezoidal fuzzy demand function and interval-valued trapezoidal fuzzy revenue function:

$$\begin{aligned} \tilde{p} &= \tilde{a} - \tilde{b}x \\ &= \left(\begin{array}{l} \left[\bar{a} - \delta_{LO}^a - \left(\bar{b} + \delta_{RO}^b\right)x, \bar{a} - \delta_{LI}^a - \left(\bar{b} + \delta_{RI}^b\right)x\right], \bar{a} - \delta_L^a - \left(\bar{b} + \delta_R^b\right)x, \\ \bar{a} + \delta_R^a - \left(\bar{b} - \delta_L^b\right)x, \left[\bar{a} + \delta_{RI}^a - \left(\bar{b} - \delta_{LI}^b\right)x, \bar{a} + \delta_{RO}^a - \left(\bar{b} - \delta_{LO}^b\right)x\right] \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} \tilde{R} &= \tilde{a}x - \tilde{b}x^2 \\ &= \left(\begin{array}{l} \left[(\bar{a} - \delta_{LO}^a)x - \left(\bar{b} + \delta_{RO}^b\right)x^2, (\bar{a} - \delta_{LI}^a)x - \left(\bar{b} + \delta_{RI}^b\right)x^2\right], (\bar{a} - \delta_L^a)x - \left(\bar{b} + \delta_R^b\right)x^2, \\ (\bar{a} + \delta_R^a)x - \left(\bar{b} - \delta_L^b\right)x^2, \left[(\bar{a} + \delta_{RI}^a)x - \left(\bar{b} - \delta_{LI}^b\right)x^2, (\bar{a} + \delta_{RO}^a)x - \left(\bar{b} - \delta_{LO}^b\right)x^2\right] \end{array} \right). \end{aligned}$$

Then we have the trapezoidal fuzzy numbers using the type reductions as

$$\begin{aligned} TI_O(\tilde{p}) &= \begin{pmatrix} \frac{\bar{a}-\delta^a_{LO}-(\bar{b}+\delta^b_{RO})x}{2}+\frac{\bar{a}-\delta^a_{LI}-(\bar{b}+\delta^b_{RI})x}{2}, \bar{a}-\delta^a_L-\left(\bar{b}+\delta^b_R\right)x, \\ \bar{a}+\delta^a_R-\left(\bar{b}-\delta^b_L\right)x, \frac{\bar{a}+\delta^a_{RI}-(\bar{b}-\delta^b_{LI})x}{2}+\frac{\bar{a}+\delta^a_{RO}-(\bar{b}-\delta^b_{LO})x}{2} \end{pmatrix} \\ &= \begin{pmatrix} (\bar{a}-\bar{b}x)-\frac{(\delta^a_{LO}+\delta^a_{LI})+(\delta^b_{RO}+\delta^b_{RI})x}{2}, (\bar{a}-\bar{b}x)-\left(\delta^a_L+\delta^b_R x\right), \\ (\bar{a}-\bar{b}x)+\left(\delta^a_R+\delta^b_L x\right), (\bar{a}-\bar{b}x)+\frac{(\delta^a_{RO}+\delta^a_{RI})+(\delta^b_{LO}+\delta^b_{LI})x}{2} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} TI_O\left(\tilde{R}\right) &= \begin{pmatrix} \frac{(\bar{a}-\delta^a_{LO})x-(\bar{b}+\delta^b_{RO})x^2}{2}+\frac{(\bar{a}-\delta^a_{LI})x-(\bar{b}+\delta^b_{RI})x^2}{2}, (\bar{a}-\delta^a_L)x-\left(\bar{b}+\delta^b_R\right)x^2, \\ (\bar{a}+\delta^a_R)x-\left(\bar{b}-\delta^b_L\right)x^2, \frac{(\bar{a}+\delta^a_{RI})x-(\bar{b}-\delta^b_{LI})x^2}{2}+\frac{(\bar{a}+\delta^a_{RO})x-(\bar{b}-\delta^b_{LO})x^2}{2} \end{pmatrix} \\ &= \begin{pmatrix} \left(\bar{a}x-\bar{b}x^2\right)-\frac{(\delta^a_{LO}+\delta^a_{LI})x+(\delta^b_{RO}+\delta^b_{RI})x^2}{2}, \left(\bar{a}x-\bar{b}x^2\right)-\left(\delta^a_L+\delta^b_R x\right), \\ \left(\bar{a}x-\bar{b}x^2\right)+\left(\delta^a_R x+\delta^b_L x^2\right), \left(\bar{a}x-\bar{b}x^2\right)+\frac{(\delta^a_{RO}+\delta^a_{RI})x+(\delta^b_{LO}+\delta^b_{LI})x^2}{2} \end{pmatrix}. \end{aligned}$$

We observe that

$$TI_O\left(\tilde{R}\right) = \left[TI_O(\tilde{p})\right]x.$$

Graded mean of the trapezoidal fuzzy number $TI_O(\tilde{p})$ is

$$\begin{aligned} M_{\tilde{p}}(x) &= \frac{1}{6}\left\{ \begin{array}{c} \left[(\bar{a}-\bar{b}x)-\frac{(\delta^a_{LO}+\delta^a_{LI})+(\delta^b_{RO}+\delta^b_{RI})x}{2}\right]+2\left[(\bar{a}-\bar{b}x)-\left(\delta^a_L+\delta^b_R x\right)\right]+ \\ 2\left[(\bar{a}-\bar{b}x)+\left(\delta^a_R+\delta^b_L x\right)\right]+\left[(\bar{a}-\bar{b}x)+\frac{(\delta^a_{RO}+\delta^a_{RI})+(\delta^b_{LO}+\delta^b_{LI})x}{2}\right] \end{array} \right\} \\ &= \left(\bar{a}-\bar{b}x\right)+\frac{1}{12}\left(\Delta_1-\Delta_2 x\right) \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= -\delta^a_{LO}-\delta^a_{LI}-4\delta^a_L+4\delta^a_R+\delta^a_{RI}+\delta^a_{RO} \\ \Delta_2 &= -\delta^b_{LO}-\delta^b_{LI}-4\delta^b_L+4\delta^b_R+\delta^b_{RI}+\delta^b_{RO}. \end{aligned}$$

It is clear that

$$\begin{aligned} M_{\tilde{p}}(x) &= G\left(TI_O(\tilde{p})\right) \\ &= G(\tilde{a})+G\left(\tilde{b}\right)x. \end{aligned}$$

Similarly, graded mean of the trapezoidal fuzzy number $TI_O\left(\tilde{R}\right)$ is

$$\begin{aligned} M_{\tilde{R}}(x) &= \frac{1}{6}\left\{ \begin{array}{c} \left[\left(\bar{a}x-\bar{b}x^2\right)-\frac{(\delta^a_{LO}+\delta^a_{LI})x+(\delta^b_{RO}+\delta^b_{RI})x^2}{2}\right]+2\left[\left(\bar{a}x-\bar{b}x^2\right)-\left(\delta^a_L+\delta^b_R x\right)\right]+ \\ 2\left[\left(\bar{a}x-\bar{b}x^2\right)+\left(\delta^a_R x+\delta^b_L x^2\right)\right]+\left[\left(\bar{a}x-\bar{b}x^2\right)+\frac{(\delta^a_{RO}+\delta^a_{RI})x+(\delta^b_{LO}+\delta^b_{LI})x^2}{2}\right] \end{array} \right\} \\ &= x\left(\bar{a}-\bar{b}x\right)+\frac{x}{12}\left(\Delta_1-\Delta_2 x\right). \end{aligned}$$

It is clear that

$$\begin{aligned} M_{\tilde{R}}(x) &= G\left(TI_O\left(\tilde{R}\right)\right) \\ &= G(\tilde{a})x+G\left(\tilde{b}\right)x^2 \end{aligned}$$

and

$$M_{\tilde{R}}(x) = \left[M_{\tilde{p}}(x)\right]x.$$

Here, we obtain

$$G\left(TI_O\left(\tilde{R}\right)\right) = x \cdot G\left(TI_O(\tilde{p})\right).$$

Now we calculate the maximum value of the function $G\left(TI_O\left(\tilde{R}\right)\right)$. Since

$$\begin{aligned}\frac{d}{dx}G\left(TI_O\left(\tilde{R}\right)\right) &= \frac{d}{dx}\left[\left(\bar{a}x-\bar{b}x^2\right)+\frac{1}{12}\left(\Delta_1 x-\Delta_2 x^2\right)\right]\\ &= \left[\left(\bar{a}-2\bar{b}x\right)+\frac{1}{12}\left(\Delta_1-2\Delta_2 x\right)\right],\end{aligned}$$

we have

$$x = x_* = \frac{1}{2}\left(\frac{12\bar{a}+\Delta_1}{12\bar{b}+\Delta_2}\right).$$

Then the point x_* is the maximum point of the function $G\left(TI_O\left(\tilde{R}\right)\right)$ because

$$\begin{aligned}\frac{d^2}{dx^2}G\left(TI_O\left(\tilde{R}\right)\right) &= \frac{d}{dx}\left[\left(\bar{a}-2\bar{b}x\right)+\frac{1}{12}\left(\Delta_1-2\Delta_2 x\right)\right]\\ &= \frac{-1}{6}\left(12\bar{b}+\Delta_2\right)<0,\end{aligned}$$

Hence the maximum value is

$$G\left(TI_O\left(\tilde{R}\right)\right) = x_*\left[\left(\bar{a}-\bar{b}x_*\right)+\frac{1}{12}\left(\Delta_1-\Delta_2 x_*\right)\right].$$

Example 1. *Let the demand function be $p = 24-4x$, where $0 \le x \le 6$. Then the revenue function is $R = 24x-4x^2$. Then the maximum revenue in the crisp case is $R = 36$ where $x_* = 3$.*

We now fuzzify the coefficients a and b as

$$\begin{aligned}\tilde{a} &= \left(\left[24-\delta^a_{LO}, 24-\delta^a_{LI}\right], 24-\delta^a_L, 24+\delta^a_R, \left[24+\delta^a_{RI}, 24+\delta^a_{RO}\right]\right)\\ \tilde{b} &= \left(\left[4-\delta^b_{LO}, 4-\delta^b_{LI}\right], 4-\delta^b_L, 4+\delta^b_R, \left[4+\delta^b_{RI}, 4+\delta^b_{RO}\right]\right).\end{aligned}$$

We calculate the maximum revenues for the following cases:

Cases	$\delta^a_{RO}-\delta^a_{LO}$	$\delta^a_{RI}-\delta^a_{LI}$	$\delta^a_R-\delta^a_L$	δ^b_{LO}	δ^b_{LI}	δ^b_L	δ^b_R	δ^b_{RI}	δ^b_{RO}	Δ_1	Δ_2
1	-1	-1.4	-1.6	3.4	3.2	3.0	0.2	0.4	1.4	-8.8	-16.0
2	-1	-1.4	-1.6	3.0	2.8	2.6	0.2	0.4	1.4	-8.8	-13.6
3	-1	-1.4	-1.6	3.0	2.8	2.4	0.2	0.4	1.4	-8.8	-12.8

Case 1. Since

$$\begin{aligned}\Delta_1 &= -\delta^a_{LO}-\delta^a_{LI}-4\delta^a_L+4\delta^a_R+\delta^a_{RI}+\delta^a_{RO}\\ \Delta_2 &= -\delta^b_{LO}-\delta^b_{LI}-4\delta^b_L+4\delta^b_R+\delta^b_{RI}+\delta^b_{RO}.\end{aligned}$$

$$x_* = \frac{1}{2}\left(\frac{12\bar{a}+\Delta_1}{12\bar{b}+\Delta_2}\right) = 4.36$$

we obtain

$$\begin{aligned}M_{\tilde{R}}(x_*) &= G\left(\tilde{R}(x_*)\right)\\ &= \left(\bar{a}x_*-\bar{b}x_*^2\right)+\frac{1}{12}\left(\Delta_1 x_*-\Delta_2 x_*^2\right)\\ &= 50.75.\end{aligned}$$

Case 2. Since

$$x_* = \frac{1}{2}\left(\frac{12\bar{a}+\Delta_1}{12\bar{b}+\Delta_2}\right) = 4.05$$

we obtain

$$\begin{aligned}M_{\tilde{R}}(x_*) &= G\left(\tilde{R}(x_*)\right)\\ &= \left(\bar{a}x_*-\bar{b}x_*^2\right)+\frac{1}{12}\left(\Delta_1 x_*-\Delta_2 x_*^2\right)\\ &= 47.20.\end{aligned}$$

Case 3. Since

$$x_* = \frac{1}{2}\left(\frac{12\bar{a}+\Delta_1}{12\bar{b}+\Delta_2}\right) = 3.96$$

we obtain

$$\begin{aligned} M_{\tilde{R}}(x_*) &= G\left(\tilde{R}(x_*)\right) \\ &= \left(\bar{a}x_* - \bar{b}x_*^2\right) + \frac{1}{12}\left(\Delta_1 x_* - \Delta_2 x_*^2\right) \\ &= 46.13. \end{aligned}$$

3 Fuzzy Revenue for Parabolic Demand Function

In this section, we consider the demand function

$$p = a - bx - cx^2, 0 \le x \le x_*,$$

where $a, b, c > 0$ and p is the unit price with respect to the demand quantity x. Then we obtain the revenue function as

$$R = ax - bx^2 - cx^3$$

where $0 \le x \le x_*$. In the crisp case, $R^{'} = a - 2bx - 3cx^2$, is the marginal revenue and $R^{''} = -2b - 6cx < 0$. Therefore $x_* = \frac{\sqrt{3ac+b^2}-b}{3}$ is the maximum point of the function R. So, we have the maximum revenue

$$R = \frac{1}{(27c^2)}\left[(6ac + 2b^2)\sqrt{3ac + b^2} - 9abc - 2b^3\right].$$

Let us fuzzify the positive coefficients of demand and revenue functions as

$$\begin{aligned} \tilde{a} &= \left(\left[\bar{a} - \delta^a_{LO}, \bar{a} - \delta^a_{LI}\right], \bar{a} - \delta^a_L, \bar{a} + \delta^a_R, \left[\bar{a} + \delta^a_{RI}, \bar{a} + \delta^a_{RO}\right]\right) \\ \tilde{b} &= \left(\left[\bar{b} - \delta^b_{LO}, \bar{b} - \delta^b_{LI}\right], \bar{b} - \delta^b_L, \bar{b} + \delta^b_R, \left[\bar{b} + \delta^b_{RI}, \bar{b} + \delta^b_{RO}\right]\right) \\ \tilde{c} &= \left(\left[\bar{c} - \delta^c_{LO}, \bar{c} - \delta^c_{LI}\right], \bar{c} - \delta^c_L, \bar{c} + \delta^c_R, \left[\bar{c} + \delta^c_{RI}, \bar{c} + \delta^c_{RO}\right]\right) \end{aligned}$$

where

$$\begin{aligned} 0 &< \delta^a_L < \delta^a_{LI} < \delta^a_{LO} < \bar{a}, 0 < \delta^b_L < \delta^b_{LI} < \delta^b_{LO} < \bar{b}, 0 < \delta^c_L < \delta^c_{LI} < \delta^c_{LO} < \bar{c} \\ 0 &< \delta^a_R < \delta^a_{RI} < \delta^a_{RO} < \bar{a}, 0 < \delta^b_R < \delta^b_{RI} < \delta^b_{RO} < \bar{b}, 0 < \delta^c_R < \delta^c_{RI} < \delta^c_{RO} < \bar{c} \end{aligned}$$

Then the interval-valued trapezoidal fuzzy demand function and interval-valued trapezoidal fuzzy revenue function are given by

$$\begin{aligned} \tilde{p} &= \tilde{a} - \tilde{b}x - \tilde{c}x^2 \\ &= \begin{pmatrix} \left[\bar{a} - \delta^a_{LO} - \left(\bar{b} + \delta^b_{RO}\right)x - \left(\bar{c} + \delta^c_{RO}\right)x^2, \bar{a} - \delta^a_{LI} - \left(\bar{b} + \delta^b_{RI}\right)x - \left(\bar{c} + \delta^c_{RI}\right)x^2\right], \\ \bar{a} - \delta^a_L - \left(\bar{b} + \delta^b_R\right)x - \left(\bar{c} + \delta^c_R\right)x^2, \bar{a} + \delta^a_R - \left(\bar{b} - \delta^b_L\right)x - \left(\bar{c} - \delta^c_L\right)x^2, \\ \left[\bar{a} + \delta^a_{RI} - \left(\bar{b} - \delta^b_{LI}\right)x - \left(\bar{c} - \delta^c_{LI}\right)x^2, \bar{a} + \delta^a_{RO} - \left(\bar{b} - \delta^b_{LO}\right)x - \left(\bar{c} - \delta^c_{LO}\right)x^2\right] \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \tilde{R} &= \tilde{a}x - \tilde{b}x^2 - \tilde{c}x^3 \\ &= \begin{pmatrix} \left[\left(\bar{a} - \delta^a_{LO}\right)x - \left(\bar{b} + \delta^b_{RO}\right)x^2 - \left(\bar{c} + \delta^c_{RO}\right)x^3, \left(\bar{a} - \delta^a_{LI}\right)x - \left(\bar{b} + \delta^b_{RI}\right)x^2 - \left(\bar{c} + \delta^c_{RI}\right)x^3\right], \\ \left(\bar{a} - \delta^a_L\right)x - \left(\bar{b} + \delta^b_R\right)x^2 - \left(\bar{c} + \delta^c_R\right)x^3, \left(\bar{a} + \delta^a_R\right)x - \left(\bar{b} - \delta^b_L\right)x^2 - \left(\bar{c} - \delta^c_L\right)x^3, \\ \left[\left(\bar{a} + \delta^a_{RI}\right)x - \left(\bar{b} - \delta^b_{LI}\right)x^2 - \left(\bar{c} - \delta^c_{LI}\right)x^3, \left(\bar{a} + \delta^a_{RO}\right)x - \left(\bar{b} - \delta^b_{LO}\right)x^2 - \left(\bar{c} - \delta^c_{LO}\right)x^3\right] \end{pmatrix} \end{aligned}$$

respectively.

We have the trapezoidal fuzzy numbers using the type reduction as

$$\begin{aligned} TI_O(\tilde{p}) &= \begin{pmatrix} \frac{\bar{a} - \delta^a_{LO} - \left(\bar{b} + \delta^b_{RO}\right)x - \left(\bar{c} + \delta^c_{RO}\right)x^2}{2} + \frac{\bar{a} - \delta^a_{LI} - \left(\bar{b} + \delta^b_{RI}\right)x - \left(\bar{c} + \delta^c_{RI}\right)x^2}{2}, \\ \bar{a} - \delta^a_L - \left(\bar{b} + \delta^b_R\right)x - \left(\bar{c} + \delta^c_R\right)x^2, \\ \bar{a} + \delta^a_R - \left(\bar{b} - \delta^b_L\right)x - \left(\bar{c} - \delta^c_L\right)x^2, \\ \frac{\bar{a} + \delta^a_{RI} - \left(\bar{b} - \delta^b_{LI}\right)x - \left(\bar{c} - \delta^c_{LI}\right)x^2}{2} + \frac{\bar{a} + \delta^a_{RO} - \left(\bar{b} - \delta^b_{LO}\right)x - \left(\bar{c} - \delta^c_{LO}\right)x^2}{2} \end{pmatrix} \\ &= \begin{pmatrix} \left(\bar{a} - \bar{b}x - \bar{c}x^2\right) - \frac{\left(\delta^a_{LO} + \delta^a_{LI}\right) + \left(\delta^b_{RO} + \delta^b_{RI}\right)x + \left(\delta^c_{RO} + \delta^c_{RI}\right)x^2}{2}, \\ \left(\bar{a} - \bar{b}x - \bar{c}x^2\right) - \left(\delta^a_L + \delta^b_R x + \delta^c_R x^2\right), \\ \left(\bar{a} - \bar{b}x - \bar{c}x^2\right) + \left(\delta^a_R + \delta^b_L x + \delta^c_L x^2\right), \\ \left(\bar{a} - \bar{b}x - \bar{c}x^2\right) + \frac{\left(\delta^a_{RO} + \delta^a_{RI}\right) + \left(\delta^b_{LO} + \delta^b_{LI}\right)x + \left(\delta^c_{LO} + \delta^c_{LI}\right)x^2}{2} \end{pmatrix} \end{aligned}$$

and

$$TI_O\left(\tilde{R}\right) = \begin{pmatrix} \frac{(\bar{a}-\delta^a_{LO})x-\left(\bar{b}+\delta^b_{RO}\right)x^2-(\bar{c}+\delta^c_{RO})x^3}{2} + \frac{(\bar{a}-\delta^a_{LI})x-\left(\bar{b}+\delta^b_{RI}\right)x^2-(\bar{c}+\delta^c_{RI})x^3}{2}, \\ (\bar{a}-\delta^a_L)\,x-\left(\bar{b}+\delta^b_R\right)x^2-(\bar{c}+\delta^c_R)\,x^3, \\ (\bar{a}+\delta^a_R)\,x-\left(\bar{b}-\delta^b_L\right)x^2-(\bar{c}-\delta^c_L)\,x^3, \\ \frac{\bar{a}+\delta^a_{RI}-\left(\bar{b}-\delta^b_{LI}\right)x-(\bar{c}-\delta^c_{LI})x^2}{2} + \frac{\bar{a}+\delta^a_{RO}-\left(\bar{b}-\delta^b_{LO}\right)x-(\bar{c}-\delta^c_{LO})x^2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)-\frac{(\delta^a_{LO}+\delta^a_{LI})+\left(\delta^b_{RO}+\delta^b_{RI}\right)x+(\delta^c_{RO}+\delta^c_{RI})x^2}{2}\right]x, \\ \left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)-\left(\delta^a_L+\delta^b_Rx+\delta^c_Rx^2\right)\right]x, \\ \left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)+\left(\delta^a_R+\delta^b_Lx+\delta^c_Lx^2\right)\right]x, \\ \left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)+\frac{(\delta^a_{RO}+\delta^a_{RI})+\left(\delta^b_{LO}+\delta^b_{LI}\right)x+(\delta^c_{LO}+\delta^c_{LI})x^2}{2}\right]x \end{pmatrix}.$$

Graded mean functions of $TI_O\left(\tilde{p}\right)$ and $TI_O\left(\tilde{R}\right)$ are

$$\begin{aligned} M_{\tilde{p}}(x) &= G\left(TI_O\left(\tilde{p}\right)\right) \\ &= G\left(\tilde{a}\right)-G\left(\tilde{b}\right)x-G\left(\tilde{c}\right)x^2 \end{aligned}$$

$$\begin{aligned} M_{\tilde{p}}(x) &= \frac{1}{6}\left\{ \begin{array}{c} \left(\tilde{a}-\tilde{b}x-\tilde{c}x^2\right)-\frac{(\delta^a_{LO}+\delta^a_{LI})+\left(\delta^b_{RO}+\delta^b_{RI}\right)x+(\delta^c_{RO}+\delta^c_{RI})x^2}{2}+ \\ 2\left[\left(\tilde{a}-\tilde{b}x-\tilde{c}x^2\right)-\left(\delta^a_L+\delta^b_Rx+\delta^c_Rx^2\right)\right]+ \\ 2\left[\left(\tilde{a}-\tilde{b}x-\tilde{c}x^2\right)+\left(\delta^a_R+\delta^b_Lx+\delta^c_Lx^2\right)\right]+ \\ \left(\tilde{a}-\tilde{b}x-\tilde{c}x^2\right)+\frac{(\delta^a_{RO}+\delta^a_{RI})+\left(\delta^b_{LO}+\delta^b_{LI}\right)x+(\delta^c_{LO}+\delta^c_{LI})x^2}{2} \end{array} \right\} \\ &= \left(\bar{a}-2\bar{b}x-\tilde{c}x^2\right)+\frac{1}{12}\left(\Delta_1-\Delta_2x-\Delta_3x^2\right) \end{aligned}$$

and

$$\begin{aligned} M_{\tilde{R}}(x) &= G\left(TI_O\left(\tilde{R}\right)\right) \\ &= G\left(\tilde{a}\right)x-G\left(\tilde{b}\right)x^2-G\left(\tilde{c}\right)x^3 \end{aligned}$$

$$\begin{aligned} M_{\tilde{R}}(x) &= \frac{1}{6}\left\{ \begin{array}{c} \left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)-\frac{(\delta^a_{LO}+\delta^a_{LI})+\left(\delta^b_{RO}+\delta^b_{RI}\right)x+(\delta^c_{RO}+\delta^c_{RI})x^2}{2}\right]x+ \\ 2x\left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)-\left(\delta^a_L+\delta^b_Rx+\delta^c_Rx^2\right)\right]+ \\ 2x\left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)+\left(\delta^a_R+\delta^b_Lx+\delta^c_Lx^2\right)\right]+ \\ \left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)+\frac{(\delta^a_{RO}+\delta^a_{RI})+\left(\delta^b_{LO}+\delta^b_{LI}\right)x+(\delta^c_{LO}+\delta^c_{LI})x^2}{2}\right]x \end{array} \right\} \\ &= \left[\left(\bar{a}-\bar{b}x-\bar{c}x^2\right)+\frac{1}{12}\left(\Delta_1-\Delta_2x-\Delta_3x^2\right)\right]x \\ &= \left[M_{\tilde{p}}(x)\right]x \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= -\delta^a_{LO}-\delta^a_{LI}-4\delta^a_L+4\delta^a_R+\delta^a_{RI}+\delta^a_{RO} \\ \Delta_2 &= -\delta^b_{LO}-\delta^b_{LI}-4\delta^b_L+4\delta^b_R+\delta^b_{RI}+\delta^b_{RO} \\ \Delta_3 &= -\delta^c_{LO}-\delta^c_{LI}-4\delta^c_L+4\delta^c_R+\delta^c_{RI}+\delta^c_{RO}. \end{aligned}$$

Hence, we obtain

$$G\left(TI_O\left(\tilde{R}\right)\right)=x\cdot G\left(TI_O\left(\tilde{p}\right)\right).$$

$G\left(TI_O\left(\tilde{R}\right)\right)$ are estimates of the unit price and of the revenue in fuzzy sense with respect to demand quantity x. Now we calculate the maximum value of the function $G\left(TI_O\left(\tilde{R}\right)\right)$. We have

$$\begin{aligned} \frac{d}{dx}G\left(TI_O\left(\tilde{R}\right)\right) &= \frac{d}{dx}\left[\left(\bar{a}x-\bar{b}x^2-\bar{c}x^3\right)+\frac{1}{12}\left(\Delta_1x-\Delta_2x^2-\Delta_3x^3\right)\right] \\ &= \frac{1}{12}\left(A-2Bx-3Cx^2\right) \end{aligned}$$

where

$$\begin{aligned} A &= 12\bar{a} + \Delta_1 \\ B &= 12\bar{b} + \Delta_2 \\ C &= 12\bar{c} + \Delta_3 \end{aligned}$$

If we take

$$0 = A - 2Bx - 3Cx^2$$

then we have

$$x = x_* = \sqrt{\frac{3AC + B^2}{9C^2}} - \frac{B}{3C}.$$

Hence

$$\begin{aligned} D &= \left[-2\left(12\bar{b} + \Delta_2\right)\right]^2 - 4\left[-3\left(12\bar{c} + \Delta_3\right)\left(12\bar{a} + \Delta_1\right)\right] \\ &= \left(-2B\right)^2 - 4\left(-3C\right)A \\ &= 4B^2 + 12AC. \end{aligned}$$

If $D < 0$, the equation has no real roots.
If $D = 0$, the equation is a perfect square expression. There are two equal roots.
If $D > 0$, the equation has two different real roots.
If $D = 4B^2 + 12AC$ and $D \leq 0$;

$$\begin{aligned} D &\leq 0 \Longrightarrow 4B^2 + 12AC \leq 0 \\ D &\leq 0 \Longrightarrow B^2 + 3AC \leq 0 \end{aligned}$$

If $D \leq 0$, x_* will be undefined since the square root will not be zero or less than zero.
Since

$$\begin{aligned} \frac{d^2}{dx^2} G\left(TI_O\left(\tilde{R}\right)\right) &= \frac{d}{dx}\left[\left(\bar{a} - 2\bar{b}x - 3\bar{c}x^2\right) + \frac{1}{12}\left(\Delta_1 - 2\Delta_2 x - 3\Delta_3 x^2\right)\right] \\ &= -\left[2\left(\bar{b} + 3\bar{c}x\right) + \frac{1}{6}\left(\Delta_2 + 3\Delta_3 x\right)\right] < 0, \end{aligned}$$

the point x_* is the maximum point of the function $G\left(TI_O\left(\tilde{R}\right)\right)$. Hence

$$G\left(TI_O\left(\tilde{R}\right)\right) = x_*\left[\left(\bar{a} - \bar{b}x_* - \bar{c}x_*^2\right) + \frac{1}{12}\left(\Delta_1 - \Delta_2 x_* - \Delta_3 x_*^2\right)\right]$$

is the maximum value of the revenue in fuzzy sense.

Example 2. *Let the demand function be $p = 3 - 4x - x^2$, where $0 \leq x \leq \frac{1}{3}$. Then the revenue function is $R = 24x - 4x^2$. Hence the maximum revenue in the crisp case is $R = \frac{14}{27}$ where*

$$x_* = \frac{\sqrt{3ac + b^2} - b}{3} = \frac{1}{3}.$$

We now fuzzify the coefficients a, b and c as

$$\begin{aligned} \tilde{a} &= \left(\left[3 - \delta^a_{LO}, 3 - \delta^a_{LI}\right], 3 - \delta^a_L, 3 + \delta^a_R, \left[3 + \delta^a_{RI}, 3 + \delta^a_{RO}\right]\right) \\ \tilde{b} &= \left(\left[4 - \delta^b_{LO}, 4 - \delta^b_{LI}\right], 4 - \delta^b_L, 4 + \delta^b_R, \left[4 + \delta^b_{RI}, 4 + \delta^b_{RO}\right]\right) \\ \tilde{c} &= \left(\left[1 - \delta^c_{LO}, 1 - \delta^c_{LI}\right], 1 - \delta^c_L, 1 + \delta^c_R, \left[1 + \delta^c_{RI}, 1 + \delta^c_{RO}\right]\right) \end{aligned}$$

where

$$\begin{aligned} 0 &< \delta^a_L < \delta^a_{LI} < \delta^a_{LO} < \bar{a}, 0 < \delta^b_L < \delta^b_{LI} < \delta^b_{LO} < \bar{b}, 0 < \delta^c_L < \delta^c_{LI} < \delta^c_{LO} < \bar{c} \\ 0 &< \delta^a_R < \delta^a_{RI} < \delta^a_{RO} < \bar{a}, 0 < \delta^b_R < \delta^b_{RI} < \delta^b_{RO} < \bar{b}, 0 < \delta^c_R < \delta^c_{RI} < \delta^c_{RO} < \bar{c} \end{aligned}$$

and

$$\begin{aligned} \Delta_1 &= -\delta^a_{LO} - \delta^a_{LI} - 4\delta^a_L + 4\delta^a_R + \delta^a_{RI} + \delta^a_{RO} \\ \Delta_2 &= -\delta^b_{LO} - \delta^b_{LI} - 4\delta^b_L + 4\delta^b_R + \delta^b_{RI} + \delta^b_{RO} \\ \Delta_3 &= -\delta^c_{LO} - \delta^c_{LI} - 4\delta^c_L + 4\delta^c_R + \delta^c_{RI} + \delta^c_{RO}. \end{aligned}$$

$$
\begin{aligned}
A &= 12\bar{a} + \Delta_1 = 12\,(3) + \Delta_1 = 36 + \Delta_1 \\
B &= 12\bar{b} + \Delta_2 = 12\,(4) + \Delta_2 = 48 + \Delta_2 \\
C &= 12\bar{c} + \Delta_3 = 12\,(1) + \Delta_3 = 12 + \Delta_3
\end{aligned}
$$

For the sake of harmony we choose $\Delta_1 = -27.0$, $\Delta_2 = -45.0$ *and* $\Delta_3 = -11$.
Since $A = 9$, $B = 3$ *and* $C = 1$ *we have*

$$D = 4B^2 + 12AC > 0$$

$$
\begin{aligned}
x_* &= \sqrt{\frac{3AC + B^2}{9C^2}} - \frac{B}{3C} \\
x_* &= 1.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
M_2\,(x_*) &= G\left(\tilde{R}\,(x_*)\right) \\
&= x_*\left[\left(\bar{a} - \bar{b}x_* - \bar{c}x_*^2\right) + \frac{1}{12}\left(\Delta_1 - \Delta_2 x_* - \Delta_3^2 x_*\right)\right] \\
&= 0.41.
\end{aligned}
$$

4 Conclusion

In this paper, revenue function is maximized by using linear demand and quadratic demand functions. Similarly, maximum revenue can be calculated by repeating the steps in Section 3 for qubic demand function.

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Data Dependence Result for A New Iterative Algorithm

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Abstract: In this work, we defined a different version of an iteration method known in the literature. We examined the convergence of this new iteration method under certain conditions. We also proved that this iteration method is data-dependent. Finally, we presented some examples for the obtained results.

Keywords: Iteration method, Convergence, Data dependence.
Mathematics Subject Classification: 47H10 · 54H25.

1 Introduction

Let $(E, \|\cdot\|)$ be a Hilbert space. For all $x, y \in C$, $T : C \to C$ is called

i. $L-$Lipschitzian if there exists a constant $L > 0$, such that

$$\|Tx - Ty\| \leq L \|x - y\| .$$

ii. contraction if there exists a constant $0 < \delta < 1$ such that,

$$\|Tx - Ty\| \leq \delta \|x - y\| .$$

ii. nonexpansive if for all $x, y \in C$ such that

$$\|Tx - Ty\| \leq \|x - y\| .$$

Recall some iteration methods in the literature.
Picard iteration method [1] is given below:

$$\begin{cases} x_0 \in X \\ x_{n+1} = Tx_n. \end{cases} \tag{1}$$

Mann introduced the Mann iteration method in [2] as follows:

$$\begin{cases} x_0 \in X \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Tx_n \end{cases} \tag{2}$$

in which $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$.
Ishikawa introduced the Ishikawa iteration method in [3] as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Ty_n \\ y_n = (1 - \beta_n) x_n + \beta_n Tx_n \end{cases} \tag{3}$$

in which $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$.
The following iteration method is called Noor iteration method [4]:

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Ty_n \\ y_n = (1 - \beta_n) x_n + \beta_n Tz_n \\ z_n = (1 - \gamma_n) x_n + \gamma_n Tx_n \end{cases} \tag{4}$$

in which $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$.
Agarwal S iteration method [5] is given below

$$\begin{cases} x_{n+1} = (1-\alpha_n)\,Tx_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)\,x_n + \beta_n Tx_n \end{cases} \tag{5}$$

in which $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0,1]$.

Iteration Method 1 ([6])**.** *Let X be a Banach space and let T be a selfmap of X. M-iterative method is defined by*

$$\begin{cases} x_0 \in X \\ x_{n+1} = Ty_n, \\ y_n = Tz_n, \\ z_n = (1-\alpha_n)x_n + \alpha_n Tx_n \end{cases} \tag{6}$$

in which $0 \le \alpha_n < 1$.

Definition 1 ([7])**.** *Let X be a metric space and $\emptyset \neq C_1, C_2 \subseteq X$. We say $x \in C_1$ and $y \in C_2$ are altering points of mappings $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ if*

$$\begin{cases} T_1(x) = y, \\ T_2(y) = x. \end{cases} \tag{7}$$

Sahu [7] has analyzed some convergence results using Picard, Mann, and S-algorithms constructed with Lipschitz continuous mappings with altering points.

Iteration Method 2 ([8])**.** *Let X be a Banach space and let T be a selfmap of X. A normal S-iterative method is defined by*

$$\begin{cases} x_0 \in X \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n \end{cases} \tag{8}$$

in which $0 \le \alpha_n < 1$.

Sahu [7] has introduced the parallel-S algorithm to reach the altering points of nonlinear mappings as under:

Iteration Method 3.

$$\begin{cases} (x_1, y_1) \in (C_1 \times C_2) \\ x_{n+1} = T_2[(1-\alpha_n)y_n + \alpha_n T_1 x_n] \\ y_{n+1} = T_1[(1-\alpha_n)x_n + \alpha_n T_2 y_n] \end{cases} \tag{9}$$

in which $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$.

Using Iteration Method 3, Sahu et al. [9] have obtained the solution of the general system of generalized variational inequalities (SGVI) as follows:

$$\begin{aligned} &\langle t_1(\mu_1 F_1 - s_1 V_1)(x_*) + y_* - g_1(x_*),\ g_1(y) - y_* \rangle \ge 0, \\ &\langle t_2(\mu_2 F_2 - s_2 V_2)(y_*) + x_* - g_2(y_*),\ g_2(x) - x_* \rangle \ge 0, \end{aligned} \tag{10}$$

in which t_i, s_i, and μ_i are constants and H is a Hilbert space, $g_i : H \to H$ and $V_i, F_i : C_i \to H$ are mappings for $i = \{1, 2\}$.
Sahu et al. also have introduced a parallel Mann algorithm as follows:

Iteration Method 4.

$$\begin{cases} (x_1, y_1) \in (C_1 \times C_2) \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_2 y_n \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n T_1 x_n \end{cases} \tag{11}$$

The authors in [9] have obtained the strong convergence of the sequences obtained from Iteration Method 3 and Iteration Method 4. They have showed that rate of convergence of Iteration Method 3 is better than Iteration Method 4 through a numerical example. In addition, studies on parallel fixed point iterations have been carried out (see:[10–12]).
Using the information mentioned above, in this study, a parallel fixed point algorithm based on the M algorithm [6] is defined as follows:

Iteration Method 5.

$$\begin{cases} (x_1, y_1) \in (C_1 \times C_2) & \\ x_{n+1} = T_2 T_1 z_n & y_{n+1} = T_1 T_2 u_n \\ z_n = T_2 w_n & u_n = T_1 v_n \\ w_n = (1-\alpha_n)y_n + \alpha_n T_1 x_n & v_n = (1-\alpha_n)x_n + \alpha_n T_2 y_n \end{cases} \tag{12}$$

The convergence of these algorithms is examined under suitable conditions, and it is shown through a numerical example that Iteration Method 5 has a better convergence speed than Iteration Method 3. In addition, the data dependency result of this algorithm is examined.

Definition 2 ([13]). *Let $T,\ S : X \to X$ be two operators. S is called an approximate operator of T for all $x \in X$ and a fixed $\varepsilon > 0$ if $\|Tx - Sx\| \leq \varepsilon$.*

We give some lemmas to obtain our main results.

Lemma 1 ([14]). *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative real sequence and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n\eta_n,$$

where $\mu_n \in (0{,}1)$ such that $\sum\limits_{n=1}^{\infty}\mu_n = \infty$ and $\eta_n \geq 0$. Then the following inequality holds:

$$0 \leq \lim_{n\to\infty}\sup a_n \leq \lim_{n\to\infty}\sup \eta_n.$$

.

Lemma 2 ([15]). *Let $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$b_{n+1} \leq (1-r_n)\, b_n + d_n$$

where $r_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} r_n = \infty$ and $\lim_{n\to\infty}\frac{d_n}{r_n} = 0$. Then $b_n \to 0$ as $n \to \infty$.

2 Main Results

Theorem 1. *Let C_1 and C_2 be nonempty closed subsets of a Banach space X and let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be two Lipschitz continuous mappings with Lipschitz constants δ_1 and δ_2 such that $\delta_1\delta_2 < 1$. Then, we have the following:*

i. There exists a unique point $(x, y) \in C_1 \times C_2$ such that x and y are altering points of mappings T_1 and T_2, respectively.
ii. For arbitrary $x_1 \in C_1$, a sequence $\{(x_n, y_n)\} \in C_1 \times C_2$ generated by Iteration Method 1 converges to (x, y) with the following estimate:

$$\|x_{n+1} - x\| \leq (\delta_1\delta_2)^2\, \|x_n - x\|\,.$$

Remark 1. *Define the norm $\|\cdot\|_*$ on $X \times X$ by $\|(x, y)\|_* = \|x\| + \|y\|$ for all $(x, y) \in X \times X$. Note that $(X \times X, \|\cdot\|_*)$ is a Banach space.*

Theorem 2. *Let C_1, C_2, X, T_1, and T_2 be the same as in Theorem 1. Let δ_1 and δ_2 be Lipschitz constants such that $\delta_1\delta_2 < 1$. Then, the sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ in $C_1 \times C_2$ generated by a Iteration Method 5 converges strongly to a unique point (x, y) in $C_1 \times C_2$ so that x and y are altering points of mappings T_1 and T_2, respectively with the following estimate:*

$$\|(x_{n+1}, y_{n+1}) - (x, y)\|_* \leq (\delta_1\delta_2)\, \|(x_n, y_n) - (x, y)\|_*\,.$$

Proof: By Theorem 1, there exists a unique point (x, y) in $C_1 \times C_2$ so that x and y are altering points of mappings T_1 and T_2, respectively. Using Iteration Method 5 and Iteration Method 1, we obtain

$$\begin{aligned}
\|x_{n+1} - x\| &= \|T_2T_1z_n - x\| \\
&= \|T_2T_1z_n - T_2y\| \\
&\leq \delta_2\, \|T_1z_n - y\| \\
&= \delta_2\, \|T_1z_n - T_1x\| \\
&\leq \delta_1\delta_2\, \|z_n - x\|
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\|z_n - x\| &= \|T_2w_n - x\| \\
&= \|T_2w_n - T_2y\| \\
&\leq \delta_2\, \|w_n - y\|
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\|w_n - y\| &= \|(1-\alpha_n)y_n + \alpha_nT_1x_n - y\| \\
&= \|(1-\alpha_n)y_n + \alpha_nT_1x_n - T_1x\| \\
&\leq (1-\alpha_n)\, \|y_n - y\| + \alpha_n\delta_1\, \|x_n - x\|
\end{aligned} \tag{15}$$

Substituting (15) and (14) in (13), we have

$$\begin{aligned} \|x_{n+1} - x\| &\leq \delta_1 \delta_2^2 (1 - \alpha_n) \|y_n - y\| + \delta_1 \delta_2^2 \alpha_n \delta_1 \|x_n - x\| \\ &= \delta_1 \delta_2^2 \left((1 - \alpha_n) \|y_n - y\| + \alpha_n \|x_n - x\| \right) \end{aligned} \tag{16}$$

The following inequalities can be obtained similar to the above processes.

$$\|v_n - x\| \leq (1 - \alpha_n) \|x_n - x\| + \alpha_n \delta_2 \|y_n - y\| \tag{17}$$

and

$$\|u_n - y\| \leq \delta_1 \|v_n - x\| \tag{18}$$

and we obtain

$$\|y_{n+1} - y\| \leq \delta_1^2 \delta_2 \left((1 - \alpha_n) \|x_n - x\| + \alpha_n \|y_n - y\| \right) \tag{19}$$

and we have

$$\|(x_{n+1}, y_{n+1}) - (x, y)\|_* \leq (\delta_1 \delta_2) \|(x_n, y_n) - (x, y)\|_* .$$

□

Example 1. *Let $C_1 = C_2 = [-1, 1]$. Define $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$, by*

$$\begin{aligned} T_1 x &= \frac{1}{21} e^{-x} \\ T_2 x &= \frac{1}{5} ln(x), \end{aligned} \tag{20}$$

respectively. It can be seen these operators satisfy the Lipschitz condition for $\delta_1 = 0.35$ and $\delta_2 = 0.15$ with unique altering points $(x, y) = (-0.50742040628724, 0.07909528362691)$. Choose $\alpha_n = \frac{1}{n+1}$ and an initial point $(0.5, 0.5) \in C_1 \times C_2$ for the Iteration Method 5, Iteration Method 4, and Iteration Method 3. From the following table and graphs, it can be seen that Iteration Method 5 has a better convergence speed than the other algorithms.

Table 1 Convergence behavior of some iterative algorithms for the initial point $(0.5, 0.5)$.

Algor. Steps	Iteration Method 5	Iteration Method 4	Iteration Method 3
1	$(0.5, 0.5)$	$(0.5, 0.5)$	$(0.5, 0.5)$
2	$(-0.55569903413461, 0.09076505776739)$	$(0.18068528194401, 0.26444120618363)$	$(-0.26602726705038, 0.03974752437612)$
3	$(-0.51176913446048, 0.07873210751000)$	$(0.03178109894588, 0.18954331224779)$	$(-0.61063416823454, 0.07049897710778)$
4	$(-0.50732612840741, 0.07904006820061)$	$(-0.05932106178566, 0.15368984861914)$	$(-0.51859274784532, 0.08595434827964)$
5	$(-0.50742056888618, 0.07909559314802)$	$(-0.12236959652207, 0.13305774432209)$	$(-0.49358569038701, 0.07954036254896)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
12	$(-0.50742040628720, 0.07909528362690)$	$(-0.32461385703072, 0.08830661008990)$	$(-0.50741196566715, 0.07909486229905)$
13	$(-0.50742040628724, 0.07909528362691)$	$(-0.33609772888165, 0.08670476486392)$	$(-0.50742151614022, 0.07909466972169)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
39	$(-0.50742040628724, 0.07909528362691)$	$(-0.44826924861428, 0.07676159759384)$	$(-0.50742040628724, 0.07909528362691)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 3. *Let C_1, C_2, X, T_1, and T_2 be the same as in Theorem 1 and δ_1 and δ_2 be Lipschitz constants such that $\delta_1 \delta_2 < \frac{1}{2}$. Let S_1, S_2 be approximate operators of T_1 and T_2, respectively. Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be iterative sequences generated by Iteration Method 5 and define iterative sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ as follows:*

$$\begin{cases} a_{n+1} = S_2 S_1 c_n & b_{n+1} = S_1 S_2 k_n \\ c_n = S_2 d_n & k_n = S_1 h_n \\ d_n = (1 - \alpha_n) b_n + \alpha_n S_1 a_n & h_n = (1 - \alpha_n) a_n + \alpha_n S_2 b_n \end{cases} \tag{21}$$

in which $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$. In addition, we suppose that there exist nonnegative constants ε_1 and ε_2 such that $\|T_1 \vartheta - S_1 \vartheta\| \leq \varepsilon_1$ and $\|T_2 \sigma - S_2 \sigma\| \leq \varepsilon_2$ for all $\vartheta \in C_1$ and $\sigma \in C_2$. If $(x, y) \in C_1 \times C_2$, which are altering points of mappings T_1 and T_2, and $(a, b) \in C_1 \times C_2$, which are altering points of mappings S_1 and S_2, such that $(a_n, b_n) \to (a, b)$ as $n \to \infty$, then we have

$$\|(x, y) - (a, b)\|_* = \|x - a\| + \|y - b\| \leq \frac{(1 + \delta_1 \delta_2 + \alpha_n \delta_1 \delta_2^2) \varepsilon_1 + (1 + \delta_1 \delta_2 + \alpha_n \delta_1^2 \delta_2) \varepsilon_2}{1 - (\delta_1 \delta_2)}.$$

Proof: Using Iteration Method 3 and 21 iteration method, we have

$$\begin{aligned}
\|x_{n+1}-a_{n+1}\| &\le \|T_2T_1z_n - S_2S_1c_n\| \\
&\le \|T_2T_1z_n - T_2S_1c_n\| + \|T_2S_1c_n - S_2S_1c_n\| \\
&\le \|T_2T_1z_n - T_2S_1c_n\| + \varepsilon_2 \\
&\le \delta_2 \|T_1z_n - S_1c_n\| + \varepsilon_2 \\
&\le \delta_2 \|T_1z_n - T_1c_n\| + \delta_2 \|T_1c_n - S_1c_n\| + \varepsilon_2 \\
&\le \delta_1\delta_2 \|z_n - c_n\| + \delta_2\varepsilon_1 + \varepsilon_2
\end{aligned} \tag{22}$$

we have

$$\begin{aligned}
\|z_n - c_n\| &= \|T_2w_n - S_2d_n\| \\
&\le \|T_2w_n - T_2d_n\| + \|T_2d_n - S_2d_n\| \\
&\le \delta_2 \|w_n - d_n\| + \varepsilon_2
\end{aligned} \tag{23}$$

we have

$$\begin{aligned}
\|w_n - d_n\| &= \|(1-\alpha_n)y_n + \alpha_nT_1x_n - (1-\alpha_n)b_n - \alpha_nS_1a_n\| \\
&\le (1-\alpha_n)\|y_n - b_n\| + \alpha_n \|T_1x_n - S_1a_n\| \\
&\le (1-\alpha_n)\|y_n - b_n\| + \delta_1\alpha_n \|x_n - a_n\| + \alpha_n\varepsilon_1
\end{aligned} \tag{24}$$

Substituting (24) and (23) in (22), we have

$$\begin{aligned}
\|x_{n+1}-a_{n+1}\| &\le \delta_1\delta_2 \|z_n - c_n\| + \delta_2\varepsilon_1 + \varepsilon_2 \\
&\le \delta_1\delta_2^2 \|w_n - d_n\| + \delta_1\delta_2\varepsilon_2 + \delta_2\varepsilon_1 + \varepsilon_2 \\
&\le \delta_1\delta_2^2(1-\alpha_n)\|y_n - b_n\| + \alpha_n\delta_1^2\delta_2^2 \|x_n - a_n\| \\
&+ \delta_1\delta_2^2\alpha_n\varepsilon_1 + \delta_1\delta_2\varepsilon_2 + \delta_2\varepsilon_1 + \varepsilon_2
\end{aligned} \tag{25}$$

By doing calculations similar to the above inequalities, we obtain

$$\begin{aligned}
\|y_{n+1}-b_{n+1}\| &\le \delta_1\delta_2 \|u_n - k_n\| + \delta_1\varepsilon_2 + \varepsilon_1 \\
&\le \delta_1^2\delta_2 \|v_n - h_n\| + \delta_1\delta_2\varepsilon_1 + \delta_1\varepsilon_2 + \varepsilon_1 \\
&\le (1-\alpha_n)\delta_1^2\delta_2 \|x_n - a_n\| + \alpha_n\delta_1^2\delta_2^2 \|y_n - b_n\| + \delta_1\delta_2\varepsilon_1 + \delta_1\varepsilon_2 + \varepsilon_1 + \alpha_n\delta_1^2\delta_2\varepsilon_2
\end{aligned} \tag{26}$$

There exists a real number $\delta \in (0,1)$ such that $1-\delta = (\delta_1 \cdot \delta_2) < 1$. Hence, we have

$$\begin{aligned}
&\|x_{n+1}-a_{n+1}\| + \|y_{n+1}-b_{n+1}\| \\
&\le (1-\delta)\left[\|x_n - a_n\| + \|y_n - b_n\|\right] + \delta\frac{(1+\delta_1\delta_2+\alpha_n\delta_1\delta_2^2)\varepsilon_1 + (1+\delta_1\delta_2+\alpha_n\delta_1^2\delta_2)\varepsilon_2}{\delta}
\end{aligned} \tag{27}$$

Denote that

$$\begin{aligned}
u_n &= \|x_n - a_n\| + \|y_n - b_n\| \\
\mu_n &= \delta \in (0,1) \\
\eta_n &= \frac{(1+\delta_1\delta_2+\alpha_n\delta_1\delta_2^2)\varepsilon_1 + (1+\delta_1\delta_2+\alpha_n\delta_1^2\delta_2)\varepsilon_2}{\delta}.
\end{aligned}$$

It is now easy to check that (27) satisfies all the requirements of Lemma 1. □

3 Conclusion

In this work, we investigate some fixed point theorems such as convergence and data dependency by using Iteration Method 5 for Lipschitz continuous mappings. We have also provided a numerical example to support the Theorem 1.

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Majorization Properties for a Subclass of Analytic Function of Complex Order

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Abstract: In this study, we investigate several majorization results for a subordination class of meromorphic functions of complex order in the punctured unit disk $\overset{\circ}{\mathcal{U}} = \{z : 0 < |z| < 1\} = \mathcal{U}/\{0\}$, defined by $q-$differential operator. Many different subclasses of analytic and meromorphic functions using the $q-$differential operator have already been defined and studied from various perspectives. Moreover, we point out some new or known consequences of our result, which is in the form of corollaries.

Keywords: Analytic functions, Majorization problem, Starlike functions of complex order, Subordination, $q-$differential operator.

1 Introduction and Definitions

Let $\mathcal{M}$ represent the class of meromorphic funtions f in the form of

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the punctured disc $\overset{\circ}{\mathcal{U}} = \{z : 0 < |z| < 1\} = \mathcal{U}/\{0\}$, where $\mathcal{U} = \overset{\circ}{\mathcal{U}} \cup \{0\}$. For the two functions $f(z)$ and $g(z)$ belonging to the class $\mathcal{M}$, there exists a Schwartz function w, which is analytic in $\mathcal{U}$ with $|w(z)| \leq |z|$ and $w(0) = 0$, such that $f(z) = g(w(z))$, and the function f is subordinate to g, written as $f \prec g$. The following relationship holds if g is univalent:

$$f \prec g \Leftrightarrow f(0) = g(0)\,, \text{ and } f\left(\overset{\circ}{\mathcal{U}}\right) \subseteq g\left(\overset{\circ}{\mathcal{U}}\right). \tag{2}$$

Because of its use in a variety of mathematical sciences, the study of $q-$calculus (quantum calculus) has fascinated and motivated many scholars. One of the primary contributors among all the mathematicians who introduced the concept of $q-$calculus theory was Jackson [12, 13]. The formulation of this concept is widely used to investigate the natüre of different structures of function theory, such as $q-$calculus was used in other branches of mathematics.
Though the authors of the first article [11] discussed the geometric nature $q-$starlike functions, Srivastava [23] laid a solid foundation fort he use of $q-$calculus in the context of function theory. Also, in [22], Srivastava provided a brief overview of basic or $q-$calculus operators and fractional $q-$calculus operators, as well as their applications in the geometric function theory of complex analysis. Later, the authors [1, 4, 5] investigated a number of useful properties for the newly defined $q-$linear differential operator, and Mehmood and Sokol [17] discussed the Ruscheweyh $q-$differential aperator, while Srivastava et al. [24] introduced a generalized operator for meromorphic harmonic functions by using the idea of convolution.
Let $0 < q < 1$. For any nonnegative integer n, the $q-$integer number n is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n+1},\ [0]_q = 0.$$

In general, we will denote

$$[\delta]_q = \frac{1-q^\delta}{1-q},$$

for a noninteger number δ. Also, the $q-$number shifted factorial is defined by

$$[n]_q! = [n]_q[n-1]_q...[2]_q[1]_q, [0]_q! = 1.$$

Clearly,

$$\lim_{q\to 1^-} [n]_q = n \quad \text{and} \quad \lim_{q\to 1^-} [n]_q! = n!.$$

Let $a, q \in \mathbb{C}\,(|q|<1)$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then the $q-$shifted factorial $(a;q)_n$ is defined by

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{j=1}^{n}\left(1 - aq^{j-1}\right),\ n \in \mathbb{N}.$$

Let $x \in \mathbb{C} - \{-n : n \in \mathbb{N}_0\}$. Then $q-$gamma function is as follows:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0<q<1.$$

In a subset of $\mathbb{C}$, the $q-$derivative (or $q-$difference) operator $D_q f$ of function f is defined by

$$(D_q f)(z) = \left\{ \begin{array}{ll} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0 \\ f'(0), & z = 0 \end{array} \right\} \tag{3}$$

provided that $f'(0)$ exists. We can easily observe from the definition of (3) that $(D_q f)(z)_{\lim_{q\to 1^-}} = f'(z)$.
Suppose that $q \in [0,1]$, then $q-$analog derivative of f as

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)},\ (z \in \mathcal{U}),$$

$$(D_q f)(z) = -\frac{1}{qz^2} + \sum_{n=1}^{\infty} [n]_q a_n z^n.$$

In 1967, Mac Gregor [16] introduced the Notion of majorization as follows.

Definition 1. *Let complex-valued functions f and g be analytic in $\mathcal{U}$. We say that f is majorized by g in $\mathcal{U}$ and write*

$$f(z) \ll g(z)\ \ (z \in \mathcal{U}) \tag{4}$$

if there exists a function $\varphi(z)$ (complex-valued function in $\mathcal{U}$) satisfying

$$|\varphi(z)| \leq 1 \text{ and } f(z) = \varphi(z)\, g(z)\ \ (z \in \mathcal{U}). \tag{5}$$

Majorization (4) is closely related to the concept of quasi-subordination between analytic functions in $\mathcal{U}$. Several researchers have published articles on this topic; for example, Tang et al. [27] gave the concept of majorization for subclasses of starlike functions based on the sine and cosine functions, Arif et al. [6] discussed majorization for various new defined classes, Cho et al. [8] obtained coefficient estimates for majorization, and Tang and Deng [26] defined the majorization problem connected with Liu-Owa integral operator and exponential function. This concept is also defined for $p-$valent function by Altıntaş and Srivastava [2] and for complex order by Altıntaş et al. [3].
The basic goal of this article is to examine and explain the idea of majorization in the context of the meromorphic function. Many researchers have shown their interest in this site. Goyal and Goswami, Dhuria and Mathur [9, 10] studied this concept for majorization for meromorphic function with the integral operator, Tang et al. [27] discussed it for meromorphic sin and cosine functions, Bulut et al, Tang et al, and Janani [7, 14, 25] explained this concept for meromorphic multivalent functions, Rasheed et al. [20] investigated a majorization problem for the class of meromorphic spiral-like functions related with a convolution operator, and Panigrahi and El-Ashwah [19] defined majorization for subclasses of multivalent meromorphic functions through iterations and combinations of the Liu–Srivastava operator and Cho–Kwon–Srivastava operator and much more. In addition, there are several other articles on this topic [10].
Here is the definition of our main function.

Definition 2. *A function $f(z) \in \mathcal{M}$ is said to be in the class $\mathcal{MS}^q(\gamma)$ of meromorphic functions of complex order $\gamma \neq 0$ in $\overset{\circ}{\mathcal{U}}$, if*

$$1 - \frac{1}{\gamma}\left[\frac{zqD_q f(z)}{f(z)} + 1\right] \prec \psi(z).$$

Now, we are going to choose a particular function instead of $\psi(z)$. This choice is

$$\psi(z) = \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,$$

and by applying the above-mentioned concept, we now consider the following class:

$$\mathcal{MS}^q_{\mathcal{J}}(\gamma) = \left\{ f(z) \in \mathcal{M} : 1 - \frac{1}{\gamma}\left[\frac{zD_q f(z)}{f(z)} + 1\right] \prec \frac{1+Az}{1+Bz} \right\}.$$

This class is related with well-known the Janowski class [15]. In the present work, we discussed majorization problem for the above-defined class of $\mathcal{MS}^q_{\mathcal{J}}(\gamma)$.

2 Main Results

We state the following $q-$analogue of the result given by Nehari [18] and Salvakumaran et al. [21].

Lemma 1 (See [28]). *If the function $\varphi(z)$ is analytic and $|\varphi(z)| < 1$ in $\mathcal{U}$, then*

$$|D_q\varphi(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}. \tag{6}$$

Theorem 1. *Let $-1 \leq B < A \leq 1$, the function $f(z) \in \mathcal{M}$ and suppose $g \in \mathcal{MS}^q_{\mathcal{J}}(\gamma)$ if $f(z)$ is majorized by$g(z)$ in $\overset{\circ}{\mathcal{U}}$, i.e.,*

$$f(z) \ll g(z).$$

Then, for $|z| \leq r_1$,

$$|qzD_q f(z)| \leq |qzD_q g(z)|, \tag{7}$$

where r_1 is the smallest positive root of the following equation:

$$\left(1-r^2\right)(1-|\gamma(A-B)+B|\,r) - 2rq(1+|B|\,r) = 0. \tag{8}$$

Proof: Since $g \in \mathcal{MS}^q_{\mathcal{J}}(\gamma)$, we readily obtained from definition (7) that

$$1-\frac{1}{\gamma}\left[\frac{zqD_q g(z)}{g(z)}+1\right] \prec \psi(z),$$

$z \in \overset{\circ}{\mathcal{U}}$ and

$$\psi(z) = \frac{1+Az}{1+Bz}.$$

By Lemma 1, there exists a bounded analytic function w in $\mathcal{U}$ and

$$1-\frac{1}{\gamma}\left[\frac{zqD_q g(z)}{g(z)}+1\right] = \frac{1+Aw(z)}{1+Bw(z)}, \tag{9}$$

with $w(\infty) = \infty$. From (9), we obtain

$$\frac{zqD_q g(z)}{g(z)} = -\frac{[\gamma(A-B)+B]\,w(z)+1}{1+Bw(z)}, \tag{10}$$

where $w(z)$ is the well-known class of bounded anlytic functions in $\mathcal{U}$ such that

$$|w(z)| \leq |z| \quad (z \in U). \tag{11}$$

From (10) and making use of (11), we obtain

$$|g(z)| \leq \frac{1+|B|\,|z|}{1-|\gamma(A-B)+B|\,|z|}\,|zqD_q g(z)|. \tag{12}$$

Since $f(z)$ is majorized by $g(z)$ in $\overset{\circ}{\mathcal{U}}$, from (5),

$$f(z) = \varphi(z)\, g(z).$$

By applying $q-$derivative on the previous equation write z as in [28] and then multiplying by qz, we have

$$qzD_q f(z) = qzD_q\varphi(z)\, g(z) + qz\varphi(z)\, D_q g(z) = qzD_q g(z)\left[\varphi(z) + \frac{D_q\varphi(z)\, g(z)}{D_q g(z)}\right]. \tag{13}$$

Noting that $\varphi(z)$ is the Schwartz function, so $\mathrm{Re}(\varphi(z)) > 0$ in $\overset{\circ}{\mathcal{U}}$, $\varphi(z) \neq 0$ for all $z \in \overset{\circ}{\mathcal{U}}$, satisfies the $q-$analogue result given by [18] proved in Lemma 1.

Now, using (12) and (6) in (13), we have

$$|qzD_qf(z)| \le |qzD_qg(z)| \left[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \frac{rq(1+|B||z|)}{1-|\gamma(A-B)+B||z|} \right].$$

Let us take $|z| = r < 1$ and $|\varphi(z)| = \zeta, (0 \le \zeta \le 1)$; we obtain

$$|qzD_qf(z)| \le Y(r,\zeta) |qzD_qg(z)|.$$

We define

$$Y(r,\zeta) = \zeta + \frac{rq\left(1-\zeta^2\right)(1+|B|r)}{(1-r^2)(1-|\gamma(A-B)+B|r)}, \ (0 \le \zeta \le 1, \ 0 < r < 1).$$

To determine r_1, it is sufficient to choose

$$r_1 = \max\{r \in [0,1) : Y(r,\zeta) \le 1, \ \forall\zeta \in [0,1]\},$$

equivalently,

$$r_1 = \max\left\{r \in [0,1) : Y^*(r,\zeta) \ge 0, \ \forall\zeta \in [0,1]\right\},$$

where

$$Y^*(r,\zeta) = \left(1-r^2\right)(1-|\gamma(A-B)+B|r) - rq(1+\zeta)(1+|B|r).$$

Clearly, when $\zeta = 1$, the above function $Y^*(r,\zeta)$ assumes its minimum value, namely,

$$\min\left\{Y^*(r,\zeta) : \zeta \in [0,1]\right\} = Y^*(r,1) = \psi^*(r),$$

where

$$\psi^*(r) = \left(1-r^2\right)(1-|\gamma(A-B)+B|r) - 2rq(1+|B|r).$$

Next, we obtained the following inequalities:

$$\psi^*(0) = 1 > 0 \text{ and } \psi^*(1) = -2q(1+|B|) < 0,$$

there exists r_1 such that $\psi^*(r) \ge 0$ for all $r \in [0, r_1]$, where r_1 is the smallest positive root of (8). The proof of Theorem 1 is completed. □

Putting $A = 1$ and $B = -1$ in Theorem, we get the following result.

Corollary 1. *Let the function $f(z) \in \mathcal{M}$ and suppose $g \in \mathcal{MS}_{\mathcal{J}}^q(\gamma)$ if $f(z)$ is majorized by $g(z)$ in $\overset{\circ}{\mathcal{U}}$, i.e.,*

$$f(z) \ll g(z).$$

Then, for $|z| \le r_2$,

$$|qzD_qf(z)| \le |qzD_qg(z)|,$$

where r_2 is the smallest positive root of the following equation:

$$(1-r)(1-|2\gamma-1|r) - 2rq = 0.$$

3 Conclusion

By making use of $q-$differential operators, many distinct subclasses of analytic and meromorphic functions have already been defined and investigated from numerous perspectives. The object of this paper is to investigate a majorization problem for a certain subclass of meromorphic functions defined in the punctured unit disk $\overset{\circ}{\mathcal{U}} = \{z : 0 < |z| < 1\} = \mathcal{U}/\{0\}$, defined by $q-$differential operator. In recent years, many authors have studied and investigated majorization results for different subclasses of analytic functions. Researchers can investigate majorization problems by defining well-known or different new subclasses.

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Fekete-Szegö Problem for Certain Subclasses of Analytic Functions Associated with The Combination of Differential Operators

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Abstract: In this work, we introduce and study some new subclasses $\mathcal{S}_{\eta}^{m}(\vartheta,\varsigma)$ and $\mathcal{C}_{\eta}^{m}(\vartheta,\varsigma)$ of the class of analytic functions defined by the combination of Noor integral operator and Deniz-Özkan differential operator in the open unit disk $\mathcal{U}=\{\sigma\in\mathbb{C}:|\sigma|<1\}$, and obtain coefficient estimates and Fekete-Szegö inequalities for these new subclasses.

Keywords: Analytic functions, Fekete-Szegö problem, Noor and Deniz-Özkan differential operator, Starlike and convex functions of complex order.

1 Introduction

The class $\mathcal{A}$ is well-known family of analytic functions $\hbar$ of the form

$$\hbar(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}a_{\kappa}\sigma^{\kappa} \tag{1}$$

in the open unit disk $\mathcal{U}=\{\sigma\in\mathbb{C}:|\sigma|<1\}$. Also, let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$. It is common knowledge that for $\hbar\in\mathcal{S}$, $\left|a_3-a_2^2\right|\leq 1$. A traditional theorem of Fekete-Szegö [9] expresses that for $\hbar\in\mathcal{S}$ given by (1)

$$\left|a_3-\delta a_2^2\right|\leq\begin{cases}3-4\delta & \text{if } \delta\leq 0,\\ 1+2\exp\left(\frac{-2\delta}{1-\delta}\right) & \text{if } 0<\delta<1,\\ 4\delta-3 & \text{if } \delta\geq 1.\end{cases}$$

This inequality is sharp because there is a function in $\mathcal{S}$ that ensures equality for each δ. Pfluger [23] proved this inequality for the complex δ values as follows:

$$\left|a_3-\delta a_2^2\right|\leq 1+2\left|\exp\left(\frac{-2\delta}{1-\delta}\right)\right|.$$

Till now, a number of authors have sought to apply the forementioned inequality to broader classes of analytical functions.
The classes of starlike and convex functions of order α given by, respectively

$$\mathcal{S}^*(\alpha)=\left\{\hbar\in\mathcal{S}:\ \Re\left(\frac{\sigma\hbar'(\sigma)}{\hbar(\sigma)}\right)>\alpha,\quad 0\leq\alpha<1,\ \sigma\in\mathcal{U}\right\}$$

and

$$\mathcal{C}(\alpha)=\left\{\hbar\in\mathcal{S}:\ \Re\left(1+\frac{\sigma\hbar''(\sigma)}{\hbar'(\sigma)}\right)>\alpha,\quad 0\leq\alpha<1,\ \sigma\in\mathcal{U}\right\}.$$

In particular, the classes $\mathcal{S}^*(0)$ and $\mathcal{C}=\mathcal{C}(0)$ are the familiar classes of starlike and convex functions in $\mathcal{U}$, respectively. Nasr and Aouf [19], Wiatrowski [27], Nasr and Aouf [18] defined these classes for complex order α.

2 Material and Methods

Let $\hbar(\sigma)=\sigma+\sum\limits_{\kappa=2}^{\infty}a_\kappa\sigma^\kappa$ and $g(\sigma)=\sigma+\sum\limits_{\kappa=2}^{\infty}b_\kappa\sigma^\kappa$ be analytic functions in $\mathcal{U}$. The Hadamard product of $\hbar$ and g, denoted by $\hbar*g$ is defined by

$$(\hbar*g)(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}a_\kappa b_\kappa\sigma^\kappa=(g*\hbar)(\sigma)\quad(\sigma\in\mathcal{U}).$$

Deniz and Özkan [8] introduced the following differential operator for $\mathcal{T}_\eta^m\hbar$ as follows

$$\begin{aligned}
&\mathcal{T}_\eta^0\hbar(\sigma)=\hbar(\sigma)\\
&\mathcal{T}_\eta^1\hbar(\sigma)=\eta\sigma^3\hbar'''(\sigma)+(2\eta+1)\,\sigma^2\hbar''(\sigma)+\sigma\hbar'(\sigma)=\mathcal{T}_\eta\hbar(\sigma)\\
&\mathcal{T}_\eta^2\hbar(\sigma)=\mathcal{T}_\eta\left(\mathcal{T}_\eta^1\hbar(\sigma)\right)\\
&\vdots\\
&\mathcal{T}_\eta^m\hbar(\sigma)\ =\mathcal{T}_\eta\left(\mathcal{T}_\eta^{m-1}\hbar(\sigma)\right)\quad(m\in\mathbb{N}=\{1,2,3,\ldots\}),
\end{aligned}$$

where $\eta\geq0$. We note that

$$\mathcal{T}_\eta^m\hbar(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}\kappa^{2m}(\eta\,(\kappa-1)+1)^m a_\kappa\sigma^\kappa\ \ (m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}),\tag{2}$$

with $\mathcal{T}_\eta^m\hbar(0)=0$.
Denote by

$$\mathcal{R}^\varsigma:=\frac{\sigma}{(1-\sigma)^{\varsigma+1}}*\hbar(\sigma)\ \ (\varsigma\in\mathbb{N}_0).$$

Then implies that

$$\mathcal{R}^\varsigma\hbar(\sigma)=\frac{\sigma\left(\sigma^{\varsigma-1}\hbar(\sigma)\right)^{(\varsigma)}}{\varsigma!}\ \ (\varsigma\in\mathbb{N}_0).$$

The operator $\mathcal{R}^\varsigma\hbar$ is called Ruscheweyh derivative operator [25]. Noor [20] defined and investigated an integral operator $\mathcal{N}^\varsigma:\mathcal{A}\to\mathcal{A}$ analogous to $\mathcal{R}^\varsigma\hbar$ as follows.
Let $\hbar_\varsigma(\sigma)=\frac{\sigma}{(1-\sigma)^{\varsigma+1}},\ \ \varsigma\in\mathbb{N}_0$, and let $\hbar_\varsigma^{(-1)}$ be defined such that

$$\hbar_\varsigma(\sigma)*\hbar_\varsigma^{(-1)}(\sigma)=\frac{\sigma}{1-\sigma}.$$

Then

$$\mathcal{N}^\varsigma\hbar(\sigma)=\hbar_\varsigma^{(-1)}(\sigma)*\hbar(\sigma)=\left[\frac{\sigma}{(1-\sigma)^{\varsigma+1}}\right]^{(-1)}*\hbar(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}\frac{\Gamma(\varsigma+1)\kappa!}{\Gamma(\varsigma+\kappa)}a_\kappa\sigma^\kappa:=\zeta(\sigma).\tag{3}$$

For the function $\zeta(\sigma)$ given by (3), we define the following convolution operator:

$$\begin{aligned}
&\mathcal{K}_\eta^0\zeta(\sigma)=\zeta(\sigma),\\
&\mathcal{K}_\eta^1\zeta(\sigma)=\mathcal{K}_\eta\zeta(\sigma)=\eta\sigma^3\zeta'''(\sigma)+(2\eta+1)\,\sigma^2\zeta''(\sigma)+\sigma\zeta'(\sigma)\\
&\ =\sigma+\sum_{\kappa=2}^{\infty}\kappa^2\left(\eta\,(\kappa-1)+1\right)\frac{\Gamma(\varsigma+1)\kappa!}{\Gamma(\varsigma+\kappa)}a_\kappa\sigma^\kappa,\\
&\vdots\\
&\mathcal{K}_\eta^m\zeta(\sigma)=\mathcal{K}_\eta(\mathcal{K}_\eta^{m-1}\zeta(\sigma))\quad(m\in\mathbb{N}).
\end{aligned}$$

It can be easily seen that

$$\mathcal{K}_\eta^m\zeta(\sigma)=\sigma+\sum_{\kappa=2}^{\infty}\kappa^{2m}(\eta\,(\kappa-1)+1)^m\frac{\Gamma(\varsigma+1)\kappa!}{\Gamma(\varsigma+\kappa)}a_\kappa\sigma^\kappa,\tag{4}$$

where $m,\varsigma\in\mathbb{N}_0$ and $\eta\geq0$.
Here the letters m and ς are related to the Deniz-Özkan differential operator and the Noor integral operator, respectively.
We now define new subclasses of analytic functions using the operator $\mathcal{K}_\eta^m\zeta(\sigma)$, as follows.

Definition 1. *Let $\vartheta\in\mathbb{C}\backslash\{0\}$, and let $\hbar$ be an univalent function of the form (1). We say that $\hbar$ belongs to $\mathcal{S}_\eta^m(\vartheta,\varsigma)$ if*

$$\Re\left(1+\frac{1}{\vartheta}\left(\frac{\sigma\left(\mathcal{K}_\eta^m\zeta(\sigma)\right)'}{\mathcal{K}_\eta^m\zeta(\sigma)}-1\right)\right)>0\quad(m,\varsigma\in\mathbb{N}_0,\ \eta\geq0,\ \sigma\in\mathcal{U}),$$

where $\zeta(\sigma):=\mathcal{N}^\varsigma\hbar(\sigma)$ is given by (3).

Definition 2. *Let $\vartheta \in \mathbb{C}\setminus\{0\}$, and let $\hbar$ be an univalent function of the form* (1). *We say that $\hbar$ belongs to $\mathcal{C}_\eta^m(\vartheta,\varsigma)$ if*

$$\Re\left(1+\frac{1}{\vartheta}\frac{\sigma\left(\mathcal{K}_\eta^m\zeta(\sigma)\right)''}{\left(\mathcal{K}_\eta^m\zeta(\sigma)\right)'}\right)>0 \quad (m,\varsigma\in\mathbb{N}_0,\ \eta\geq 0,\ \sigma\in\mathcal{U}),$$

where $\zeta(\sigma):=\mathcal{N}^\varsigma\hbar(\sigma)$ is given by (3).

The following significant subclasses have been examined by numerous writers in earlier publications, taking precise values to the parameters ϑ,ς,η and ξ, for example, $\mathcal{S}_\eta^m(1,1)=\mathcal{S}_m^*(0,\eta)$, $\mathcal{C}_\eta^m(1,1)=\mathcal{C}_m(0,\eta)$ [8], $\mathcal{S}^0(1,\varsigma)=\mathcal{M}(n,0)$ Sokol and Bansal (see: [26]), $\mathcal{S}^0(\vartheta,\varsigma=n)=N_{(n)}^*$ Noor (see: [20]).
In fact, many authors have studied the Fekete-Szegö inequality for various a variety of sublasses of $\mathcal{A}$, the upper bound for $\left|a_3-\delta a_2^2\right|$ is studied by a variety of authors (see: [1, 2, 4, 5, 14–17]) and (see also recent research on this subject by [3, 6, 7, 10–13, 21, 22]).
We focus on the coefficient estimates and the Fekete-Szegö inequality for the subclasses $\mathcal{S}_\eta^m(\vartheta,\varsigma)$ and $\mathcal{C}_\eta^m(\vartheta,\varsigma)$ in this paper.

3 Results And Discussion

We denote by $\mathcal{P}$ a class of analytic function in $\mathcal{U}$ with $p(0)=1$ and $\Re p(\sigma)>0$. The following lemma is required to prove our main results.

Lemma 1 ([24]). *Let $p\in\mathcal{P}$ with $p(\sigma)=1+c_1\sigma+c_2\sigma^2+\cdots$, then $|c_n|\leq 2$, for $n\geq 1$. If $|c_1|=2$ then $p(\sigma)\equiv p_1(\sigma)=(1+\gamma_1\sigma)/(1-\gamma_1\sigma)$ with $\gamma_1=c_1/2$. Inversely, if $p(\sigma)\equiv p_1(\sigma)$. for some $|\gamma_1|=1$, then $c_1=2\gamma_1$ and $|c_1|=2$. Additionally, we have $\left|c_2-\frac{c_1^2}{2}\right|\leq 2-\frac{|c_1|^2}{2}$. Additionally, we have*

$$\left|c_2-\frac{c_1^2}{2}\right|\leq 2-\frac{|c_1|^2}{2}.$$

If $|c_1|<2$ and $\left|c_2-\frac{c_1^2}{2}\right|=2-\frac{|c_1|^2}{2}$, then $p(\sigma)\equiv p_2(\sigma)$, where

$$p_2(\sigma)=\frac{1+\sigma\frac{\gamma_2\sigma+\gamma_1}{1+\bar{\gamma}_1\gamma_2\sigma}}{1-\sigma\frac{\gamma_2\sigma+\gamma_1}{1+\bar{\gamma}_1\gamma_2\sigma}},$$

and $\gamma_1=c_1/2$, $\gamma_2=\frac{2c_2-c_1^2}{4-|c_1|^2}$. Conversely, if $p(\sigma)\equiv p_2(\sigma)$ for some $|\gamma_1|<1$ and $|\gamma_2|=1$ then $\gamma_1=c_1/2$, $\gamma_2=\frac{2c_2-c_1^2}{4-|c_1|^2}$ and $\left|c_2-\frac{c_1^2}{2}\right|2-\frac{|c_1|^2}{2}$.

Then, we present the result that follows.

Theorem 1. *Let $m,\varsigma\in\mathbb{N}_0$, $\eta\geq 0$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $\mathcal{S}_\eta^m(\vartheta,\varsigma)$, then*

$$|a_2|\leq\frac{|\vartheta|(\varsigma+1)}{[4(\eta+1)]^m}, \tag{5}$$

$$|a_3|\leq\frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6[9(2\eta+1)]^m}\max\{1,|1+2\vartheta|\}. \tag{6}$$

Consider the functions

$$\frac{\sigma\left(\mathcal{K}_\eta^m\zeta(\sigma)\right)'}{\mathcal{K}_\eta^m\zeta(\sigma)}=1+\vartheta\left(p_1(\sigma)-1\right), \tag{7}$$

$$\frac{\sigma\left(\mathcal{K}_\eta^m\zeta(\sigma)\right)'}{\mathcal{K}_\eta^m\zeta(\sigma)}=1+\vartheta\left(p_2(\sigma)-1\right) \tag{8}$$

where p_1 and p_2 are given in Lemma 1. In equalities (5) and (6) are satisfied for the functions (7) and (8), respectively.

Proof: Denote $\mathcal{K}_\eta^m\zeta(\sigma)=\sigma+\Delta_2\sigma^2+\Delta_3\sigma^3+\cdots$, then

$$\Delta_2=\frac{2[4(\eta+1)]^m}{\varsigma+1}a_2 \text{ and } \Delta_3=\frac{6[9(2\eta+1)]^m}{(\varsigma+1)(\varsigma+2)}a_3. \tag{9}$$

We get by equating the coefficients of both sides

$$\Delta_2=\vartheta c_1 \text{ and } \Delta_3=\frac{\vartheta^2c_1^2}{2}+\frac{\vartheta c_2}{2}, \tag{10}$$

so that, on account of (9) and (10)

$$a_2=\frac{\vartheta c_1(\varsigma+1)}{2[4(\eta+1)]^m} \text{ and } a_3=\frac{\vartheta\left(\vartheta c_1^2+c_2\right)(\varsigma+1)(\varsigma+2)}{12[9(2\eta+1)]^m}. \tag{11}$$

Taking (11) and Lemma 1 into account, we get

$$|a_2| = \left|\frac{\vartheta c_1 (\varsigma+1)}{2[4(\eta+1)]^m}\right| \frac{|\vartheta|(\varsigma+1)}{[4(\eta+1)]^m}, \tag{12}$$

and

$$\begin{aligned} |a_3| &= \left|\frac{\vartheta(\varsigma+1)(\varsigma+2)}{12[9(2\eta+1)]^m}\left[c_2 - \frac{c_1^2}{2} + \frac{(1+2\vartheta)c_1^2}{2}\right]\right| \\ &\le \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{12[9(2\eta+1)]^m}\left[2 - \frac{|c_1|^2}{2} + \frac{|1+2\vartheta||c_1|^2}{2}\right] \\ &= \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{12[9(2\eta+1)]^m}\left[2 + \frac{(|1+2\vartheta|-1)|c_1|^2}{2}\right] \\ &\le \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6[9(2\eta+1)]^m}\max\{1, [1+|1+2\vartheta|-1]\}. \end{aligned}$$

Thus, we have

$$|a_3| \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6[9(2\eta+1)]^m}\max\{1, |1+2\vartheta|\}.$$

We can now calculate the sharpness of the estimates in (5) and (6).
Firstly, in (5) the equality holds if $c_1 = 2$. Alternatively, we have $p(\sigma) \equiv p_1(\sigma) = (1+\sigma)/(1-\sigma)$.
As a result, the extremal function in $\mathcal{S}_\eta^m(\vartheta,\varsigma)$ is given by

$$\frac{\sigma\left(\mathcal{K}_\eta^m \zeta(\sigma)\right)'}{\mathcal{K}_\eta^m \zeta(\sigma)} = \frac{1+(2b-1)\sigma}{1-\sigma}. \tag{13}$$

Next, in (6), for first case, the equality holds if $c_1 = c_2 = 2$. Therefore, the extremal functions in $\mathcal{S}_\eta^m(\vartheta,\varsigma)$ is given by (13) and for second case, the equality holds if $c_1 = 0,\ c_2 = 2$. Equivalently, we have $p(\sigma) \equiv p_2(\sigma) = (1+\sigma^2)/(1-\sigma^2)$. Therefore, the extremal function in $\mathcal{S}_\eta^m(\vartheta,\varsigma)$ is given by

$$\frac{\sigma\left(\mathcal{K}_\eta^m \zeta(\sigma)\right)'}{\mathcal{K}_\eta^m \zeta(\sigma)} = \frac{1+(2\vartheta-1)\sigma^2}{1-\sigma^2}.$$

□

Putting $\varsigma = 1$ in Theorem 1, we get the following result.

Corollary 1. *Let $m \in \mathbb{N}_0, \eta \ge 0$ and $\vartheta \in \mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $\mathcal{S}_\eta^m(\vartheta,1)$ then*

$$|a_2| \le \frac{2|\vartheta|}{[4(\eta+1)]^m},$$

and

$$|a_3| \le \frac{|\vartheta|}{[9(2\eta+1)]^m}\max\{1, |1+2\vartheta|\}.$$

Firstly, we think functional $\left|a_3 - \delta a_2^2\right|$ for $\vartheta \in \mathbb{C}\setminus\{0\}$ and $\delta \in \mathbb{C}$.

Theorem 2. *Let $m,\varsigma \in \mathbb{N}_0$, $\eta \ge 0$, $\vartheta \in \mathbb{C}\setminus\{0\}$ and $\hbar \in \mathcal{S}_\eta^m(\vartheta,\varsigma)$. Then for $\delta \in \mathbb{C}$*

$$\left|a_3 - \delta a_2^2\right| \le \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{6[9(2\eta+1)]^m}\max\left\{1, \left|1+2\vartheta - \frac{6\delta\vartheta(\varsigma+1)[9(2\eta+1)]^m}{(\varsigma+2)[(4(\eta+1))]^{2m}}\right|\right\}.$$

There is a function $\mathcal{S}_\eta^m(\vartheta,\varsigma)$ that ensures equality for each δ.

Proof: From (11), we have

$$
\begin{aligned}
a_3-\delta a_2^2 &= \frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{12{\left[9\left(2\eta+1\right)\right]}^m}\left[c_2+\vartheta c_1^2\right]-\delta\frac{\vartheta^2c_1^2{(\varsigma+1)}^2}{4{\left[4\left(\eta+1\right)\right]}^{2m}}\\
&= \frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{12{\left[9\left(2\eta+1\right)\right]}^m}\left[c_2+\vartheta c_1^2-\frac{3\delta\vartheta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}c_1^2\right]\\
&= \frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{12{\left[9\left(2\eta+1\right)\right]}^m}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(1+2\vartheta-\frac{6\delta\vartheta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}\right)\right].
\end{aligned}
$$

Then, with the help of Lemma 1, we get

$$
\begin{aligned}
\left|a_3-\delta a_2^2\right| &\le \frac{\left|\vartheta\right|\left(\varsigma+1\right)\left(\varsigma+2\right)}{12{\left[9\left(2\eta+1\right)\right]}^m}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left|1+2\vartheta-\frac{6\delta\vartheta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}\right|\right]\\
&= \frac{\left|\vartheta\right|\left(\varsigma+1\right)\left(\varsigma+2\right)}{12{\left[9\left(2\eta+1\right)\right]}^m}\left[2+\frac{\left|c_1^2\right|}{2}\left(\left|1+2\vartheta-\frac{6\delta\vartheta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}\right|-1\right)\right]\\
&\le \frac{\left|\vartheta\right|\left(\varsigma+1\right)\left(\varsigma+2\right)}{6{\left[9\left(2\eta+1\right)\right]}^m}\max\left\{1,\left|1+2\vartheta-\frac{6\delta\vartheta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}\right|\right\}.
\end{aligned}
$$

□

For $\varsigma=1$ in Theorem 2, we get the following result.

Corollary 2. *Let $m\in\mathbb{N}_0$, $\eta\ge0$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $\mathcal{S}_\eta^m\left(\vartheta,1\right)$, then for $\delta\in\mathbb{C}$*

$$
\left|a_3-\delta a_2^2\right|\le\frac{\left|\vartheta\right|}{{\left[9\left(2\eta+1\right)\right]}^m}\max\left\{1,\left|1+2\vartheta-\frac{4\delta\vartheta{\left[9\left(2\eta+1\right)\right]}^m}{{\left[4\left(\eta+1\right)\right]}^{2m}}\right|\right\}.
$$

We consider the case where δ and ϑ are real. Then we have:

Theorem 3. *Let $m,\varsigma\in\mathbb{N}_0$, $\eta\ge0$, $\vartheta>0$ and $\hbar\in\mathcal{S}_\eta^m\left(\vartheta,\varsigma\right)$. Then for $\delta\in\mathbb{R}$ we have*

$$
\left|a_3-\delta a_2^2\right|\le\begin{cases}\frac{\vartheta(\varsigma+1)(\varsigma+2)}{6[9(2\eta+1)]^m}\left[1+2\vartheta\left(1-\frac{3\delta(\varsigma+1)[9(2\eta+1)]^m}{(\varsigma+2)[9(2\eta+1)]^{2m}}\right)\right] & \textit{if}\quad \delta\le A\le B,\\ \frac{\vartheta(\varsigma+1)(\varsigma+2)}{6[9(2\eta+1)]^m} & \textit{if}\quad A<\delta<B,\\ \frac{\vartheta(\varsigma+1)(\varsigma+2)}{6[9(2\eta+1)]^m}\left[2\vartheta\left(\frac{3\delta(\varsigma+1)[9(2\eta+1)]^m}{(\varsigma+2)[4(\eta+1)]^{2m}}-1\right)-1\right] & \textit{if}\quad \delta\ge B,\end{cases}
$$

where $A=\frac{(\varsigma+2)[4(\eta+1)]^{2m}}{3(\varsigma+1)[9(2\eta+1)]^m}$ and $B=\frac{(1+2\vartheta)(\varsigma+2)[4(\eta+1)]^{2m}}{6\vartheta(\varsigma+1)[9(2\eta+1)]^m}$. There is a function $\mathcal{S}_\eta^m\left(\vartheta,\varsigma\right)$ such that equality holds for each δ.

Proof: First, let $\delta\le\frac{(\varsigma+2)[4(\eta+1)]^{2m}}{3(\varsigma+1)[9(2\eta+1)]^m}\le\frac{(1+2\vartheta)(\varsigma+2)[4(\eta+1)]^{2m}}{6\vartheta(\varsigma+1)[9(2\eta+1)]^m}$. In this case, (11) and Lemma 1 give

$$
\begin{aligned}
\left|a_3-\delta a_2^2\right| &\le \frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{12{\left[9\left(2\eta+1\right)\right]}^m}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left(1+2\vartheta-\frac{6\delta\vartheta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}\right)\right]\\
&\le \frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{6{\left[9\left(2\eta+1\right)\right]}^m}\left[1+2\vartheta\left(1-\frac{3\delta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}\right)\right].
\end{aligned}
$$

Now, let $\frac{(\varsigma+2)[4(\eta+1)]^{2m}}{3(\varsigma+1)[9(2\eta+1)]^m}<\delta<\frac{(1+2\vartheta)(\varsigma+2)[4(\eta+1)]^{2m}}{6\vartheta(\varsigma+1)[9(2\eta+1)]^m}$. Then, using the above calculations, we obtain

$$
\left|a_3-\delta a_2^2\right|\le\frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{6{\left[9\left(2\eta+1\right)\right]}^m}.
$$

Finally, if $\delta\ge\frac{(1+2\vartheta)(\varsigma+2)[4(\eta+1)]^{2m}}{6\vartheta(\varsigma+1)[9(2\eta+1)]^m}$, then

$$
\begin{aligned}
\left|a_3-\delta a_2^2\right| &\le \frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{12{\left[9\left(2\eta+1\right)\right]}^m}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left(\frac{6\delta\vartheta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}-2\vartheta-1\right)\right]\\
&\le \frac{\vartheta\left(\varsigma+1\right)\left(\varsigma+2\right)}{6{\left[9\left(2\eta+1\right)\right]}^m}\left[2\vartheta\left(\frac{3\delta\left(\varsigma+1\right)\left[9\left(2\eta+1\right)\right]^m}{\left(\varsigma+2\right)\left[4\left(\eta+1\right)\right]^{2m}}-1\right)-1\right].
\end{aligned}
$$

□

Taking $\varsigma = 1$ in Theorem 3, we get the following result.

Corollary 3. *Let $m \in \mathbb{N}_0$, $\eta \geq 0$ and $\vartheta > 0$. If $\hbar$ of the form (1) is in $\mathcal{S}_\eta^m(\vartheta, 1)$, then for $\delta \in \mathbb{R}$*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{\vartheta}{[9(2\eta+1)]^m}\left[1 + 2\vartheta\left(1 - \frac{2\delta[9(2\eta+1)]^m}{[4(\eta+1)]^{2m}}\right)\right] & \text{if} \quad \delta \leq A \leq B, \\ \frac{\vartheta}{[9(2\eta+1)]^m} & \text{if} \quad A < \delta < B, \\ \frac{\vartheta}{[9(2\eta+1)]^m}\left[2\vartheta\left(\frac{2\delta[9(2\eta+1)]^m}{[4(\eta+1)]^{2m}} - 1\right) - 1\right] & \text{if} \quad \delta \geq B, \end{cases}$$

where $A = \frac{[4(\eta+1)]^{2m}}{2[9(2\eta+1)]^m}$ and $B = \frac{(1+2\vartheta)[4(\eta+1)]^{2m}}{4\vartheta[9(2\eta+1)]^m}$.

We now get a solution of the Fekete-Szegö inequality and coefficients bounds of functions in $\mathcal{C}_\eta^m(\vartheta, \varsigma)$.

Theorem 4. *Let $m, \varsigma \in \mathbb{N}_0$, $\eta \geq 0$, $\delta \in \mathbb{C}$ and $\vartheta \in \mathbb{C} \setminus \{0\}$. If $\hbar$ of the form (1) is in $\mathcal{C}_\eta^m(\vartheta, \varsigma)$, then*

$$|a_2| \leq \frac{|\vartheta|(\varsigma+1)}{2[4(\eta+1)]^m}, \quad |a_3| \leq \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{18[9(2\eta+1)]^m} \max\{1, |1+2\vartheta|\}.$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{18[9(2\eta+1)]^m} \max\left\{1, \left|1 + 2\vartheta - \frac{9\delta\vartheta(\varsigma+1)[9(2\eta+1)]^m}{(\varsigma+2)[4(\eta+1)]^{2m}}\right|\right\}.$$

Proof: Denote $\mathcal{K}_\eta^m \zeta(\sigma) = \sigma + \Delta_2\sigma^2 + \Delta_3\sigma^3 + \cdots$, then

$$\Delta_2 = \frac{2[4(\eta+1)]^m}{\varsigma+1} a_2 \text{ and } \Delta_3 = \frac{6[9(2\eta+1)]^m}{(\varsigma+1)(\varsigma+2)} a_3. \tag{14}$$

According to the definition of the class $\mathcal{C}_\eta^m(\vartheta, \varsigma)$, there exists $p \in \mathcal{P}$ such that

$$\frac{\sigma\left(\mathcal{K}_\eta^m\zeta(\sigma)\right)''}{\left(\mathcal{K}_\eta^m\zeta(\sigma)\right)'} = 1 + \vartheta(p(\sigma) - 1),$$

so that

$$\frac{\sigma(2\Delta_2 + 6\Delta_3\sigma + \cdots)}{1 + 2\Delta_2\sigma + 3\Delta_3\sigma^2 + \cdots} = \vartheta\left(1 + c_1\sigma + c_2\sigma^2 + \cdots\right) - \vartheta.$$

We get by equating the coefficients of both sides

$$\Delta_2 = \frac{\vartheta c_1}{2} \text{ and } 6\Delta_3 - 4\Delta_2^2 = \vartheta c_2, \tag{15}$$

so that, on account of (14) and (15)

$$a_2 = \frac{\vartheta c_1(\varsigma+1)}{4[4(\eta+1)]^m} \text{ and } a_3 = \frac{\vartheta\left(\vartheta c_1^2 + c_2\right)(\varsigma+1)(\varsigma+2)}{36[9(2\eta+1)]^m}. \tag{16}$$

From (16) and Lemma 1, we get

$$|a_2| = \left|\frac{\vartheta c_1(\varsigma+1)}{4[4(\eta+1)]^m}\right| \frac{|\vartheta|(\varsigma+1)}{2[4(\eta+1)]^m}, \tag{17}$$

and

$$\begin{aligned} |a_3| &= \left|\frac{\vartheta(\varsigma+1)(\varsigma+2)}{36[9(2\eta+1)]^m}\left[c_2 - \frac{c_1^2}{2} + \frac{(1+2\vartheta)c_1^2}{2}\right]\right| \\ &\leq \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{36[9(2\eta+1)]^m}\left[2 - \frac{|c_1|^2}{2} + \frac{|1+2\vartheta||c_1|^2}{2}\right] \\ &= \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{36[9(2\eta+1)]^m}\left[2 + \frac{(|1+2\vartheta| - 1)|c_1|^2}{2}\right] \\ &\leq \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{18[9(2\eta+1)]^m} \max\{1, [1 + |1+2\vartheta| - 1]\}. \end{aligned}$$

Thus, we have

$$|a_3| \frac{|\vartheta|(\varsigma+1)(\varsigma+2)}{18[9(2\eta+1)]^m} \max\{1, |1+2\vartheta|\}.$$

Then, with the help of Lemma 1, we get

$$\begin{aligned}\left|a_3-\delta a_2^2\right| &\le \frac{|\vartheta|\,(\varsigma+1)\,(\varsigma+2)}{36[9\,(2\eta+1)]^m}\left[2-\frac{\left|c_1^2\right|}{2}+\frac{\left|c_1^2\right|}{2}\left|1+2\vartheta-\frac{9\delta\vartheta\,(\varsigma+1)\,[9\,(2\eta+1)]^m}{(\varsigma+2)\,[4\,(\eta+1)]^{2m}}\right|\right]\\ &= \frac{|\vartheta|\,(\varsigma+1)\,(\varsigma+2)}{36[9\,(2\eta+1)]^m}\left[2+\frac{\left|c_1^2\right|}{2}\left(\left|1+2\vartheta-\frac{9\delta\vartheta\,(\varsigma+1)\,[9\,(2\eta+1)]^m}{(\varsigma+2)\,[4\,(\eta+1)]^{2m}}\right|-1\right)\right]\\ &\le \frac{|\vartheta|\,(\varsigma+1)\,(\varsigma+2)}{18[9\,(2\eta+1)]^m}\max\left\{1,\left|1+2\vartheta-\frac{9\delta\vartheta\,(\varsigma+1)\,[9\,(2\eta+1)]^m}{(\varsigma+2)\,[4\,(\eta+1)]^{2m}}\right|\right\}.\end{aligned}$$

□

Putting $\varsigma=1$ in Theorem 4, we get the following result.

Corollary 4. *Let $m\in\mathbb{N}_0$, $\eta\ge 0$, $\delta\in\mathbb{C}$ and $\vartheta\in\mathbb{C}\setminus\{0\}$. If $\hbar$ of the form (1) is in $\mathcal{C}_\eta^m(\vartheta,1)$, then*

$$|a_2|\le\frac{|\vartheta|}{[4\,(\eta+1)]^m},\;\; |a_3|\le\frac{|\vartheta|}{3[9\,(2\eta+1)]^m}\max\{1,|1+2\vartheta|\}$$

and

$$\left|a_3-\delta a_2^2\right|\le\frac{|\vartheta|}{3[9\,(2\eta+1)]^m}\max\left\{1,\left|1+2\vartheta-\frac{6\delta\vartheta[9\,(2\eta+1)]^m}{[4\,(\eta+1)]^{2m}}\right|\right\}.$$

4 Conclusion

In our present study, we have introduced and studied the coefficient problems related with each of the two new subclasses $\mathcal{S}_\eta^m(\vartheta,\varsigma)$ and $\mathcal{C}_\eta^m(\vartheta,\varsigma)$ of the class of analytic functions defined by the combination of Deniz-Özkan differential and Noor integral operators in the open unit disk. We have studied some interesting results such as the Fekete-Szegö inequalities according to the case of δ. Also, for certain values of the parameters, we re-obtain some special classes studied earlier by various authors.

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Optimal points in b- metric spaces endowed with graph

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Abstract: In this paper, we focus on the best proximity point theorems to prove their uniqueness on a b - metric space endowed with a graph. We also furnish some numerical examples to support our claims. We derive fixed point result as a result of our observations. As an application of our main result, we find the solution of a nonlinear integral equation.

Keywords: best proximity point, fixed point, graphical b-metric space, weak P - property.

1 Section title

2 Introduction and Preliminaries

For a nonself mapping $g : E_1 \to E_2$, where E_1 and E_2 are nonempty subsets of a metric space $(\aleph, b')$, the idea of a fixed point is not appropriate when the intersection of E_1 and E_2 is empty. If a mapping g has a solution and intersection of E_1 and E_2 is nonempty, then g has a fixed point. In other words, if the fixed point equation $gd = d$, has no exact solution for nonself mappings, then it is fascinating to find an approximate solution d such that the error $b'(d, gd)$ is minimum. In view of the fact that $b'(d, gd) \geq b'(E_1, E_2)$ for all $d \in E_1$, an optimal approximate solution is an element d for which the error $b'(d, gd)$ attains the least possible value $b'(E_1, E_2)$. The existence of such point d, known as the best proximity point of a nonself mapping g, satisfying the condition

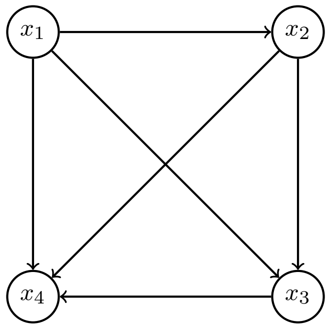
$$b'(d, gd) = b'(E_1, E_2) = \inf\{b'(d, f) : d \in E_1, f \in E_2\}.$$

In contemporary years, many authors studied the best proximity point problems in metric space or normed space (see [1–3, 6, 7] and references cited therein). Recently, the best proximity point and fixed point theory have crucial role in graph theory. Jachymski [11] considered metric spaces with the structure of a graph as a part where the symmetry condition is preserved in relation to the fixed point theory of contractive-type mappings. In 2017, Shukla et al. [14] gave the notion of graphical metric space in which the triangular inequality replace by weaker condition. Notably, the triangular inequality is fulfilled by only those points positioned on some path involved in graphical structure related with the space. In 2019, Chuensupantharat et al. [4] introduced the notion of a graphical b-metric which is generalization of b- metric spaces and prove some interesting results on fixed point theory appeared in the graphical b-metric space. Graph theory have various applications in allied sciences, such as computer science and engineering (see [4, 8, 12, 13, 15] and references cited therein).
Motivated by the importance of graph theory and its application, we focus on the best proximity point theorems in b - metric spaces endowed with a graph in this paper. We derive fixed point result as a result of our observations which appeared in the literature. We also furnish some numerical examples to support our claims. As an application of our main result, we find the solution of a nonlinear integral equation.
Let $E_1, E_2 \in CB(\aleph)$ where $CB(\aleph)$ be the families of all nonempty closed and bounded subsets of a b- metric space $(\aleph, b)$. Define

$$\begin{aligned}
&B(E_1, E_2) = \sup\{b(d, E_2) : d \in E_1\}, \text{ where}\\
&b(d, E_2) = \inf\{b(d, f) : f \in E_2\},\\
&E_{1_0} = \{d \in E_1 : \text{there exists some } f \in E_2 \text{ such that } b(d, f) = d(E_1, E_2)\},\\
&E_{2_0} = \{f \in E_2 : \text{there exists some } d \in E_1 \text{ such that } b(d, f) = d(E_1, E_2)\},\\
&H_b = \max\{B(E_1, E_2), B(E_2, E_1)\},
\end{aligned}$$

which is known as the Pompeiu- Hausdorff b- metric induced by b-metric.
We recall some basic concepts of graph theory which we will use later.
Let $(\aleph, b)$ be a graphical b- metric space and $\triangle = \aleph \times \aleph$. A graph G is determined by the given of a pair (V, E), where $V = V(G)$ is a

Fig. 1



set of vertices coinciding with $\aleph$ and $E = E(G)$ the set of its edges such that $\triangle \subset E(G)$. Additionally, we assume that graph G does not contain parallel edges. Let G^{-1} be the graph defined as follows: $E(G^{-1}) = \{(e, f) \in \aleph \times \aleph : (f, e) \in E(G)\}$ and $V(G^{-1}) = V(G)$. It is understandable that G^{-1} obtain from graph G by reversing the direction of its edges.
Czerwik [5], proposed a generalization of metric spaces by relaxing the triangle inequality in a way that allows the extension of fixed point theory to cover also these badly behaved function spaces. The resulting notion of b- metric spaces created a new direction in which fixed point theory could be developed. In 2019, Chuensupantharat et al. [4] introduced the notion of a graphical b-metric which is generalization of b-metric spaces as follows:

Definition 2.1. Let $\aleph$ be a nonempty set endowed with graph $G, l \geq 1$ and $b : \aleph \times \aleph \to [0, \infty)$ be a function satisfies the following conditions:

(B1) $b(d, f) \geq 0$;
(B2) $b(d, f) = 0$ if and only if $d = f$;
(B3) $b(d, f) = b(f, d)$;
(B4) $(dQf)_G, e \in (dQf)_G \Rightarrow b(d, f) \leq l[b(d, e) + b(e, f)]$;

for all $d, e, f \in \aleph$, $(dQf)_G$ represents a path leading from d to f in G and e is a vertex lying on the path $(dQf)_G$ in graph G. Then b is a graphical b- metric and the pair $(\aleph, b)$ is a called a graphical b- metric space.

Example 2.1. *Let $\aleph = \{1, 2, 3, 4\}$ and $b : \aleph \times \aleph$ be defined as:*

$$b(d, f) = \begin{cases} 0, \text{ if } d = f \\ 5x', \text{ if } d, f \in \{1, 3\} \text{ and } d \neq f \\ 2x', \text{ if } d \text{ or } f \notin \{1, 3\} \text{ and } d \neq f \end{cases}$$

where $x' > 0$ be a constant including graph $G(V(G), E(G)); V(G) = \aleph, E(G) = \aleph \times \aleph$. Then $(\aleph, b)$ is a $(\aleph, b)$ is a called a graphical b-metric space with $l = \frac{5}{4} > 1$ but not a graphical metric space as

$$b(1, 3) = 5x' > 4x' = b(1, 2) + b(2, 3).$$

Definition 2.2. Let (E_1, E_2) be a pair of nonempty subsets of a metric space $(\aleph, b')$ with $E_{1_0} \neq \emptyset$. Then the pair (E_1, E_2) is said to have weak P-property (see [9]) if and only if for any $d_1, d_2 \in E_1$ and $f_1, f_2 \in E_2$,

$$\begin{aligned} &b'(d_1, f_1) = b'(E_1, E_2), \\ &b'(d_2, f_2) = b'(E_1, E_2), \quad \text{then} \\ &b'(d_1, d_2) \leq b'(f_1, f_2). \end{aligned}$$

Definition 2.3. Let E_1, E_2 be two nonempty subsets of a b-metric space $(\aleph, b)$. The b-metric is called sequentially continuous [10] if every $d \in E_1$, $f \in E_2$ and every sequence $d_n \in E_1$, $f_n \in E_2$ such that $d_n \to d$, $f_n \to f$ we have $b(d_n, f_n) \to b(d, f)$.

Definition 2.4. Let e and f be two vertices in a graph G. A path in G from e to f of length k; $(k \in \mathbb{N} \cup \{0\})$ is a sequence $(d_i)_{i=1}^{k}$ of distinct vertices such that $d_0 = e$ $d_k = f$ and $(d_i, d_{i+1}) \in E(G)$ for $i = 1, 2, \cdots, k$. We denote

$$[e]_G^k = \{f \in \aleph : \text{ there is a path in } G \text{ of length } k \text{ from } e \text{ to } f\}.$$

If there is a path between any two vertices of a graph G, we say that G is connected.

3 Main results

Throughout the paper, we consider $(\aleph, b)$ to be a graphical b- metric space endowed with directed graph G. Additionally, we assume that graph G does not contain parallel edges such that $\aleph = V(G)$.

Definition 3.1. Let E_1 and E_2 be two nonempty subsets of a graphical b- metric space $(\aleph, b)$. A mapping $g : E_1 \to CB(E_2)$ is said to be G_b-contraction if for all $d, f \in E_1$ with $(d, f) \in E(G)$

(i) $H_b(gd, gf) \leq \frac{\delta}{l^2} b(d, f)$ for some $\delta \in [0, 1)$;
(ii) $b(d_1, w) = b(E_1, E_2)$ and $b(f_1, c) = b(E_1, E_2)$ then $d_1, f_1 \in E(G); c \in gd$ and $w \in gf$.

Theorem 1. Let $(\aleph, b)$ be a complete graphical b- metric space, E_1, E_2 be two nonempty closed subsets of $(\aleph, b)$ such that (E_1, E_2) has the weak P-property. Let $g : E_1 \to CB(E_2)$ be a continuous G_b-contraction such that $g(d) \subseteq E_{2_0}$ for each $d \in E_{1_0}$ and b be a sequentially continuous. Assume the following condition $(\mathfrak{A})$: d_0 and d_1 exist in E_{1_0} such that there is a path in E_{1_0} between them and $b(d_1, c_0) = b(E_1, E_2)$ where $c_0 \in gd_0 \subseteq E_{2_0}$. Then, there the sequence $\{d_n\} : n \in \mathbb{N}$ exists with $b(d_{n+1}, gd_n) = b(E_1, E_2)$ and g has a unique best proximity point.

Proof: two points d_0 and d_1 in E_{1_0} exist such that $b(d_1, c_0) = b(E_1, E_2)$, and a path $(e_0^i)_{i=0}^k$ in G between them exists such that the sequence $(e_0^i)_{i=0}^k$ contains points of E_{1_0}. Subsequently, $e_0^0 = d_0$, $e_0^k = d_1$ and $(e_0^{i-1}, e_0^i) \in E(G)$ for all $1 \leq i \leq k$. Given that $e_0^1 \in E_{1_0}$, $w_0^1 \in g(e_0^1) \subseteq E_{2_0}$ and from the definition of E_{1_0}, $e_1^1 \in E_{1_0}$ exists such that $b(e_1^1, w_0^1) = b(E_{1_0}, E_{2_0})$. By proceeding this way, for $i = 2 \cdots, k$, $e_1^i \in E_{1_0}$ exists such that $b(e_1^i, w_0^i) = b(E_{1_0}, E_{2_0})$. Since $(e_0^i)_{i=0}^k$ is a path in G, then $(e_0^0, e_0^1) = (d_0, e_0^1) \in E(G)$. From the above, we have $b(d_1, c_0) = b(E_{1_0}, E_{2_0})$ and $b(e_1^1, w_0^1) = b(E_{1_0}, E_{2_0})$. g is a G_b-contraction; consequently, $(d_1, e_1^1) \in E(G)$. In the same way, we obtain

$$(d_2, e_2^1) \in E(G) \text{ and } (e_2^{i-1}, e_2^i) \in E(G) \text{ for all } i = 1, 2, \cdots, k.$$

Let $d_3 = e_2^k$. Then, $(e_2^i)_{i=0}^k$ is a path from $d_2 = e_2^0$ and $d_3 = e_2^k$. By repeating this process, for all $n \in \mathbb{N}$, we create a path $(e_0^i)_{i=0}^k$ from $d_n = e_n^0$ and $d_{n+1} = e_n^k$, which gives us a sequence $\{d_n\}$ where $d_{n+1} \in [d_n]_G^k$, This shows that sequence $\{d_n\}$ is connected and $b(d_{n+1}, gd_n) = b(E_1, E_2)$ such that

$$b(e_{n+1}^i, e_n^i) = b(E_1, E_2) \text{ for all } i = 1, 2, \cdots, k. \tag{1}$$

By (1) and weak P-property we obtain

$$b(e_n^{i-1}, e_n^i) \leq b(w_{n-1}^{i-1}, w_{n-1}^i) \leq H_b(ge_{n-1}^{i-1}, ge_{n-1}^i) \text{ for all } i = 1, 2, \cdots, k.$$

Given that g is a G_b-contraction, for all $n \in \mathbb{N}$, $(e_{n-1}^{i-1}, e_{n-1}^i) \in E(G)$, we get

$$H_b(ge_{n-1}^{i-1}, ge_{n-1}^i) \leq \frac{\delta}{l^2} b(e_{n-1}^{i-1}, e_{n-1}^i) \text{ for all } i = 1, 2, \cdots, k.$$

Using induction, we get

$$b(e_n^{i-1}, e_n^i) \leq H_b(ge_{n-1}^{i-1}, ge_{n-1}^i) \leq \frac{\delta^n}{l^{2n}} b(e_0^{i-1}, e_0^i) \text{ for all } i = 1, 2, \cdots, k.$$

By the triangular inequality, we obtain

$$\begin{aligned}
b(d_{n+1}, d_n) =& b(e_n^0, e_n^k) \\
\leq& lb(e_n^0, e_n^1) + lb(e_n^1, e_n^k) \\
\leq& lb(e_n^0, e_n^1) + l[l(b(e_n^1, e_n^2) + b(e_n^2, e_n^k))] \\
=& lb(e_n^0, e_n^1) + l^2 b(e_n^1, e_n^2) + l^2 b(e_n^2, e_n^k) \\
\leq& lb(e_n^0, e_n^1) + l^2 b(e_n^1, e_n^2) + \cdots + l^k b(e_n^{k-1}, e_n^k) \\
\leq& lH_b(ge_{n-1}^0, ge_{n-1}^1) + l^2 H_b(ge_{n-1}^1, ge_{n-1}^2) + \cdots + l^k H_b(ge_{n-1}^{k-1}, ge_{n-1}^k) \\
\leq& l(\frac{\delta}{l^2}) b(e_{n-1}^0, e_{n-1}^1) + l^2 (\frac{\delta}{l^2}) b(e_{n-1}^1, e_{n-1}^2) + \cdots + l^k (\frac{\delta}{l^2}) b(e_{n-1}^{k-1}, e_{n-1}^k) \\
\leq& l \frac{\delta^n}{l^{2n}} b(e_0^0, e_0^1) + l^2 \frac{\delta^n}{l^{2n}} b(e_0^1, e_0^2) + \cdots + l^k \frac{\delta^n}{l^{2n}} b(e_0^{k-1}, e_0^k) \\
\leq& \frac{\delta^n}{l^{2n-1}} \left(b(e_0^0, e_0^1) + lb(e_0^1, e_0^2) + \cdots + l^{k-1} b(e_0^{k-1}, e_0^k) \right) \\
\leq& \frac{\delta^n}{l^{2n-1}} \beta,
\end{aligned} \tag{2}$$

where $\beta = \sum_{i=1}^k l^{i-1} b(e_0^{i-1}, ge_0^i)$.

Now, we claim that the sequence $\{d_n\}$ is a Cauchy. For each $m, n \in \mathbb{N}, m > n$ and by the triangular inequality we have

$$\begin{aligned}
b(d_n, d_m) \leq & l[b(d_n, d_{n+1}) + b(d_{n+1}, d_m)] \\
\leq & lb(d_n, d_{n+1}) + l[lb(d_{n+1}, d_{n+2}) + lb(d_{n+2}, d_m)] \\
= & lb(d_n, d_{n+1}) + l^2 b(d_{n+1}, d_{n+2}) + l^2 b(d_{n+2}, d_m) \\
\leq & lb(d_n, d_{n+1}) + l^2 b(d_{n+1}, d_{n+2}) + \cdots + l^{m-n} b(d_{m-1}, d_m) \\
\leq & l\left(\frac{\delta^n}{l^{2n-1}}\beta\right) + (l)^2\left(\frac{\delta^n}{l^{2(n+1)-1}}\beta\right) + \cdots + (l)^{m-n}\left(\frac{\delta^n}{l^{2(m-1)-1}}\beta\right) \\
\leq & \left(\frac{\delta^n}{l^{2n-2}}\beta\right)(1 + \frac{\delta}{l} + \cdots + \frac{\delta}{l}^{m-n-1} l^{m-n-1}) \\
\leq & \left(\frac{\delta^n}{l^{2n-2}}\beta\right)\sum_{i=1}^{\infty}\left(\frac{\delta}{l}\right)^{i-1} \\
\leq & \left(\frac{\delta^n}{l^{2n-2}}\beta\right)\left(\frac{l}{l-\delta}\right),
\end{aligned}$$

then $\lim_{m,n\to\infty} b(d_n, d_m) = 0$. Therefore, $\{d_n\}$ is a Cauchy sequence and there exits $d \in E_1$ such that $\lim_{n\to\infty} d_n = d$. Since g is a continuous we have $gd_n \to gd$ as $n \to \infty$. Also b is a sequentially continuous we get

$$b(d_{n+1}, gd_n) = b(E_1, E_2).$$

Taking $n \to \infty$, we obtain

$$b(d, gd) = b(E_1, E_2).$$

Hence d is a best proximity point of g. Suppose that d_1 and d_2 two best proximity point of g so that

$$\begin{aligned}
b(d_1, gd_1) = & b(E_1, E_2) \\
b(d_2, gd_2) = & b(E_1, E_2).
\end{aligned}$$

By weak P- property we have

$$b(d_1, d_2) \leq b(gd_1, gd_2) \leq \delta b(d_1, d_2).$$

This shows that $b(d_1, d_2) \leq \delta b(d_1, d_2); \delta < 1$, which is contradiction. This implies g has a unique best proximity point. □

Next, we prove a result for single valued mapping in a graphical b- metric space.

Definition 3.2. Let E_1 and E_2 be two nonempty subsets of a graphical b- metric space $(\aleph, b)$. A mapping $g : E_1 \to E_2$ is said to be G_b-contraction if for all $d, f \in E_1$ with $(d, f) \in E(G)$

(i) $b(gd, gf) \leq \frac{\delta}{l^2} b(d, f)$ for some $\delta \in [0, 1)$;
(ii) $b(d_1, gf) = b(E_1, E_2)$ and $b(f_1, gd) = b(E_1, E_2)$ then $d_1, f_1 \in E(G)$.

Theorem 2. Let $(\aleph, b)$ be a complete graphical b- metric space, E_1, E_2 be two nonempty closed subsets of $(\aleph, b)$ such that (E_1, E_2) has the weak P-property. Let $g : E_1 \to (E_2)$ be a continuous G_b-contraction such that $g(E_{1_0}) \subseteq E_{2_0}$ and b be a sequentially continuous. Assume the following condition $(\mathfrak{A})$: d_0 and d_1 exist in E_{1_0} such that there is a path in E_{1_0} between them and $b(d_1, gd_0) = b(E_1, E_2)$. Then, there the sequence $\{d_n\} : n \in \mathbb{N}$ exists with $b(d_{n+1}, gd_n) = b(E_1, E_2)$ and g has a unique best proximity point.

Example 3.1. *Consider* $\aleph = \mathbb{R}^2$ *with metric*

$$b(d, f) = \left|d_1' - f_1'\right|^3 + \left|d_2' - f_2'\right|^3,$$

for all $d = (d_1', d_2'), f = (f_1', f_2') \in \mathbb{R}^2$ *is a* b- *metric space with* $l = 4$. *Consider a graph* G *with* $V(G) = \aleph$ *and* $E(G) = \{(d, f) \in \aleph \times \aleph : b(d, f) < 2\}$. *Then* $(\aleph, b)$ *is a graphical* b- *metric space.*

Suppose $E_1 = \left\{(\frac{1}{2}, d_1') : 0 \leq d_1' \leq 1\right\}$ *and* $E_2 = \left\{(0, d_1') : 0 \leq d_1' \leq 1\right\}$, *such that* $b(E_1, E_2) = \frac{1}{8}$.

Define $g : E_1 \to E_2$ by

$$g(d_1') = \begin{cases} \{(0,1)\}, d_1' = (\frac{1}{2}, 1), \\ \left\{(0, \frac{u}{4}) : 0 \leq u \leq d_1'\right\}, \text{ otherwise} \end{cases}$$

for all $d_1' \in E_1$. If $e_1 = (\frac{1}{2}, d_1')$ and $e_2 = (\frac{1}{2}, d_2')$ in E_1, for $d_1', d_2' \in [0, \frac{1}{2}]$. Then

$$ge_1 = \left\{(0, \frac{d'}{4}) : 0 \leq d' \leq d_1'\right\},$$

and

$$ge_2 = \left\{(0, \frac{d'}{4}) : 0 \leq d' \leq d'_2\right\}.$$

We can see that g is a continuous map. Since $E_{1_0} = E_1$ and $E_{2_0} = E_2$ then $g(E_{1_0}) \subseteq E_{2_0}$ for each $d_1' \in E_{1_0}$. Also the pair (E_1, E_2) satisfies weak P - property. Next, we prove that g is G_b contraction. Let $\delta = \frac{1}{3}$. Take $e_1 = (\frac{1}{2}, d'_1)$ and $e_2 = (\frac{1}{2}, d'_2)$ in E_{1_0} where $0 \leq d_1', d_2' \leq \frac{1}{2}$. Consider

$$\begin{aligned} H_b(ge_1, ge_2) &= (0-0)^3 + \left(\frac{d_1'}{4} - \frac{d_2'}{4}\right)^3 \\ &= \frac{1}{64}(d_1' - d_2')^3 \\ &< \frac{1}{48}(d_1' - d_2')^2. \end{aligned}$$

It implies that

$$H_b(ge_1, ge_2) \leq \frac{\delta}{l^2} b(e_1, e_2),$$

for all $e_1, e_2 \in E_1$. Now, $e_1, e_2 \in E_1$ and $(h, i) \in E(G)$ such that

$$\begin{aligned} b(e_1, gh) &= b(E_1, E_2) = \frac{1}{8} \\ b(e_2, gi) &= b(E_1, E_2) = \frac{1}{8}. \end{aligned}$$

By the weak P- property we have $b(e_1, e_2) \leq b(gh, gi) < \delta b(h, i) < b(h, i)$. Since $(h, i) \in E(G)$ then $b(h, i) < 2$, which gives $b(e_1, e_2) < 2$ therefore $(e_1, e_2) \in E(G)$. Hence g is a G_b contraction. Let $d_0 = (\frac{1}{2}, \frac{1}{2})$, $d_1 = (\frac{1}{2}, \frac{1}{8})$ and $k = 1$. Since $b(d_0, d_1) = \frac{27}{512} < 2$, then the pair $(d_0, d_1) \in E(G)$. By definition of g we have, $gd_0 = (0, \frac{1}{8})$, and we obtain $b(d_1, gd_0) = \frac{1}{8} = b(E_1, E_2)$. Thus, condition $(\mathfrak{A})$ holds. All conditions of Theorem 1 are satisfied and g has a best proximity point $(\frac{1}{2}, 1)$.

Example 3.2. *Consider $\aleph = \mathbb{R}^2$ with metric*

$$b(d, f) = \left|d_1' - f_1'\right|^2 + \left|d_2' - f_2'\right|^2,$$

for all $d = (d_1', d_2'), f = (f_1', f_2') \in \mathbb{R}^2$ is a b- metric space. Consider a graph G with $V(G) = \aleph$ and $E(G) = \{(d, f) \in \aleph \times \aleph : b(d, f) < 49\}$. Then $(\aleph, b)$ is a graphical b- metric space with $l = 2$.

Suppose

$$E_1 = \left\{(d_1', d_2') : {d_1'}^2 + {d_2'}^2 = 3^2 \text{ and } d_2' \geq 0\right\}$$

and

$$E_2 = \left\{(d_1', d_2') : {d_1'}^2 + {d_2'}^2 = 1^2 \text{ and } d_2' \geq 0\right\}$$

, such that $b(E_1, E_2) = 4$.

Define $g : E_1 \to E_2$ by $g(d_1', d_2') = \frac{(d_1', d_2')}{3}$, for all $d_1', d_2' \in E_1$. We can see that g is a continuous map. Since $E_{1_0} = E_1$ and $E_{2_0} = E_2$ then $g(E_{1_0}) \subseteq E_{2_0}$ for each $d_1' \in E_{1_0}$. Also the pair (E_1, E_2) satisfies weak P - property. Next, we prove that g is G_b contraction. Let $\delta = \frac{1}{2}$.

Take $e_1 = (d'_1, f'_1)$ and $e_2 = (d'_2, f'_2)$ in E_{1_0}. Consider

$$\begin{aligned}
b(ge_1, ge_2) =& \left(\frac{d'_1}{3} - \frac{d'_2}{3}\right)^2 + \left(\frac{f'_1}{3} - \frac{f'_2}{3}\right)^2 \\
=& \frac{1}{3^2}\left(d'_1 - d'_2\right)^2 + \frac{1}{3^2}\left(f'_1 - f'_2\right)^2 \\
=& \frac{1}{9}\left(d'_1 - d'_2\right)^2 + \left(f'_1 - f'_2\right)^2 \\
=& \frac{1}{9} b(e_1, e_2) \\
<& \frac{1}{8} b(e_1, e_2) = \frac{\delta}{l^2} b(e_1, e_2).
\end{aligned}$$

It implies that

$$b(ge_1, ge_2) \leq \frac{\delta}{l^2} b(e_1, e_2),$$

for all $e_1, e_2 \in E_1$. Now, $e_1, e_2 \in E_1$ and $(h, i) \in E(G)$ such that

$$\begin{aligned}
b(e_1, gh) =& b(E_1, E_2) = 4 \\
b(e_2, gi) =& b(E_1, E_2) = 4.
\end{aligned}$$

By the weak P- property we have $b(e_1, e_2) \leq b(gh, gi) < \delta b(h, i) < b(h, i)$. Since $(h, i) \in E(G)$ then $b(h, i) < 49$, which gives $b(e_1, e_2) < 49$ therefore $(e_1, e_2) \in E(G)$. Hence g is a G_b contraction. Let $d_0 = (3, 0)$, $d_1 = (2\sqrt{2}, 1)$ and $k = 1$. Since $b(d_0, d_1) = 1.2 < 49$, then the pair $(d_0, d_1) \in E(G)$. By definition of g we get, $gd_0 = (\frac{2\sqrt{2}}{3}, \frac{1}{3})$, and we obtain $b(d_1, gd_0) = 4 = b(E_1, E_2)$. Thus, condition $(\mathfrak{A})$ holds. All conditions of Theorem 2 are satisfied and g has a best proximity point $(0, 3)$.

Corollary 3.1. [4] Consider a complete graphical b- metric space $(\aleph, b)$ and a continuous self-mapping $g : \aleph \to \aleph$ such that for all $d, f \in \aleph$, if $(d, f) \in E(G)$ then $(gd, gf) \in E(G)$ and $b(gd, gf) \leq \frac{\delta}{l^2} b(d, f); \delta \in [0, 1)$. If b is a sequentially continuous then, g has a fixed point.

Example 3.3. *Consider $\aleph = \{0\} \cup \{\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \cdot\}$ with metric defined as*

$$b(a, d) = |a - d|^3, \tag{3}$$

for all $a, d \in \aleph$ is a b- metric space. Consider a graph G with $V(G) = \aleph$ and $E(G) = \{(d, f) \in \aleph \times \aleph : d \leq f\}$. Then $(\aleph, b)$ is a graphical b- metric space with $l = 4$. Define $g : \aleph \to \aleph$ by

$$g(a) = \frac{a}{4}, \tag{4}$$

Next we have to show that g satisfies conditions of Corollary 3.1.
Let $\delta = \frac{1}{2}$. Take z_1, z_2 in E_{1_0}, then

$$\begin{aligned}
b(gz_1, gz_2) =& \left|\frac{z_1}{4} - \frac{z_2}{4}\right|^3 \\
=& \frac{1}{64}(|z_1 - z_2|^3 \\
=& \frac{1}{64} b(z_1, z_2) \\
<& \frac{1}{32} b(z_1, z_2) = \frac{\delta}{l^2} b(z_1, z_2)
\end{aligned}$$

This implies

$$b(gz_1, gz_2) \leq \frac{\delta}{l^2} b(z_1, z_2)$$

for all $z_1, z_1 \in E_{1_0}$. Since $(z_1, z_2) \in E(G)$ then $b(gz_1, gz_2) < b(z_1, z_2)$. Therefore $b(gz_1, gz_2) \in E(G)$. Hence all conditions of Corollary 3.1, are satisfied and g has a fixed point say 0. As the following figure **??**, *takes the weighted graph for $n = 6$, in which the weight of any edge (z_1, z_2) is equal to the value of $b(z_1, z_2)$.*

4 Application to integral equations

In this section, we obtain the solution of integral equation as an application of our obtained results.
If we take $E_1 = E_2 = \aleph$ in Theorem 1, we obtain the solution of nonlinear integral equation.

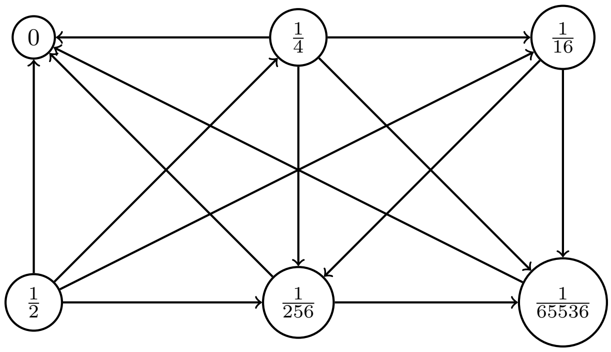


Fig. 2: Weighted graph for $n = 6$ where weight of edge $(z_1, z_2) = b(z_1, z_2)$
.

Theorem 3. Let $\mathtt{C}[0,1]$ be the set of all continuous functions on closed interval $[0,1]$, with metric defined by

$$b(e,f) = \sup_{s\in[0,1]} |e(s) - f(s)|^3 \tag{5}$$

for all $e, f \in \mathtt{C}[0,1]$ is a b- metric space with $l = 4$. Suppose that

$$\Im = \left\{ d \in \mathtt{C}[0,1] : \inf_{u\in[0,1]} e(u) > 0 \text{ and } u \leq 1; u \in [0,1] \right\}.$$

Consider the graph G with the vertex and edge set given as below:

$$E(G) = \left\{ (e,f) \in \mathtt{C}[0,1] \times \mathtt{C}[0,1] : e, f \in \Im, e(u) \leq f(u), \text{ for all } u \in [0,1] \right\},$$

and $V(G) = \mathtt{C}[0,1]$. Then $(\aleph, b)$ is a graphical b- metric space. Consider the nonlinear integral equation

$$e(s) = \upsilon(s) + \int_0^1 \omega(s,h,e(h))dh, \tag{6}$$

where $s \in [0,1]$, $\upsilon : [0,1] \to \mathbb{R}$, $\omega : [0,1] \times [0,1] \times a[0,1] \to \mathbb{R}$ for each $a \in \mathtt{C}[0,1]$. Suppose that the following statements hold:

(i) υ is continuous on $[0,1]$ with $\inf_{u\in[0,1]} \upsilon(u) > 0$ and $\omega(s,h,e(h))$ is integrable with respect to h on $[0,1]$ such that $\inf_{u\in[0,1]} \omega(s,h,e(h))(u) > 0$,
(ii) $ge \in \mathtt{C}[0,1]$ for all $e \in \mathtt{C}[0,1]$, where $ge(s) = \upsilon(s) + \int_0^1 \omega(s,h,e(h))dh$ for all $s \in [0,1]$,
(iii) for all $h, s \in [0,1]$ and $e, f \in \mathtt{C}[0,1]$, $|\omega(s,h,e(h)) - \omega(s,h,f(h))| \leq \frac{\delta}{l^2} (|e(h) - f(h)|)$; $\delta \in [0,1)$,

Then nonlinear integral equation (6) has a solution in $\mathtt{C}[0,1]$.

Proof: Define a mapping $g : \mathtt{C}[0,1] \to \mathtt{C}[0,1]$ by

$$ge(s) = \upsilon(s) + \int_0^1 \omega(s,h,e(h))dh,$$

for all $e \in \mathtt{C}[0,1]$ and for all $s \in [0,1]$. Then g is a continuous mapping.
Now, we claim that g is a G_b contraction. For all $h, s \in [0,1]$ and $e, f \in \mathtt{C}[0,1]$, we have

$$\begin{aligned} b(ge(h), gf(h)) &= \sup_{h\in[0,1]} \left| \int_0^1 \omega(s,h,e(h))dh - \int_0^1 \omega(s,h,f(h))dh \right|^3 \\ &\leq \sup_{h\in[0,1]} \int_0^1 |\omega(s,h,e(h)) - \omega(s,h,f(h))|^3 dh \\ &\leq \sup_{h\in[0,1]} \frac{\delta}{l^2} \left(|e(h) - f(h)|^3 \right) \int_0^1 d(h) \\ &= \frac{\delta}{l^2} \sup_{h\in[0,1]} \left(|e(h) - f(h)|^3 \right) \\ &= \frac{\delta}{l^2} b(e,f). \end{aligned}$$

It implies that

$$b(ge, gf) \leq \frac{\delta}{l^2} b(e, f).$$

Let $(e, f) \in E(G)$ with $e, f \in \mathtt{C}$. By definition of $E(G)$ we get, $e(u) \leq f(u)$ for all $u \in [0, 1]$, and by condition (i), we have $\inf_{u\in[0,1]} ge(u) > 0, \inf_{u\in[0,1]} gf(u) > 0$. Since $e(u) \leq f(u)$ for all $u \in [0, 1]$ then by definition of g we have $ge(u) \leq gf(u)$ for all $u \in [0, 1]$. This shows $(ge, gf) \in E(G)$ Therefore, we conclude all the hypotheses of Corollary 3.1 are satisfied. Thus, equation (6) has a solution $e \in \mathtt{C}[0, 1]$. □

Example 4.1. *Consider a integral equations below:*

$$e(s) = 1 + \sin s + \int_0^1 \left(\frac{e(h)}{4} + 1 \right) dh,$$

for all $s \in [0, 1]$ *and*

$$ge(s) = 1 + \sin s + \int_0^1 \left(\frac{e(h)}{4} + 1 \right) dh.$$

Consider the graph G with the vertex and edge set given as below:

$$E(G) = \left\{ (e, f) \in \mathtt{C} \times \mathtt{C} : e, f \in \Im, e(u) \leq f(u), \textit{for all } u \in [0, 1] \right\},$$

and $V(G) = \mathtt{C}$. *For the hypothesis (iii) of Theorem* 3*, we can write*

$$\begin{aligned} & |\omega(s, h, e(h)) - \omega(s, h, f(h))|| \\ =& \left| \frac{e(h)}{64} - \frac{f(h)}{64} \right| \\ =& \frac{1}{64} |e(h) - f(h)| \\ <& \frac{1}{32} |e(h) - f(h)| \\ =& \frac{\delta}{l^2} |e(h) - f(h)|, \end{aligned}$$

where $\delta = \dfrac{1}{2}, l^2 = 16$. *All the hypotheses of Theorem* 3 *is satisfied. Therefore, given problem has a unique solution.*

5 Conclusion

In this paper, we focus on the best proximity point theorems to prove its uniqueness on a b - metric space endowed with a graph that is more general than fixed point. We derive fixed point result as a result of our observations which appeared in the literature. We also furnish some numerical examples to support our claims. As an application of our main result we find the solution of nonlinear integral equation.

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Conflict of Interest The authors declare that they have not any conflict of interest.

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Fractional Mathieu-Duffing System with the stability condition and chaos control

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Abstract: In this paper, we introduce the chaotic fractional Mathieu-Duffing system and state a theorem to analyze stability of the system base on the Lyapunov second method. Next, we eliminate the chaotic behaviors of the system by means of feedback controller and presented theorem. We further present numerical simulations and reveal chaotic and asymptotic stability behaviors of the system to verify the theoretical analysis.

Keywords: Chaos, Fractional Mathieu-Duffing System, Stability.

1 Introduction

In the past few decades many researchers pointed out that several areas of physics, control engineering and signal processing may be precisely described with the help of fractional calculus (integral and derivative of non integer order). In the control theory, the stability analysis is one of the most important problems in the control of fractional systems and Lyapunov direct method is an available technique to stabilize chaotic system. Further, with the development of fractional calculus, some researchers proposed the Prabhakar (regularized Prabhakar) fractional integral and derivative which are the generalization of the Riemann-Liouville (Caputo) fractional integral and derivative [1]. The Prabhakar fractional derivative has important applications in the mathematics, economics, time-evolution of polarization processes, fractional Poisson process, fractional Maxwell model in linear viscoelasticity, generalized model of particle deposition in porous media, generalized reaction-diffusion equations and describing anomalous relaxation of Havriliak-Negami models in the field of dielectric materials. Further, it has wider stability region and faster convergence speed over integer derivative. So, study on the generalized fractional systems defined based on the Prabhakar fractional derivative is of the great importance.

In the current research, we consider the following chaotic fractional Mathieu-Duffing system

$$\begin{cases} {}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}x(t) = y(t), \\ {}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}y(t) = (a\sin\theta t + b)x(t) - cx^3(t) - \nu y(t), \end{cases} \tag{1}$$

with the initial condition (x_0, y_0). Where ${}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}$ is the regularized Prabhakar fractional derivative and $\gamma, \mu \in (0,1)$, $\omega < 0$, $0 < \rho < 2$, $\rho\gamma < \mu$ are its parameters. Further, $(x, y) \in \mathbb{R}^2$ is the state variable, and the parameters a, b, ν, c, θ are positive.

Our aim in this study is to analyze the dynamical behaviors consisting of the stability, chaotic behaviors and chaos control of chaotic fractional Mathieu-Duffing system. For this purpose, we state a theorem for the stability of system with the the regularized Prabhakar fractional derivative. Due to the difficulty of obtaining the analytical solution for the system, we mention a numerical algorithm for solving the chaotic fractional Mathieu-Duffing system. Then, by means of a feedback controller and presented theorem, we stabilize the chaotic orbits to the origin. Further, present numerical simulations and reveal chaotic and asymptotic stability behaviors of the system to verify the theoretical analysis.

2 Preliminaries

Definition 1. *[1] Let $f \in L^1[0,b]$, $0 < t < b \leq \infty$, and $m-1 < \mu < m$ $(m \in \mathbb{N})$. The Prabhakar derivative is defined by*

$$D^{\gamma}_{\rho,\mu,\omega,0+}f(t) = \frac{d^m}{dt^m}E^{-\gamma}_{\rho,m-\mu,\omega,0+}f(t), \qquad \rho,\mu,\omega,\gamma \in \mathbb{C}, \Re(\rho), \Re(\mu) > 0, \tag{2}$$

also, the regularized Prabhakar derivative for $f \in AC^m[o,b]$ ($AC^m[0,b] = \{f : [0,b] \to \mathbb{R} : \frac{d^{m-1}}{dt^{m-1}}f(t) \in AC[0,b]\}$) is given by

$$\begin{aligned} {}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}f(t) &= E^{-\gamma}_{\rho,m-\mu,\omega,0+}\frac{d^m}{dx^m}f(t) \\ &= D^{\gamma}_{\rho,\mu,\omega,0+}f(t) - \sum_{k=0}^{m-1} t^{k-\mu}E^{-\gamma}_{\rho,k-\mu+1}(\omega t^{\rho})f^{(k)}(0+). \end{aligned} \tag{3}$$

where $E^{\gamma}_{\rho,\mu,\omega,0+}$ *is the Prabhakar integral with generalized Mittag-Leffler function* $E^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} \frac{z^k}{k!}$ *in its kernel defined as follows*

$$E^{\gamma}_{\rho,\mu,\omega,0+} f(t) = \int_0^t (t-\xi)^{\mu-1} E^{\gamma}_{\rho,\mu}\big(\omega(t-\xi)^{\rho}\big) f(\xi) d\xi. \tag{4}$$

Lemma 1. *[2] Let* $\gamma, \rho, \mu, \nu, \sigma, \omega \in \mathbb{C}$ $(\Re(\rho), \Re(\mu), \Re(\nu) > 0)$, *then*

$$\int_0^t (t-\eta)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-\eta)^{\rho}) \eta^{\nu-1} E^{\sigma}_{\rho,\nu}(\omega\eta^{\rho}) d\eta = t^{\mu+\nu-1} E^{\gamma+\sigma}_{\rho,\mu+\nu}(\omega t^{\rho}).$$

Lemma 2. *If* $f(t) \in C(a,b) \cap L(a,b)$, *then* ${}^{C}D^{\gamma}_{\rho,\mu,\omega,a^+} E^{\gamma}_{\rho,\mu,\omega,a^+} f(t) = f(t)$. *If further* $f(t)$ *and its fractional Prabhakar derivatives belong to* $C(a,b) \cap L(a,b)$, *we have for* $m-1 < \mu \leq m$

$$E^{\gamma}_{\rho,\mu,\omega,a^+} {}^{C}D^{\gamma}_{\rho,\mu,\omega,a^+} f(t) = f(t) - \sum_{j=0}^{m-1} f^{(j)}(a)(t-a)^j.$$

Proof: The proof is straightforward, following the proof of Lemma 5 in [3], by using Lemma 2.2 in [4]. □

3 Stability theorem for the generalized fractional system

We now introduce the generalized fractional system with regularized Prabhakar derivative and present a theorem for stability of such system. Consider the following system

$${}^{C}D^{\gamma}_{\rho,\mu,\omega,a+} X(t) = f(t, X(t)), \tag{5}$$

where $\gamma, \rho, \mu, \omega \in (0,1)$, $X(t) \in \mathbb{R}^n$ is a state vector and $f(t, X(t)) \in \mathbb{R}^n$ satisfies a Lipschitz condition.

Theorem 1. *Let* $\gamma, \rho, \mu, \omega \in (0,1)$. *If there is a positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *such that system* (5) *satisfies*

$$X^T(t) P^C D^{\gamma}_{\rho,\mu,\omega,a+} X(t) \leq 0, \qquad \forall X(t) \in \mathbb{R}^n, \tag{6}$$

then system (5) *is stable.*

Proof: We consider $V(t) = \frac{1}{2} X^T(t) P X(t)$ as a Lyapunov function candidate. By using Lemma (2) and the Prabhakar integral definition, we have

$$\begin{aligned} V'(t) = X^T(t) P X'(t) &= X^T(t) P \lim_{\Delta t\to 0} \frac{X(t) - X(t-\Delta t)}{\Delta t} \\ &= X^T(t) P \lim_{\Delta t\to 0} \frac{E^{\gamma}_{\rho,\mu,\omega,t-\Delta t^+} {}^{C}D^{\gamma}_{\rho,\mu,\omega,a^+} X(t)}{\Delta t} \\ &= X^T(t) P \lim_{\Delta t\to 0} \frac{\int_{t-\Delta t}^{t} (t-\xi)^{\mu-1} E^{\gamma}_{\rho,\mu}\big(\omega(t-\xi)^{\rho}\big) {}^{C}D^{\gamma}_{\rho,\mu,\omega,a^+} X(\xi) d\xi}{\Delta t}. \end{aligned}$$

Since $f(t, X(t))$ satisfies a Lipschitz condition, $f(\xi, X(\xi)) = f(t, X(t))$ when $\Delta t \to 0$ and $\xi \in (t-\Delta t, t]$. So,

$$\begin{aligned} X^T(t) P^C D^{\gamma}_{\rho,\mu,\omega,a^+} X(\xi) = X^T(t) P f(\xi, X(\xi)) &= X^T(t) P f(t, X(t)) \\ &= X^T(t) P^C D^{\gamma}_{\rho,\mu,\omega,a+} X(t) \leq 0. \end{aligned}$$

For $\gamma, \rho, \mu, \omega \in (0,1)$ and $t > \xi$, we have $(t-\xi)^{\mu-1} > 0$ and $E^{\gamma}_{\rho,\mu}\big(\omega(t-\xi)^{\rho}\big) > 0$. Therefore,

$$V'(t) = \lim_{\Delta t\to 0} \frac{\int_{(t-\Delta t)}^{t} (t-\xi)^{\mu-1} E^{\gamma}_{\rho,\mu}\big(\omega(t-\xi)^{\rho}\big) X^T(t) P^C D^{\gamma}_{\rho,\mu,\omega,a^+} X(\xi) d\xi}{\Delta t} \leq 0. \tag{7}$$

According to the Lyapunov second method, the above inequality implies that system (5) is asymptotically stable. □

4 Numerical simulation

We proposed a numerical method to solve the generalized fractional dynamical system by transforming the original system into a system of ordinary differential equations of first order. According to the Prabhakar derivative definition and Lemma 1 we have

$$
\begin{aligned}
D^{\gamma}_{\rho,\mu,\omega,0+}f(t) &= t^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega t^{\rho}\big)f(0) + t^{1-\mu}E^{-\gamma}_{\rho,2-\mu}\big(\omega t^{\rho}\big)f'(0) \\
&+ \int_0^t (t-\xi)^{1-\mu}E^{-\gamma}_{\rho,2-\mu}\big(\omega(t-\xi)^{\rho}\big)f''(\xi)d\xi.
\end{aligned} \tag{8}
$$

By using the Taylor expansion of the generalized Mittag-Leffler function and the binomial extension, the third term of the right hand side of (8) can be written as

$$
\begin{aligned}
&\int_0^t (t-\xi)^{1-\mu}E^{-\gamma}_{\rho,2-\mu}\big(\omega(t-\xi)^{\rho}\big)f''(\xi)d\xi \\
&= \sum_{k=0}^{\infty}\frac{(-\gamma)_k\omega^k}{\Gamma(\rho k-\mu+2)k!}t^{\rho k-\mu+1}\int_0^t f''(\xi)\Big(1+\sum_{p=1}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)p!}\big(\frac{\xi}{t}\big)^p\Big)d\xi.
\end{aligned} \tag{9}
$$

Let $V_n(f^p), n\in\mathbb{N}$, denote the nth moment of the function f^p, where $f^p(p\in\mathbb{N})$ is the pth (integer) derivative of f, i.e., $V_n(f^{(p)})(t) = \int_0^t f^{(p)}(\xi)\xi^n d\xi, \quad n\in\mathbb{N},\ t\geq 0$. Integrating term by term of series in (9), we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(-\gamma)_k\omega^k}{\Gamma(\rho k-\mu+2)k!}t^{\rho k-\mu+1}\int_0^t f''(\xi)\Big(1+\sum_{p=1}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)p!}\big(\frac{\xi}{t}\big)^p\Big)d\xi \\
&= \sum_{k=0}^{\infty}\frac{(-\gamma)_k\omega^k}{\Gamma(\rho k-\mu+2)k!}t^{\rho k-\mu+1}\sum_{p=1}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)p!}f'(t) + \sum_{k=0}^{\infty}\frac{(-\gamma)_k\omega^k t^{\rho k-\mu}(f(t)-f(0))}{\Gamma(\rho k-\mu+1)k!} \\
&- \sum_{k=0}^{\infty}\frac{(-\gamma)_k\omega^k t^{\rho k}}{\Gamma(\rho k-\mu+2)k!}\sum_{p=2}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)(p-1)!}\Big(\frac{f(t)}{t^{\mu}}+\frac{\widehat{V}_p(f)(t)}{t^{p+\mu-1}}\Big),
\end{aligned}
$$

where $\widehat{V}_p(f)(t) = -(p-1)\int_0^t \xi^{p-2}f(\xi)d\xi, \quad p=2,3,\cdots$. Therefore

$$
\begin{aligned}
D^{\gamma}_{\rho,\mu,\omega,0+}f(t) &= \sum_{k=0}^{\infty}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!}\Bigg[\frac{f'(t)}{t^{\mu-1}}\big(1+\sum_{p=1}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)p!}\big) \\
&- \Big(\frac{(\mu-\rho k-1)}{t^{\mu}}f(t) + \sum_{p=2}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)(p-1)!}\big(\frac{f(t)}{t^{\mu}}+\frac{\widehat{V}_p(f)(t)}{t^{p+\mu-1}}\big)\Big)\Bigg].
\end{aligned}
$$

Obviously, the relation between ${}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}f(t)$ and $f'(t)$ are defined as follows

$$
\begin{aligned}
{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}f(t) &= \sum_{k=0}^{\infty}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!}\Bigg[\frac{f'(t)}{t^{\mu-1}}\big(1+\sum_{p=1}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)p!}\big) \\
&- \Big(\frac{(\mu-\rho k-1)}{t^{\mu}}(f(t)+f(0)) + \sum_{p=2}^{\infty}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)(p-1)!}\big(\frac{f(t)}{t^{\mu}}+\frac{\widehat{V}_p(f)(t)}{t^{p+\mu-1}}\big)\Big)\Bigg].
\end{aligned} \tag{10}
$$

We approximate ${}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}f(t)$ by using the first M terms in the sum appearing in (10) as follows

$$
\begin{aligned}
{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}f(t) &\simeq \sum_{k=0}^{M}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!}\Bigg[\frac{f'(t)}{t^{\mu-1}}\big(1+\sum_{p=1}^{M}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)p!}\big) \\
&- \Big(\frac{(\mu-\rho k-1)}{t^{\mu}}(f(t)+f(0)) + \sum_{p=2}^{M}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)(p-1)!}\big(\frac{f(t)}{t^{\mu}}+\frac{\widehat{V}_p(f)(t)}{t^{p+\mu-1}}\big)\Big)\Bigg].
\end{aligned} \tag{11}
$$

We can rewrite equation (11) as

$$
\begin{aligned}
{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}f(t) &\simeq \Omega f'(t) + \Phi f(t) - Q_2 t^{-\mu}f(0) \\
&- \sum_{k=0}^{M}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!}\sum_{p=2}^{M}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)(p-1)!}\frac{\widehat{V}_p(f)(t)}{t^{p+\mu-1}},
\end{aligned}
$$

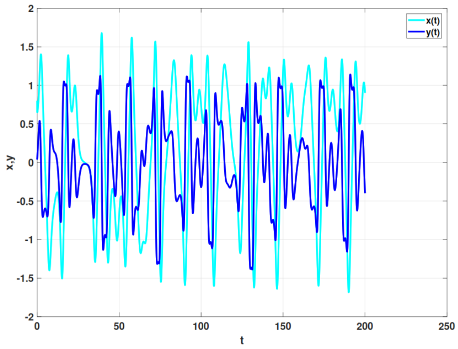

Fig. 1: Numerical value of $x(t)$, $y(t)$ of system (1) for $\gamma = 0.02$, $\rho = 0.9$, $\omega = 0.03$, $\mu = 0.95$, $h = 0.01$, $M = 5$, $x(0) = 0.9$ and $y(0) = 0.03$.

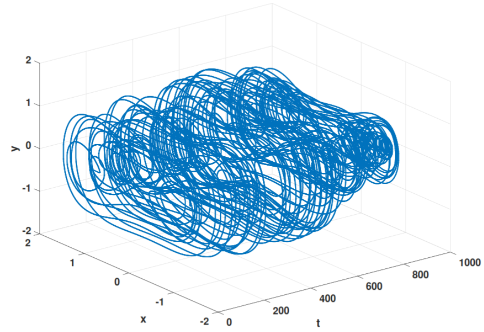

Fig. 2: Numerical value of $x(t)$, $y(t)$ of system (1) for $\gamma = 0.02$, $\rho = 0.9$, $\omega = 0.03$, $\mu = 0.95$, $h = 0.1$, $M = 5$ and different initial conditions.

where

$$\Omega = (Q_1 + R_1)t^{1-\mu}, \quad \Phi = (Q_2 - R_2)t^{-\mu},$$
$$Q_1 = \sum_{k=0}^{M} \frac{(-\gamma)_k(\omega t^\rho)^k}{\Gamma(\rho k - \mu + 2)k!}, \quad R_1 = \sum_{k=0}^{M} \frac{(-\gamma)_k(\omega t^\rho)^k}{\Gamma(\rho k - \mu + 2)k!} \sum_{p=1}^{M} \frac{\Gamma(p - \rho k + \mu - 1)}{\Gamma(-\rho k + \mu - 1)p!},$$
$$Q_2 = \sum_{k=0}^{M} \frac{(-\gamma)_k(\omega t^\rho)^k}{\Gamma(\rho k - \mu + 1)k!}, \quad R_2 = \sum_{k=0}^{M} \frac{(-\gamma)_k(\omega t^\rho)^k}{\Gamma(\rho k - \mu + 2)k!} \sum_{p=2}^{M} \frac{\Gamma(p - \rho k + \mu - 1)}{\Gamma(-\rho k + \mu - 1)(p-1)!}.$$

Finally, we can rewrite the system (5) as

$$x'(t) = \frac{1}{\Omega}\Bigg[Ax(t) + f(t, x(t)) - \Phi x(t) + Q_2 t^{-\mu} x(0) + \sum_{k=0}^{M} \frac{(-\gamma)_k(\omega t^\rho)^k}{\Gamma(\rho k - \mu + 2)k!} \sum_{p=2}^{M} \frac{\Gamma(p - \rho k + \mu - 1)}{\Gamma(-\rho k + \mu - 1)(p-1)!} \frac{\widehat{V}_p(x)(t)}{t^{p+\mu-1}}\Bigg],$$

with the initial condition $x(t_0) = x_0$.

By performing simulations for $a = 0.5$, $b = 1$, $\nu = 0.2$, $c = 1$, $\theta = 1$, $\gamma = 0.02$, $\rho = 0.9$, $\mu = 0.95$, $\omega = 0.03$, the system possesses a chaotic behavior. This behavior is shown in Figs. 1-2. Numerical solution of the system is obtained by using the well known fourth order Runge-Kutta method with initial conditions $x(0) = 0.8$, $y(0) = 0.05$, $h = 0.1$ and $M = 5$. Figs. 1-2 indicate the chaotic behavior of system (1).

5 Controlling chaos

We now design a linear feedback controller for the system (1) and get

$$\begin{cases} {}^C D^{\gamma}_{\rho,\mu,\omega,0+} x(t) = y(t) + k_1 x(t), \\ {}^C D^{\gamma}_{\rho,\mu,\omega,0+} y(t) = (0.5 \sin t + 1)x(t) - x^3(t) - 0.2y(t) + k_2 y(t), \end{cases} \tag{12}$$

where k_1 and k_2 are the control parameters. We intend to find suitable values of the parameters k_1 and k_2 such that the chaotic fractional Mathieu-Duffing system (1) becomes stable. Using Theorem 1 for the positive definite matrix $P = I$, leads to

$$\begin{aligned} X^T(t)P^C D^{\gamma}_{\rho,\mu,\omega,a+} X(t) &= x(t)^C D^{\gamma}_{\rho,\mu,\omega,a+} x(t) + y(t)^C D^{\gamma}_{\rho,\mu,\omega,a+} y(t) \\ &= x(t)y(t) + k_1 x^2(t) + (0.5 \sin t + 1)x(t)y(t) - x^3(t)y(t) - 0.2y^2(t) + k_2 y^2(t) \\ &\le (k_1 + \frac{1}{2} + \frac{0.5|\sin t| + 1}{2})x^2(t) + (k_2 + 0.3\frac{0.5|\sin t| + 1}{2})y^2(t) + \frac{1}{2}x^4 + \frac{1}{2}x^2y^2 \\ &\le (k_1 + 1.25)x^2(t) + (k_2 + 1.05)y^2(t) + \frac{1}{2}x^4 + \frac{1}{2}x^2y^2. \end{aligned} \tag{13}$$

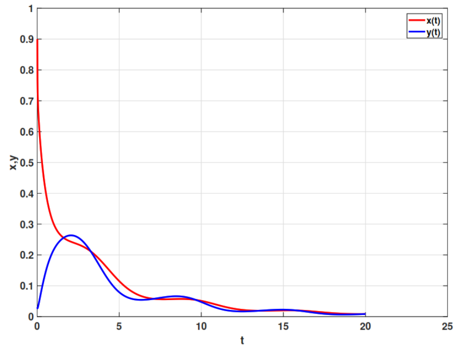

Fig. 3: Numerical value of $x(t)$, $y(t)$ of system (12) for $\gamma = 0.02$, $\rho = 0.9$, $\omega = 0.03$, $\mu = 0.95$, $h = 0.01$, $M = 5$, $x(0) = 0.9$ and $y(0) = 0.03$.

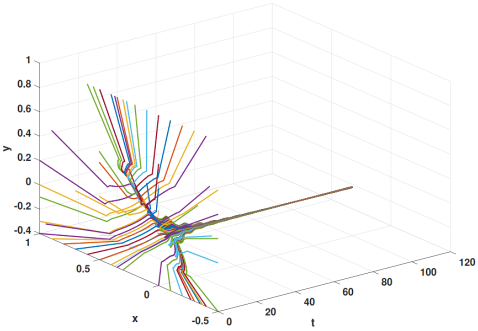

Fig. 4: Numerical value of $x(t)$, $y(t)$ of system (12) for $\gamma = 0.02$, $\rho = 0.9$, $\omega = 0.03$, $\mu = 0.95$, $h = 0.1$, $M = 5$ and different initial conditions.

To stabilize the system (12), we take $k_1 + 1.25 \leq -\frac{1}{2}x^4 - \frac{1}{2}x^2y^2$ and $k_2 + 1.05 \leq -\frac{1}{2}x^4 - \frac{1}{2}x^2y^2$. Because, even for cases of $k_1 + 1.25 = -\frac{1}{2}x^4 - \frac{1}{2}x^2y^2$ and $k_2 + 1.05 = -\frac{1}{2}x^4 - \frac{1}{2}x^2y^2$, we have the following trivial relation

$$
\begin{aligned}
&(k_1 + 1.25)x^2(t) + (k_2 + 1.05)y^2(t) + \frac{1}{2}x^4 + \frac{1}{2}x^2y^2 \leq 0 \\
&\Leftrightarrow -(\frac{1}{2}x^4 + \frac{1}{2}x^2y^2)(x^2 + y^2) \leq -(\frac{1}{2}x^4 + \frac{1}{2}x^2y^2) \\
&\Leftrightarrow (x^2 + y^2) \geq 0.
\end{aligned}
$$

So, by taking the control parameters $k_1 \leq -1.25 - \frac{1}{2}x^4 - \frac{1}{2}x^2y^2$ and $k_2 \leq -1.05 - \frac{1}{2}x^4 - \frac{1}{2}x^2y^2$, the system (12) becomes stable. The numerical values and the phase portrait of system (12) indicate the asymptotic stability behavior of (12). We fix the parameters $\gamma = 0.02$, $\rho = 0.9$, $\omega = 0.03$, $M = 5$, $k_1 = -1.25 - \frac{1}{2}x^4 - \frac{1}{2}x^2y^2$ and $k_2 = -1.05 - \frac{1}{2}x^4 - \frac{1}{2}x^2y^2$.
The results is depicted in Fig. 3 for $\mu = 0.95$, $h = 0.01$, $x(0) = 0.9$, $y(0) = 0.03$, and the numerical results for $\mu = 0.95$, $h = 0.1$ and different initial conditions is depicted in Fig. 4.

6 Conclusion

In this work, we formulate a theorem for stability of fractional system with the regularized Prabhakar fractional derivative and applied the presented theorem for refusing and controlling chaos of the system. We have shown the chaotic behaviors of the system by numerical simulations. Figs. 1-2 have indicated chaotic attractors of the system. Then, to stabilize these system, we have employed the linear feedback controller and have obtained the control parameters by using the Lyapunov stability theorem. Figs. 3-4 have indicated the asymptotic stability behaviors of system with the obtained control parameters and the fixed parameters of system.

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DRBEM solutions of singularly perturbed MHD flow in a square duct with variably conducting and no-slip/slipping walls

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Abstract: In this study, the Dual Reciprocity Boundary Element Method (DRBEM) solutions of singularly perturbed magnetohydrodynamic (MHD) flow equations are investigated in a square duct with variably conducting and either slipping or no-slip walls. The MHD flow equations governed by the velocity $V(x, y)$ of the fluid and the induced magnetic field $B(x, y)$ are coupled, and convection-diffusion type including the so-called Hartmann number (Ha) as a coefficient of the convection terms. When Ha is large due to the high intensity of the external magnetic field, the MHD flow equations become convection dominated. That is, the coefficients of the diffusion terms are very small giving the singularly perturbed MHD flow equations whose numerical solutions are difficult to be found especially on the thin boundary layer regions. These singularly perturbed MHD equations are solved with DRBEM using Shishkin mesh which consists of the transition points depending on Ha and the number of nodes taken on each side of the duct. DRBEM numerical results show that, the well-known behaviors of $V(x, y)$ and $B(x, y)$ are deduced for large values of Ha such as 500, 700, and 1000. That is, the flattening flow and boundary layers formation are observed. For variably conducting and no-slip walls, for a fixed Ha and increasing wall conductivity c, the velocity $V(x, y)$ decreases in magnitude whereas the profiles of $B(x, y)$ become perpendicular to the duct walls. On the other hand, for variably conducting and slipping walls, the slip length α has such an impact on the velocity $V(x, y)$ that, its magnitude increases with an increase in α for each fixed Ha. Moreover, it is seen that the induced magnetic field $B(x, y)$ profiles is not much effected from the increase in the slip length.

Keywords: DRBEM, Hartmann number, MHD flow, Shishkin mesh, singular perturbation, slip length, transition point

1 Introduction

The field of magnetohydrodynamics (MHD) investigates the interaction between electrically conducting fluids and magnetic fields or electric currents, combining principles from fluid mechanics and electrodynamics [2]. It provides a framework for understanding and analyzing the complex interactions between charged particles and magnetic fields within conducting fluids, such as plasmas, liquid metals, and ionized gases. MHD has a wide range of applications across various scientific and engineering domains. Numerous industries and engineering disciplines benefit from the applications of magnetohydrodynamics which explores the behavior of electrically conducting fluids under the influence of magnetic fields. Examples include MHD flow meters, accelerators, blood pressure measurements, electromagnetic pumps, and MHD generators-reactors. Hartmann's extensive research [5] on the flow of electrically conducting fluids between parallel planes in the presence of a transverse magnetic field has found widespread application in various areas, contributing to the understanding of magnetohydrodynamics and its practical implications. There are some studies [9], [14] in which Finite Difference Method (FDM) is used to solve the MHD flow problem. Moreover, the boundary element method (BEM) which discretizes only the boundary of the problem domain [3], [15] is an alternative to the basic domain discretization methods such as FDM and Finite Element Method (FEM) [10], [16] for solving the MHD duct flow problems . The resulting system of equations are quite small in size compared to FEM and FDM discretized system of equations sizes. Both BEM and dual reciprocity boundary element method (DRBEM) [8], [11], [13] have been used for solving MHD duct flow problems in different geometries with several types of boundary conditions.

The objective of this study is to investigate the solutions to the singularly perturbed magnetohydrodynamic (MHD) flow equations in a square duct with either slipping or no-slip walls of varying conductivity utilizing the Dual Reciprocity Boundary Element Method (DRBEM). Since the MHD flow equations are convection-diffusion type equations with the Hartmann number (Ha) as the coefficient of the convection terms, for large Ha they become convection dominated with small diffusion term coefficients, making the numerical solutions challenging, particularly in thin boundary layer regions. To address this challenge, a Shishkin mesh [7] is employed, which incorporates transition points dependent on Ha and determines the number of nodes on each side of the duct. Using the DRBEM, the singularly perturbed MHD flow equations are solved numerically. The numerical results reveal that, the well-known behaviors of the velocity $V(x, y)$ and the induced magnetic field $B(x, y)$ are seen for large Ha values such as 500, 700, and 1000. These are the flattening flow and the boundary layer formation.

2 Mathematical formulation

The magnetohydrodynamics (MHD) concerns with the flow of an electrically conducting fluid under the influence of an external magnetic field. As a result of some physical laws which are Ohm's and Ampere's laws, Maxwell's equations, continuity and momentum equations, MHD flow equations are obtained. The governing MHD flow equations [1], [2] in the cross-section of a pipe (duct) are given as

$$\begin{aligned} \nabla^2 V + Ha\frac{\partial B}{\partial x} &= -1 \\ \nabla^2 B + Ha\frac{\partial V}{\partial x} &= 0 \end{aligned} \quad \text{in} \quad \Omega = \{(x,y) : -1 \le x, y \le 1\}, \tag{1}$$

where $Ha = LB_0\sqrt{\sigma/\nu\rho}$ is the Hartmann number L, σ, ρ, ν being the characteristic length, electrical conductivity, density and kinematic viscosity of the fluid, respectively. $V(x, y)$ and $B(x, y)$ are the velocity and induced magnetic field in the pipe-axis direction, respectively. The problem is considered with the boundary condition

$$V + \alpha\partial V/\partial n = 0 \quad \text{and} \quad B + c\partial B/\partial n = 0,$$

where c is the conductivity ratio and α is the slipping length. When c approaches to infinity, we have electrically perfectly conducting walls whereas they become insulated walls when it tends to zero. The walls of the duct are no-slip for $\alpha = 0$, however, the velocity of the fluid slips at the boundary if $\alpha \neq 0$.

3 Shishkin mesh construction

The Shishkin mesh stands out from other meshes due to its unique transition parameters at which the mesh changes from coarse to fine structure [17]. These transition parameters are defined based on the characteristics of the velocity field components, that is, the coefficient of the term $\nabla = (\partial/\partial x, \partial/\partial y)$ in the problem (1), specifically changing behavior observed in regular or parabolic boundary layers, or a combination of both. When considering the MHD duct flow problem (1), the lines $y = 0$ and $y = 1$ (representing the side walls) exhibit parabolic boundary layers, while the lines $x = 0$ and $x = 1$ (representing the Hartmann walls) demonstrate regular boundary layers. This distinction in the type of boundary layers guides the specification of the transition parameters within the Shishkin mesh. In conclusion, the Shishkin mesh is constructed by using four transition points [7] two as τ_1 and $1 - \tau_1$ in x- direction and two as τ_2 and $1 - \tau_2$ in y- direction, where

$$\tau_1 = \min\left\{\frac{1}{2}, C\epsilon \ln M\right\} \quad \text{and} \quad \tau_2 = \min\left\{\frac{1}{4}, \sqrt{\epsilon}\ln M\right\}. \tag{2}$$

with $\epsilon = 1/Ha$, M the number of mesh points on one side of the duct. Since there are regular and parabolic boundary layers on the Hartmann and side walls, respectively, we have the meshes $\Omega^M_{\tau_1}$ and $\Omega^M_{\tau_2}$ on the x- and y-axes. These meshes are obtained by dividing the sub-intervals $(0, \tau_i)$ and $(1 - \tau_i, 1)$ into $M/4$ equal mesh elements while $(\tau_i, 1 - \tau_i)$ into $M/2$ equal mesh elements to get totally M mesh elements on each side for $i = 1, 2$. Totally, the Shishkin mesh is obtained as $\Omega^M_{\tau_1,\tau_2} = \Omega^M_{\tau_1} \times \Omega^M_{\tau_2}$. As the Hartmann number (Ha) increases, it becomes necessary to increase the value of the parameter M. Additionally, it is crucial to choose the constant C carefully so that the transition points on the x-axis adequately cover the width of the Hartmann layers, which are of the order $O(\epsilon)$, where $\epsilon = 1/Ha$. Furthermore, based on the theory of boundary layers in MHD duct flow [1], it is established that the thickness of the Hartmann layers is of order $\epsilon = 1/Ha$, while the thickness of the side layers is of order $\sqrt{\epsilon} = 1/\sqrt{Ha}$. This is why the construction of the Shishkin mesh with four transition parameters is theoretically appropriate.

4 DRBEM application

The dual reciprocity boundary element method (DRBEM) is a numerical technique that converts differential equations into boundary integral equations. It uses the fundamental solution of the Laplace equation for Poisson's type equations [4], given by $u^* = (\ln(1/r))/2\pi$, where r represents the distance between the source and field points. In the MHD problem (1), all terms except the Laplacian are moved to the right-hand side and treated as the inhomogeneity part, which is approximated by using a radial basis function with N boundary and L interior nodes. By applying Green's first identity twice and discretizing the resulting integrals, the coupled equations are transformed into matrix-vector equations

$$\begin{aligned} \mathbf{H}V - \mathbf{G}\frac{\partial V}{\partial n} &= (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}(-1 - Ha\frac{\partial B}{\partial x}), \\ \mathbf{H}B - \mathbf{G}\frac{\partial B}{\partial n} &= (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}(Ha\frac{\partial V}{\partial x}). \end{aligned} \tag{3}$$

Defining the matrices $\mathbf{D} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}})\mathbf{F}^{-1}$ and $\mathbf{E} = \mathbf{D}\frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}$ of size $(N + L) \times (N + L)$, the enlarged system is obtained as

$$\bar{\mathbf{H}}\begin{Bmatrix} V \\ B \end{Bmatrix} = \bar{\mathbf{G}}\begin{Bmatrix} \partial V/\partial n \\ \partial B/\partial n \end{Bmatrix} + \bar{\mathbf{D}}\begin{Bmatrix} -1 \\ 0 \end{Bmatrix}, \tag{4}$$

where

$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{H} & 0 \\ 0 & \mathbf{H} \end{bmatrix} + Ha\begin{bmatrix} 0 & \mathbf{E} \\ \mathbf{E} & 0 \end{bmatrix}, \quad \bar{\mathbf{G}} = \begin{bmatrix} \mathbf{G} & 0 \\ 0 & \mathbf{G} \end{bmatrix}, \quad \bar{\mathbf{D}} = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix}$$

are the matrices of size $(2N+2L)\times(2N+2L)$. Here, **H** and **G** are the DRBEM matrices of size $(N+L)\times(N+L)$ and their entries are defined as

$$\mathrm{H}_{ij} = \int_{\Gamma_j} q^* \mathrm{d}\Gamma_j, \quad \mathrm{H}_{ii} = c_i, \quad \mathrm{G}_{ij} = \int_{\Gamma_j} u^* \mathrm{d}\Gamma_j, \quad \text{and} \quad \mathrm{G}_{ii} = \frac{l}{2\pi}\left(\ln\left(\frac{2}{l}\right)+1\right),$$

where l is the length of the elements and $c_i = 1/2$ since we have a unit square domain. The matrix **F** in the DRBEM approach is the coordinate matrix of size $(N+L)\times(N+L)$, consisting of the columns f_j. It is symmetric and non-singular [6]. Additionally, the matrices $\hat{\mathbf{U}}$ and $\hat{\mathbf{Q}}$ are of size $(N+L)\times(N+L)$ and are obtained by taking each vector $\hat{\mathbf{u}}_j$ and $\hat{\mathbf{q}}_j = \frac{\partial \hat{\mathbf{u}}_j}{\partial n}$ as columns, respectively. The radial basis functions are connected to the Laplace operator through the equation $\nabla^2 \hat{\mathbf{u}}_j = f_j$. Since we have variably conducting walls, the induced magnetic field is always unknown on the boundary and so for the normal derivative we use the condition $\dfrac{\partial B}{\partial n} = -\dfrac{1}{c}B$. If the walls are no-slip, then the velocity is known but its normal derivative is unknown on the boundary. However, when the slip exits on the walls, the velocity is unknown on the boundary. In this case, we use the condition $\dfrac{\partial V}{\partial n} = -\dfrac{1}{\alpha}V$. Then, all the conditions for the boundary and interior nodes are inserted into the matrix vector equation. Finally, each unknown on the right hand side of the system (4) is carried to the left hand side by shuffling the corresponding columns of the matrices $\bar{\mathbf{H}}$ and $\bar{\mathbf{G}}$. This results in a linear system $\bar{\mathbf{A}}\mathbf{x} = \bar{\mathbf{d}}$, where **x** represents the vector of unknowns. By solving this linear system as a whole, the DRBEM gives the advantage of obtaining simultaneous solutions for both the unknowns V and B. This feature enhances the method's effectiveness and computational efficiency.

5 Numerical results

DRBEM solutions of singularly perturbed coupled MHD duct flow problem are considered by using Shishkin mesh. The velocity and induced magnetic profiles are obtained for large values of Ha such as 500, 700, and 1000 for variably conducting and slip/no-slip walls. We use $N = 4M$ constant boundary elements in the middle of each sub-interval and $L = M^2$ interior nodes. Totally, $N+L$ nodes are obtained from Shishkin mesh. The radial basis function is chosen as $f = 9r$ with the particular solution $\hat{u} = 1 + r^3$ [13] since $\nabla^2 \hat{u} = f$. To evaluate the entries of the matrices **H** and **G** the Gauss-Legendre numerical integration is used with 8 nodes [12]. The numerical results are deduced in MATLAB by using a high performance computer (HPC).

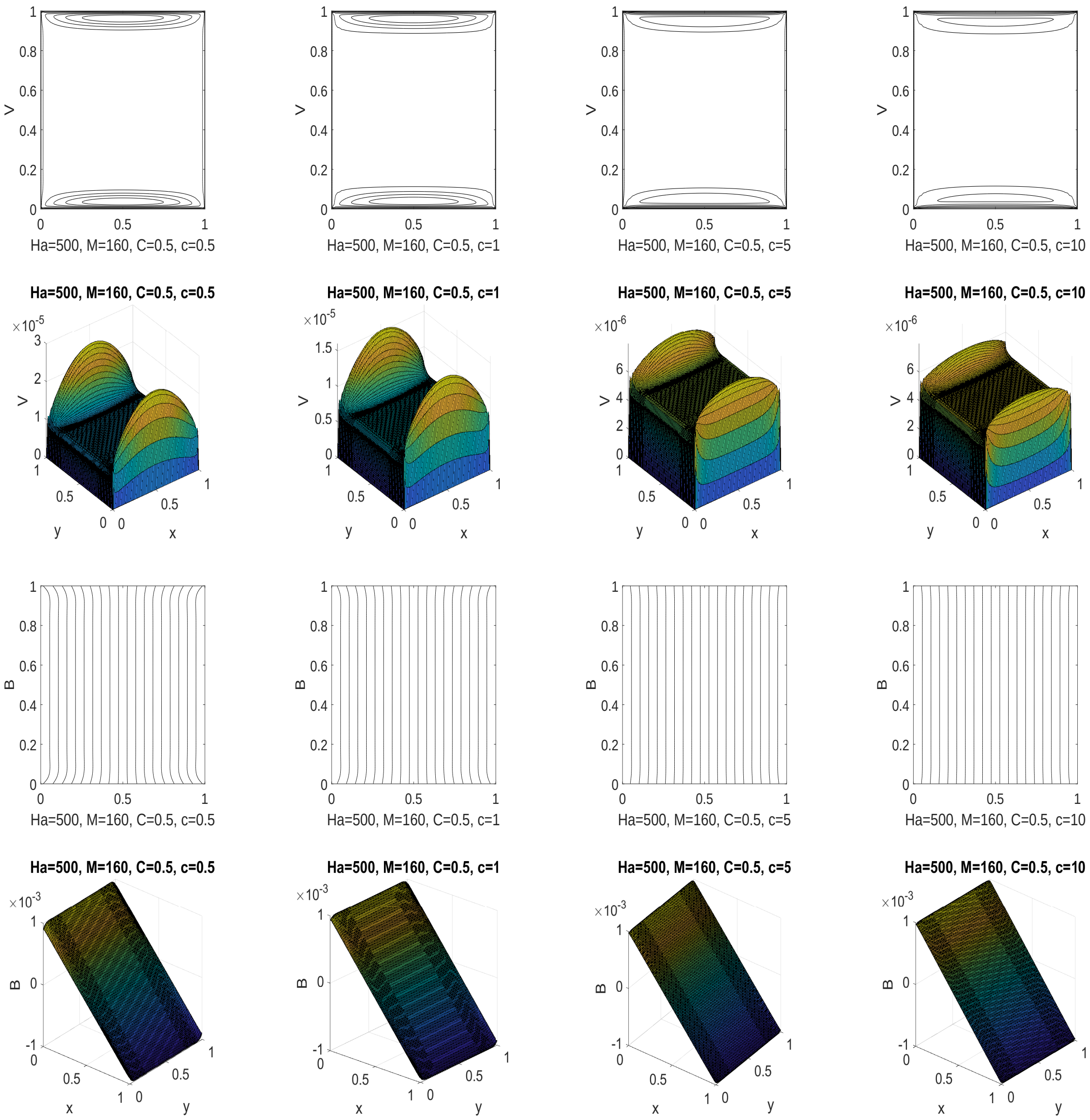

Fig. 1: Velocity and the induced magnetic field profiles for no-slip and variably conducting walls when $Ha = 500$.

Fig 1, 2, and 3 give the simulations of the velocity and the induced magnetic field in both contours and level curves for the no-slip and variably conducting boundary condition case when $Ha = 500$, 700, and 1000, respectively. It is commonly seen that when the conductivity parameter c increases, the induced magnetic field becomes perpendicular to the side walls trying to behave as if nearly electrically perfectly conducting. On the other hand, this increase in c results in a decrease in the velocity magnitude. Moreover, for a fixed conductivity parameter c, when the Hartmann number increases, both the velocity and the induced magnetic field magnitudes drop, which is the well-known characteristic of the MHD flow problem. Fig 4 shows the visualizations of the velocity and the induced magnetic field for the slip and variably conducting case for $Ha = 500$ and 700. It is observed that the fluid flows in terms of two loops aligned the side walls. Furthermore, when the slipping length α rises, the velocity magnitude increases as well for each Hartmann number, which is the expected behavior in MHD duct flow [18].

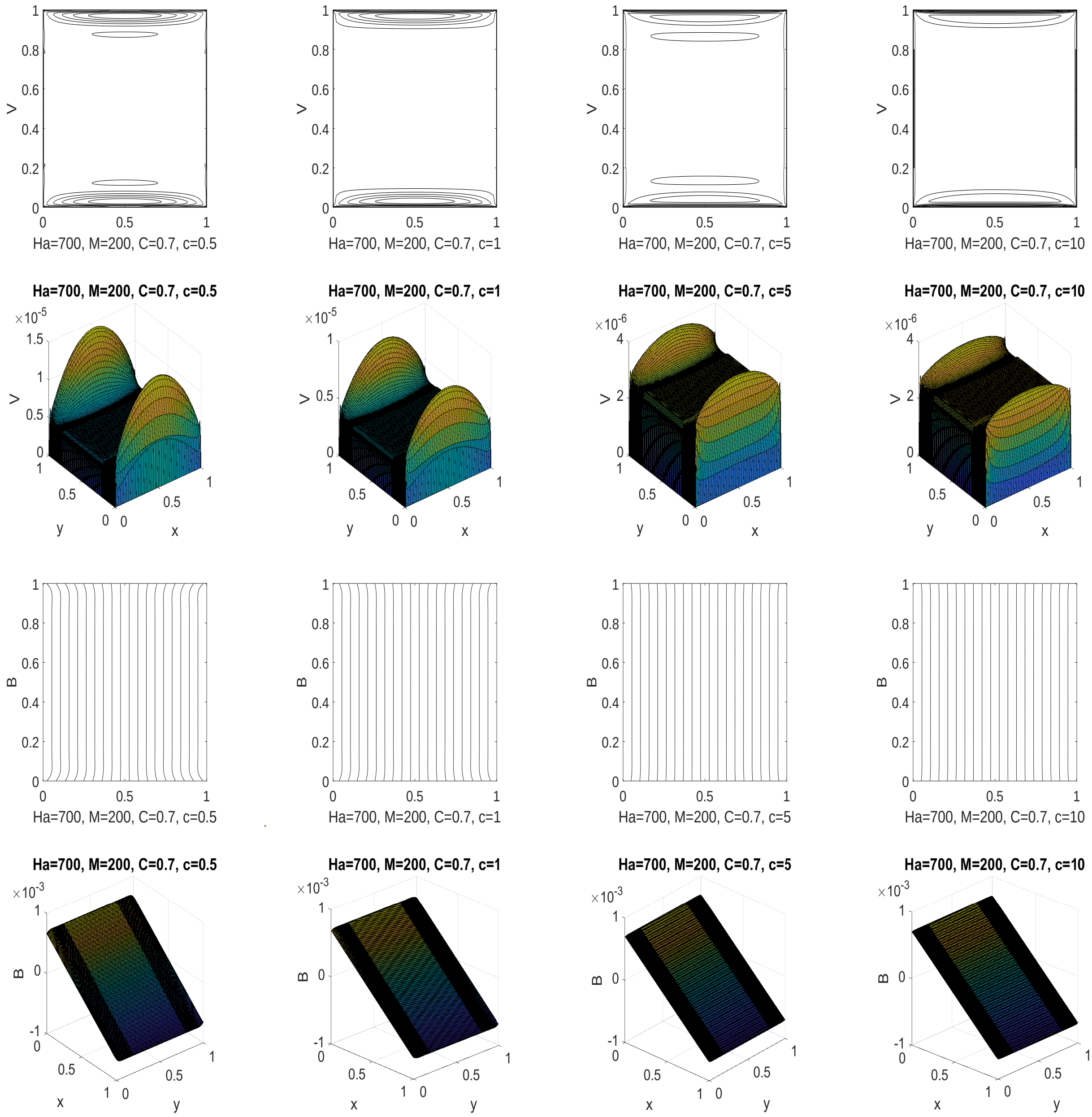

Fig. 2: Velocity and the induced magnetic field profiles for no-slip and variably conducting walls when $Ha = 700$.

6 Conclusion

In this study, the DRBEM is employed to obtain the solutions of singularly perturbed coupled MHD flow equations within a square duct when the walls are either no-slip or slipping and have variable conductivity. To handle the singular perturbation, the DRBEM is adapted to the Shishkin mesh. The numerical results demonstrate that, for large values of the Hartmann number Ha, the expected behaviors of the velocity and the induced magnetic field are observed. This includes the presence of a flattening flow and the formation of boundary layers. The study also analyzes the effects of the conductivity parameter and the slipping length. For variably conducting and no-slip walls, increasing the wall conductivity c for a fixed Ha leads to a decrease in the magnitude of the velocity V. At the same time, the profiles of the induced magnetic field B become perpendicular to the duct walls. Conversely, for variably conducting and slipping walls, the slip length α has a notable impact on the velocity V that, increasing α for a fixed Ha results in an increase in the magnitude of the velocity V.

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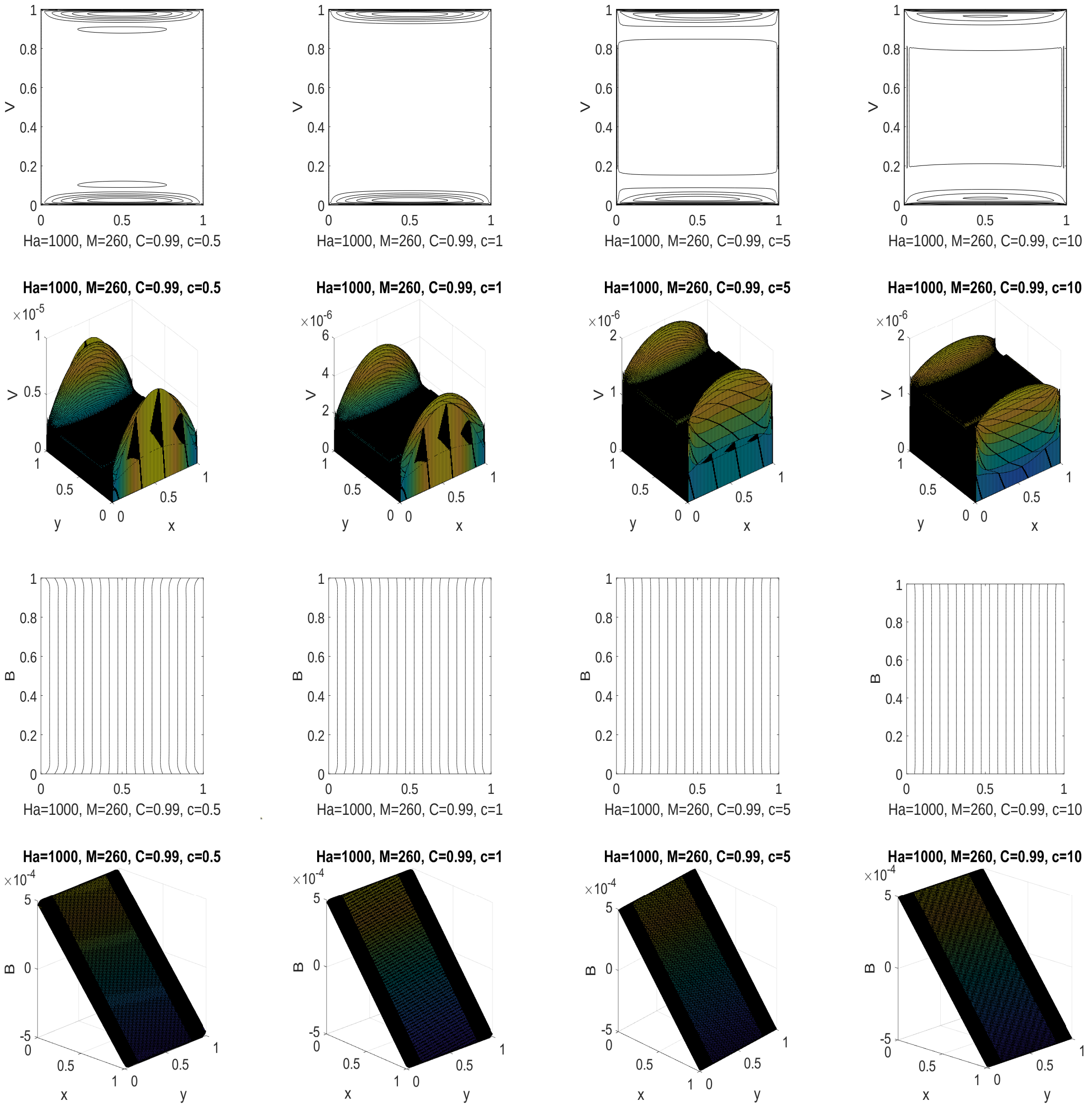

Fig. 3: Velocity and the induced magnetic field profiles for no-slip and variably conducting walls when $Ha = 1000$.

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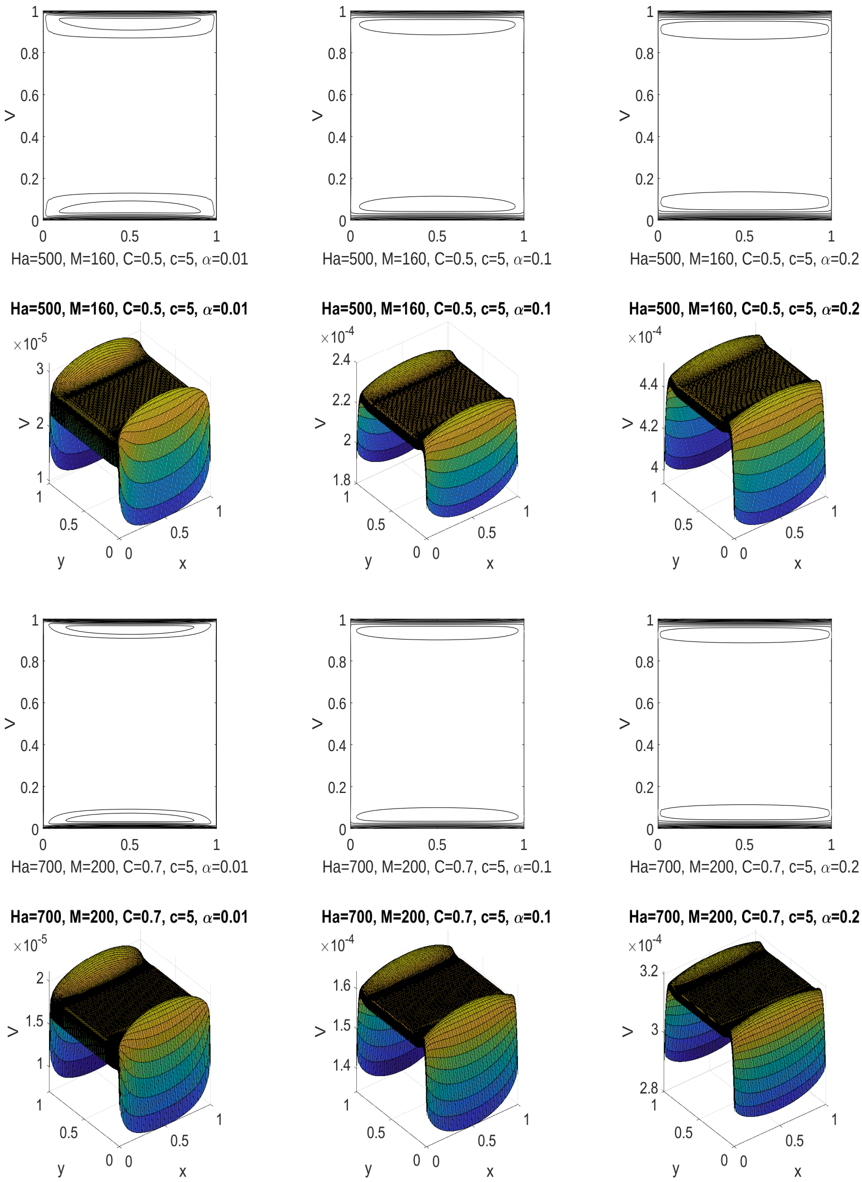

Fig. 4: Velocity and the induced magnetic field profiles for slip and variably conducting walls when $Ha = 500$ and 700.

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A Study on T-Magnetic Curves

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Abstract: In this study, we handled T-magnetic curves in three dimensional Kenmotsu manifolds. We obtained necessary and sufficient conditions of T-magnetic and T-magnetic Legendre curves to be biharmonic, f-harmonic and f-biharmonic. We introduced some non-existence theorems.

Keywords: Kenmotsu manifold, Magnetic curves.

1 Introduction

In this section, the literature is briefly discussed.

- A geometric point of view to the magnetic fields in 3-dimensional Sasakian manifolds was given by Cabrerizo, Fernandez and Gomez in 2009, [1].
- After this, many authors began to work on magnetic curves in different types of manifolds. Some basic articles can be listed as; [2–6].
- Unlike previous studies, Perktaş et al. studied biharmonicity and biminimality conditions, Bozdağ et al. studied f-harmonicity, f-biharmonicity, bi-f-harmonicity and f-biminimality conditions of magnetic curves in 3-dimensional normal almost paracontact metric manifolds, respectively in [7, 8].
- On the other hand, harmonic and biharmonic maps between Riemannian manifolds investigated by Eells and Sampson in 1964, [9].
- f-harmonic maps between Riemannian manifolds were defined by Lichnerowicz in 1970 and studied by Lemaire and Eells in 1978, [10].
- Then f-biharmonic maps between Riemannian manifolds are defined by Lu in 2013-2015, [11, 12]. And Ou gave complete classification of f-biharmonic curves in 3D Euclidean space and characterization of f-biharmonic curves in n-dimensional space forms in 2014, [13].
- Finally, Sarkar et. all studied Legendre curves in 3-dimensional trans-Sasakian manifolds in 2014, [14].
- In this study, we focus on biharmonicity, f-harmonicity and f-biharmonicity of a T-magnetic curve in Kenmotsu manifolds.

2 Preliminaries

In this section, we give a brief review of basic facts of this presentation.

Definition 1. *Let (M, g) and $(\bar{M}, \bar{g})$ be Riemannian manifolds, then a harmonic map $\phi : (M, g) \to (\bar{M}, \bar{g})$ is defined as the critical point of the energy functional*

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g,$$

where v_g is the volume element of (M, g). Then by using Euler-Lagrange equation $\tau(\phi)$ of the energy functional $E(\phi)$, where it is the tension field of map ϕ, a map called as harmonic if

$$\tau(\phi) := trace\nabla d\phi = 0. \quad (1)$$

Here ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{M}}$ of $\bar{M}$ and the pull-back connection ∇^{ϕ}, [15].

Biharmonic maps, which can be considered as a natural generalization of harmonic maps, are defined as below.

Definition 2. *A map $\phi : (M, g) \to (\bar{M}, \bar{g})$ is defined as a biharmonic map if it is a critical point, for all variations, of the bienergy functional*

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g.$$

Then the Euler-Lagrange equation $\tau_2(\phi)$, for the bienergy functional $E_2(\phi)$, where $\tau_2(\phi)$ is the bitension field of map ϕ equals to

$$\tau_2(\phi) = trace(\nabla^{\phi}\nabla^{\phi} - \nabla^{\phi}_{\nabla})\tau(\phi) - trace(R^{\bar{M}}(d\phi, \tau(\phi))d\phi) = 0,$$

if ϕ is a biharmonic map. Here $R^{\bar{M}}$ is the curvature tensor field of $\bar{M}$, [15].

Definition 3. *A map $\phi : (M, g) \to (\bar{M}, \bar{g})$ is said to be an f-harmonic if it is critical point of f-energy functional,*

$$E_f(\phi) = \frac{1}{2}\int_M f|d\phi|^2 dv_g,$$

where $f \in C^{\infty}(M, \mathbb{R})$ is a positive smooth function. Then the f-harmonic map equation obtained by using Euler-Lagrange equation as follows;

$$\tau_f(\phi) = f\tau(\phi) + d\phi(gradf) = 0, \tag{2}$$

where $\tau_f(\phi)$ is the f-tension field of the map ϕ.
f-harmonic maps are generalizations of harmonic maps, [15, 16].

Definition 4. *A map $\phi : (M, g) \to (\bar{M}, \bar{g})$ is said to be an f-biharmonic if it is critical point of the f-bienergy functional*

$$E_{2,f}(\phi) = \frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g.$$

The Euler-Lagrange equation for the f-biharmonic map is given by

$$\tau_{2,f}(\phi) = f\tau_2(\phi) + \Delta f\tau(\phi) + 2\nabla^{\phi}_{gradf}\tau(\phi) = 0, \tag{3}$$

where $\tau_{2,f}(\phi)$ is the f-bitension field of the map ϕ.
A f-biharmonic map turns into a biharmonic map if f is a constant, [16].

Now let recall some basic definitions about Kenmotsu manifolds and magnetic curves (see [18, 19]).

A differentiable manifold M of dimension $(2n+1)$ is called almost contact manifold with the almost contact structure (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ, a $1-$form η satisfying the following conditions:

$$\varphi^2 = -I + \eta \otimes \xi, \tag{4}$$

$$\eta(\xi) = 1, \tag{5}$$

where I denotes the identity transformation. As an immediate consequences of the conditions (4) we have $\varphi\xi = 0, \quad \eta \circ \varphi = 0$.

If a $(2n+1)$-dimensional almost contact manifold M with an almost contact structure (φ, ξ, η) admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \Gamma(TM), \tag{6}$$

then we say that M is an almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g).

From (6) it can be easily seen that

$$g(X, \varphi Y) = -g(\varphi X, Y), \tag{7}$$

$$g(X, \xi) = \eta(X), \tag{8}$$

for any $X, Y \in TM$.

An almost contact metric structure (φ, ξ, η, g) on a manifold M is called trans-Sasakian structure if there exist two functions α and β on an almost contact metric manifold M satisfying

$$(\nabla_X \varphi)Y = \alpha\,(g(X, Y)\xi - \eta(Y)X) + \beta\,(g(\varphi X, Y)\,\xi - \eta(Y)\varphi X), \tag{9}$$

for any $X, Y \in \Gamma(TM)$ and ∇ Levi-civita connection, then M is called a trans-Sasakian manifold, [18]. Finally if $\alpha = 0$ and $\beta = 1$ then a trans-Sasakian manifold M is called a Kenmotsu manifold.

For a 3-dimensional Kenmotsu manifold, the curvature tensor field equation given as below,

$$\begin{aligned} R(X, Y)Z &= (\frac{r}{2} + 2)(g(Y, Z)X - g(X, Z)Y) \\ &+ (\frac{r}{2} + 3)(g(X, Z)\eta(Y) - g(Y, Z)\eta(X))\xi \\ &+ (\frac{r}{2} + 3)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X), \end{aligned} \tag{10}$$

where r is the scalar curvature of the manifold M and $X, Y, Z \in \Gamma(TM)$.

The contact distribution of an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by

$$\{X \in \Gamma(TM) : \eta(X) = 0\}$$

and an integral curve of the contact distribution is called a Legendre curve, [20].

3 On Magnetic Curves

In this section we derive biharmonicity, f-harmonicity and f-biharmonicity conditions for T-magnetic curves and T-magnetic Legendre curves in Kenmotsu manifolds.

The Serret-Frenet vectors of a charged particle are affected by a magnetic field when this charged particle entered into this area. Then with this effect, a force called Lorentz force becomes exposed and so this charged particle begin to trace an orbit called magnetic curve. The trajectories of charged particles moving on a Riemannian manifold under the action of a magnetic field are defined as magnetic curves, in [6].

A smooth curve $\gamma : I \subset \mathbb{R} \to M$ on a 3-dimensional Kenmotsu manifold is called T-magnetic curve if satisfies

$$\nabla_T T = \varphi T \tag{11}$$

where $T = \dot{\gamma}$. Here "." denotes the differentiation with respect to the arc parameter.

Then with the help of (9), (10) and (11), we get

$$\nabla_T^2 T = -T - \eta(T)\varphi T + \eta(T)\xi, \tag{12}$$

$$\nabla_T^3 T = -2\eta(T)T + (\eta(T)^2 - 1)\varphi T - 2\eta(T)^2\xi, \tag{13}$$

$$R(\nabla_T T, T)T = R(\varphi T, T)T = \big[(\frac{r}{2}+2) - \eta(T)^2(\frac{r}{2}+3)\big]\varphi T.$$

4 Biharmonic T-Magnetic Curves in Kenmotsu Manifolds

Let $\gamma : I \subset \mathbb{R} \to M$ be a curve in M. Then with the help of bitension field, we get the biharmonicity condition as below;

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T + R(\nabla_T T, T)T \\ &= 2\eta(T)T \\ &+ \big[(\frac{r}{2}+1) - \eta(T)^2(\frac{r}{2}+2)\big]\varphi T \\ &- 2\eta(T)^2\xi \\ &= 0. \end{aligned} \tag{14}$$

From (14), we get following theorem.

Theorem 1. *Let M be a 3-dimensional Kenmotsu manifold and $\gamma : I \subset \mathbb{R} \to M$ be a T-magnetic curve. Then γ is a biharmonic T-magnetic curve iff the followings holds:*

$$\begin{cases} 2\eta(T) = 0, \\ (\frac{r}{2}+1) - \eta(T)^2(\frac{r}{2}+2) = 0, \\ -2\eta(T)^2 = 0. \end{cases} \tag{15}$$

Theorem 2. *There is no non-Legendre biharmonic T-magnetic curve in 3-dimensional Kenmotsu manifold.*

Theorem 3. *Let M be a 3-dimensional Kenmotsu manifold and $\gamma : I \subset \mathbb{R} \to M$ be a T-magnetic curve. Then γ is a biharmonic T-magnetic Legendre curve iff the scalar curvature $r = -2$.*

Proof: Since γ is a Legendre curve, $\eta(T) = 0$. By using this in 15, it is easy to get that $\frac{r}{2} + 1 = 0$. □

5 f-Harmonic T-Magnetic Curves in Kenmotsu Manifolds

In this section, we investigate the f-harmonicity condition for a T-magnetic curve in a 3-dimensional Kenmotsu manifold M.

Let $\gamma : I \subset \mathbb{R} \to M$ be a T-magnetic curve then via definition (3), the f-harmonicity condition given as below;

$$\tau_f(\gamma) = f^{'}T + f\nabla_T T = f^{'}T + f\varphi T = 0. \tag{16}$$

From (16), we get following nonexistence theorem.

Theorem 4. *There does not exist a proper f-harmonic T-magnetic curve in a Kenmotsu manifold.*

Proof: By using the condition given in (16), it is easy to see that $f^{'} = 0$ so f is a constant. This situation contradicts the definition of an f-harmonic curve. □

6 f-Biharmonic T-Magnetic Curves in Kenmotsu Manifolds

Here, we derive the f-biharmonicity condition for a T-magnetic curve in Kenmotsu manifolds. By substuting (11), (12), (13) and (14) in the equation of f-bitension field $\tau_{2,f}(\gamma)$, we obtained f-biharmonicity condition as below;

$$\begin{aligned}\tau_{2,f}(\gamma) &= f(\nabla_T^3 T + R(\nabla_T T, T)T) + 2f^{'}\nabla_T^2 T + f^{''}\nabla_T T \\ &= \big[2f\eta(T) - 2f^{'}\big]T \\ &+ \big[f[(\frac{r}{2}+1) - \eta(T)^2(\frac{r}{2}+2)] - 2f^{'}\eta(T) + f^{''}\big]\varphi T \\ &+ \big[-2f\eta(T)^2 + 2f^{'}\eta(T)\big]\xi \\ &= 0. \end{aligned} \tag{17}$$

From (17), we get following theorem.

Theorem 5. *Let M be a 3-dimensional Kenmotsu manifold and $\gamma : I \subset \mathbb{R} \to M$ be a T-magnetic curve. Then γ is a f-biharmonic T-magnetic curve iff the followings holds:*

$$\begin{cases} f\eta(T) - f^{'} = 0, \\ f[(\frac{r}{2}+1) - \eta(T)^2(\frac{r}{2}+2)] - 2f^{'}\eta(T) + f^{''} = 0, \\ -f\eta(T)^2 + f^{'}\eta(T) = 0. \end{cases} \tag{18}$$

Theorem 6. *There does not exist proper f-biharmonic T-magnetic Legendre curve in 3-dimensional Kenmotsu manifold.*

Proof: We know that from the definiton of a Legendre curve $\eta(T) = 0$ and by substutiting this to the first equation of (18), it is easy to see that $f^{'} = 0$ thus f is a constant. This situation contradicts the definition of a proper f-biharmonic curve. □

Theorem 7. *Let M be a 3-dimensional Kenmotsu manifold and $\gamma : I \subset \mathbb{R} \to M$ be a T-magnetic curve. Then γ is a non-proper f-biharmonic T-magnetic Legendre curve iff the scalar curvature $r = -2$.*

Proof: By using the properties of being a non-proper Legendre curve to the second equation of (18), then it is easy to see that $r = -2$. □

7 Conclusion

In our study, the harmonic, biharmonic and f-harmonic conditions of T-magnetic curves were investigated in Kenmotsu manifolds. Although studies on this type of special curves exist in the literature, our study is a first in Kenmotsu manifolds. For future research, it will be planned to investigate the characteristics of N-magnetic and B-magnetic curves to be harmonic, biharmonic, f-harmonic, bi-f-harmonic, etc.

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Results on eigenvalue problems of nonlinear conformable fractional differential equations

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Abstract: In this article, we consider the basic properties of nonlinear conformable fractional differential equation of order $\alpha \in (1, 2)$. Necessary conditions for existence of eigenfunctions are provided with the help of the maximum principle. Lower and upper bounds for the eigenvalues are estimated. We also convert the fractional differential equation to an equivalent integral equation in order to obtain a sufficient condition for the nonexistence of ordered solutions.

1 Introduction

We shall consider the following nonlinear eigenvalue problem of conformable fractional differential equation

$$\begin{aligned} {}_tT_\alpha u(x) + g(t)u' + h(t)u &= -\lambda k(t, u), \quad \alpha \in (1, 2], \quad t \in [0, 1], \\ u(0) = u'(0), &\quad u(1) + u'(1) = 0. \end{aligned} \tag{1}$$

where $g, h \in C\big(0, 1\big)$ is a continuous function and $k \in C^1([0, 1] \times \mathbb{R})$, and ${}_tT_\alpha$ is the conformable fractioan derivative of order α.

Recently, many research papers are devoted to fractional boundary value problems since they can be used to model many physical phenomena including engineering, physics, aviscoelasticity, electrochemistry and electromagnetics; [3, 12] and the references therein. This article studies the eigenvalues problems of nonlinear fractional differential equations involving the conformable fractional derivative. In recent years there has been growing interests on the conformable fractional differential equations [13, 15, 16].

In the literature, there exist various definitions for the fractional derivative. On the other hand, there are two fractional derivatives commonly used in the fractional differential equations. These are the Riemann-Lioville and Caputo fractional derivatives. Both definitions rely on the Riemann-Lioville fractional integral operator. However, some good properties of the classical derivatives do not hold for these fractional derivatives. Recently, a new and simple fractional derivative so-called conformable fractional derivative has been defined based on the limit process in [5].

Many works have studied the existence and uniqueness of conformable fractional boundary value problems [1, 4, 6, 7, 10, 11]. Conformable Sturm-Lioville eigenvalue problems are studied in [2], and the extremal solution with integral boundary condition has been presented in [9].

In [14], the authors apply the method of lower and upper solutions with the monotone iterative scheme to a periodic boundary value problem of impulsive conformable fractional integro-differential equations and they provide sufficient conditions for the existence of solutions. The method of lower and upper solutions with the monotone iterative schemes can produce two successive sequences approximating to the extremal solutions of nonlinear differential equations, see, e.g., [8, 17].

The organization of this paper is as follows. In Section 2, we recall the definition of the Caputo-Fabrizio fractional derivative and integration and its properties. In Section 3, the existence and uniqueness of the solutions of the problem are investigated. We give an example to demonstrate the applicability of the results in the last section.

2 Preliminaries

This section introduces some basic definitions and tools that will be used in the following analysis.

Definition 1. *[5] Let $f : [a, \infty) \to \mathbb{R}$ and $\alpha \in (0, 1]$. The conformable fractional derivative is defined as*

$${}_tT^a_\alpha\,(f)\,(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon(t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}, \quad t > 0. \tag{2}$$

If $a = 0$, we take ${}_tT_\alpha$.

The conformable fractional integral of order $\alpha \in (0,1]$ is defined as follows.

Definition 2. *[5, 15] Let $a \geq 0$ and $t \geq a$. The conformable fractional integral of order $\alpha \in (0,1)$ of a function defined on $(a,t]$ is defined as*

$$ {}_tI_a^\alpha f(t) = \int_a^t \frac{f(x)}{(x-a)^{1-\alpha}} dx, \tag{3}$$

if the integral exists.

The higher-order conformable fractional derivative and integral can be defined based on the above definitions.

Definition 3. *[15] Let $\alpha \in (n, n+1]$, and set $\beta = \alpha - n$. Then, the (left) fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order α, where $f^{(n)}(t)$ exists, is defined by $(\boldsymbol{T}_\alpha^a f)(t) = \left(T_\beta^a f^{(n)}\right)(t)$.*

When $a = 0$ we write $\boldsymbol{T}_\alpha$.

Definition 4. *[15] Let $\alpha \in (n, n+1]$ then the left conformable fraction integral of order α starting at a is defined by*

$$\left(I_\alpha^\alpha f\right)(t) = \boldsymbol{I}_{n+1}^\alpha \left((t-a)^{\beta-1} f\right) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx$$

The next lemma shows the conformable fractional derivative satisfies the following good properties which does not hold for the commonly used fractional derivative operators.

Lemma 1. *[15] Let $f, g : [a, \infty) \to \mathbb{R}$ be α-differentiable at a point $t \geq a$ with $\alpha \in (0,1)$, then*

(a) ${}_tT_\alpha^a(af + bg) = a_tT_\alpha^a(f) + b_tT_\alpha^a(g)$, *for all real constants a, b.*
(b) ${}_tT_\alpha^a(t^\mu) = \mu t^{\mu-\alpha}$ *for all $\mu \in \mathbb{R}$,*
(c) ${}_tT_\alpha^a(fg) = f_tT_\alpha^a(g) + g_tT_\alpha^a(f)$,
(d) ${}_tT_\alpha^a(\frac{f}{g}) = \dfrac{f_tT_\alpha^a(g) - g_tT_\alpha^a(f)}{g^2}$,
(e) ${}_tT_\alpha^a(fog)(t) = (t-a)^{1-\alpha} g'(t) f'(g(t))$. *(g is a function defined in the range of f and also differentiable),*

where $f_t = \frac{df}{dt}$ and $g_t = \frac{dg}{dt}$. Moreover, if f is differentiable, then ${}_tT_\alpha^a(f)(t) = (t-a)^{1-\alpha}\frac{df}{dt}$.

Theorem 1. *[15] Assume that $f : [a, \infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$\boldsymbol{T}_\alpha^a I_\alpha^a f(t) = f(t)$$

Theorem 2. *Let $\alpha \in (n, n+1]$ and $f : [a, \infty) \to \mathbb{R}$ be $(n+1)$-differentiable function for $t \geq a$. Then we have*

$$I_\alpha^a \boldsymbol{T}_\alpha^a(f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

Definition 5. *A function $v(t) \in C^2[0,1]$ is called a lower solution of the problem (1) if it satisfies*

$$P(v) = {}_tT_\alpha v(t) + g(t)v' + h(t)v + \lambda k(t, v) \geq 0, \quad t \in (0,1), 1 < \alpha < 2$$

and

$$v(0) - v'(0) \leq 0, \quad v(1) + v'(1) \leq 0$$

Analogously, a function $w(t) \in C^2[0,1]$ is called an upper solution if it satisfies the above inequalities with reversed signs.

If $v(t) \leq w(t)$, for all $t \in [0,1]$, we say that v and w are ordered lower and upper solutions.

3 Estimations of the eigenvalues

This section introduces a fundamental lemma which provides the positivity of the conformable fractional derivative. We also present some results on the lower and uppur solutions of the problem (1). Furthermore, we provide necessary conditions for the existence of eigenpairs with estimates on the bounds for the eigenvalues.

Theorem 3. *Assume that $f \in C^2[0,1]$ attains its minimum at $t_0 \in (0,1)$, then*

$${}_tT_\alpha f(t_0) \geq 0$$

Proof: From Lemma 1, one has ${}_tT_\alpha f(t_0) = t^{2-\alpha} f''(t_0)$. Since t_0 is a minimum pint, the classical calculus implies that $f''(t_0) \geq 0$. Combining this result with the fact that $t^{2-\alpha} \geq 0$ for $\alpha \in (1,2]$ yields the desired result. Thus, we complete the proof. □

Lemma 2 (Positivity Result). *Let $z(t) \in C^2[0,1],\ \mu(t,z) \in C([0,1]\times\mathbb{R})$ and $\mu(t,z)<0, \forall t \in (0,1)$. If $z(t)$ satisfies the following inequalities*

$$\begin{aligned} &{}_tT_\alpha z(t) + a(t)z'(t) + \mu(t,z)z \le 0, \quad t \in (0,1) \\ &z(0) - z'(0) \ge 0, \quad \text{and} \quad z(1) + z'(1) \ge 0. \end{aligned} \tag{4}$$

where $a(t) \in C[0,1]$, then $z(t) \ge 0$, for all $t \in [0,1]$.

Proof: Assume that, to reach a contradiction, the conclusion is false, then $z(t)$ has absolute minimum at some point t_0 with $z(t_0)<0$. Let $t_0 \in (0,1)$. Since t_0 is an extreme point of $z(t)$, one has $z'(t_0)=0$. From Theorem 3, we have $({}_tT_\alpha z)(t_0) \ge 0$. These results together with the assumption that $\mu(t_0, z(t_0)) < 0$ yield

$${}_tT_\alpha(t_0) + a(t_0)z'(t_0) + \mu(t_0, z(t_0))z(t_0) = {}_tT_\alpha(t_0) + \mu(t_0, z(t_0))z(t_0) > 0,$$

which contradicts to the assumption (4). If $t_0=0$, by the maximum principle, $z'(0^+) \ge 0$. Applying the boundary condition $z(0)-z'(0) \ge 0$, we have $z(0) \ge 0$ which contradicts to the claim that $z(t_0)<0$. Similarly, if $t_0=1$, then the maximum principle implies $z'(1^-) \le 0$. The boundary condition $z(1)+z'(1^-) \ge 0$ yields $z(1) \ge 0$ which also contradicts to the claim. As a result, we arrive at a contradiction. Thus, our claim is false and we must have $z(t) \ge 0$. Therefore, we complete the proof. □

Theorem 4. *Consider the problem (1). If $h(t) + \lambda\frac{\partial k(t,u)}{\partial u} < 0$, for all $u \in C^2[0,1]$ and $t \in (0,1)$, then we have*

(1) Any lower and upper solutions are ordered.
(2) The problem (1) possesses at most one solution.

Proof: (1) Let v and w respectively, be any lower and upper solutions of the problem (1). We then have

$$\begin{aligned} {}_tT_\alpha v(t) + g(t)v'(t) + h(t)v + \lambda k(t,v) \ge 0, \quad t \in (0,1), \\ v(0) - v'(0) \le 0, \quad v(1) + v'(1) \le 0, \end{aligned} \tag{5}$$

and

$$\begin{aligned} {}_tT_\alpha w(t) + g(t)w'(t) + h(t)w + \lambda k(t,w) \le 0, \quad t \in (0,1), \\ w(0) - w'(0) \ge 0, \quad w(1) + w'(1) \ge 0. \end{aligned} \tag{6}$$

Subtracting (5) from (6), we have

$${}_tT_\alpha(w-v) + g(t)\left(w'-v'\right) + h(t)(w-v) + \lambda(k(t,w)-k(t,v)) \le 0.$$

With the help of the mean value theorem, one has

$${}_tT_\alpha(w-v) + g(t)\left(w'-v'\right) + \left(h(t) + \lambda\frac{\partial k}{\partial u}(\xi)\right)(w-v) \le 0,$$

where $\xi = \gamma w + (1-\gamma)v$ and $0 \le \gamma \le 1$. Let $z(t) = w(t) - v(t)$ for $t \in (0,1)$. Then the function z satisfies the following inequality

$${}_tT_\alpha z + g(t)z' + \left(h(t) + \lambda\frac{\partial k}{\partial u}(\xi)\right)z \le 0,$$

subject to the boundary conditions $z(0)-z'(0) \ge 0$ and $z(1)+z'(1) \ge 0$. Using the assumption that $h(t) + \lambda\frac{\partial k}{\partial u}(\xi) < 0$ and the positivity result of Lemma 2, we obtain at once $z \ge 0$. But this implies that $w \ge v$ by the definition of z. This completes the proof of (1)

(2) Next, we shall prove (2). Let u_1 and u_2 be two solutions of the problem (1). Then these two solution must satisfy the following equation

$${}_tT_\alpha u_1 + g(t)u_1' + h(t)u_1 + \lambda k(t,u_1) = 0, \tag{7}$$

$${}_tT_\alpha u_2 + g(t)u_2' + h(t)u_2 + \lambda k(t,u_2) = 0, \tag{8}$$

subject to the boundary conditions $u_1(0) - u_1'(0) = u_2(0) - u_2'(0) = 0$, and $u_1(1) + u_1'(1) = u_2(1) + u_2'(1) = 0$. By subtracting (8) from (7), using the mean value theorem, and letting $z = u_1 - u_2$, we have

$${}_tT_\alpha z + g(t)z' + \left(h(t) + \lambda\frac{\partial k}{\partial u}(\xi)\right)z = 0 \tag{9}$$

for some ξ between u_1 and u_2, with the boudnary conditions

$$z(0) - z'(0) = 0 \quad \text{and} \quad z(1) + z'(1) = 0. \tag{10}$$

Using the our assumption that $h(t) + \lambda\frac{\partial k}{\partial u}(\xi) < 0$ and the positivity result of Lemma 2 we conclude that $z \ge 0$. On the other hand, we stress out that the function $-z$ also satisfies the equations given by (9)-(10). Again by the assumption that $h(t) + \lambda\frac{\partial k}{\partial u}(\xi) < 0$ and the positivity result of Lemma 2 we conclude that $z \geq 0$. Consequently, we msut have $z = 0$. This implies that $u_1 = u_2$ and the problem (1) has at most one solution. The proof is now completed. □

The following corollary gives analytical lower and upper bounds estimates of the eigenvalues.

Corollary 1. *Consider the eigenvalue problem (1), with $k(t, 0) = 0$. We have the following necessary conditions for the existence of a nontrivial eigenfunction.*

(1) If there exists a negative constant ξ such that $\frac{\partial k}{\partial u} \leq \xi < 0$, then $\lambda \leq \sup \left\{ -h / \frac{\partial k}{\partial u} \right\}$

(2) If there exists a positive constant μ such that $\frac{\partial k}{\partial u} \geq \mu > 0$, then $\lambda \geq \inf \left\{ -h / \frac{\partial k}{\partial u} \right\}$

Proof: (1) Assume that, to reach a contradiction, the eigenvalues λ of the problem satisfy $\lambda > \sup \left\{ -h / \frac{\partial k}{\partial u} \right\}$. Then one has $\lambda > -h / \frac{\partial k}{\partial u}$ for all $t \in [0, 1]$. Using the assumption that $\frac{\partial k}{\partial u} < 0$, we obtain $h(t) + \lambda \frac{\partial k}{\partial u} < 0$. By Theorem 4, the problem (1) possesses at most one solution. The assumption $k(t, 0) = 0$ implies that $u = 0$ is a solution. Hence, the problem (1) has only the zero solution and thus there is no nontrivial eigenfunction.

(2) Assume that $\lambda < \inf \left\{ -h / \frac{\partial k}{\partial u} \right\}$. We get $\lambda < -h / \frac{\partial k}{\partial u}$ for all $t \in [0, 1]$. By the assumption that $\frac{\partial k}{\partial u} > 0$, one must have $h(t) + \lambda \frac{\partial k}{\partial u} < 0$. T Hence, the problem (1) has only the zero solution and thus there is no nontrivial eigenfunction. □

4 Ordered solutions

A sufficient condition for non-existence of ordered solutions for the problem (1) is provided in this section. This result will be used in the next section in proving the uniqueness of the solution.

Definition 6. *Let $u_1 \neq u_2$ be two solutions of (1). We say that u_1 and u_2 are ordered solutions, if either $u_1 \leq u_2$ or $u_2 \leq u_1$ for all $t \in [0, 1]$.*

Lemma 3. *Consider the eigenvalue problem (1) with $g, h \in C[0, 1]$ and $k \in C^1([0, 1] \times \mathbb{R})$. A function $u(t) \in C^2[0, 1]$ solves the problem (1) if and only if it solves the following integral equation*

$$\begin{aligned} u(t) = &\Big(\frac{2}{3} \int_0^1 s^{\alpha-2} H(s, u) ds - \frac{1}{3} \int_0^1 s^{\alpha-1} H(s, u) ds \Big) (1 + t) \\ &- \int_0^t (t - s) s^{\alpha-2} H(s, u) ds, \end{aligned} \tag{11}$$

where $H(s, u) = g(s) u'(s) + h(s) u(s) + \lambda k(s, u)$.

Proof: Let $u(t)$ be a solution of the problem (1). Applying the operator ${}_tI_a^\alpha$ to both sides of (1) and using Theorem 2 and $H(s, u) \in C[0, 1]$, we obtain

$$\begin{aligned} u(t) &= u(0) + u'(0)t - {}_tI_a^\alpha H(t, u(t)) \\ &= u(0) + u'(0)t - \int_0^t (t - s) s^{\alpha-2} H(s, u) ds \\ &= u(0) + u'(0)t - \int_0^t (t - s) s^{\alpha-2} \big(g(s) u'(s) + h(s) u(s) + \lambda k(s, u) \big) ds. \end{aligned}$$

Differentiating the above equation yields

$$u'(t) = u'(0) - \int_0^t s^{\alpha-2} H(s, u) ds.$$

Applying the boundary conditions, we find that $u'(0) = u(0) = \frac{2}{3} \int_0^1 s^{\alpha-2} H(s, u) ds - \frac{1}{3} \int_0^1 s^{\alpha-1} H(s, u) ds$. Substituting these values in the above equation gives the desired result (11).

Conversely, assume that Let $u(t)$ be a solution of (11). Substituting

$$\nu = \frac{2}{3} \int_0^1 s^{\alpha-2} H(s, u) ds - \frac{1}{3} \int_0^1 s^{\alpha-1} H(s, u) ds$$

yields

$$u(t) = \nu (1 + t) - \int_0^t (t - s) s^{\alpha-2} H(s, u) ds. \tag{12}$$

Differentiating the above expression with respect to the variable t, we have

$$u'(t) = \nu - \int_0^t s^{\alpha-2} H(s, u) ds.$$

Now, we note that $u(0) = u'(0) = \nu$ and $u(1) + u'(1) = 0$. Applying the conformable fractional derivative operator ${}_tT_\alpha$ to both sides of (12) gives

$$ {}_tT_\alpha u(t) = {}_tT_\alpha(\nu(1+t) - \int_0^t (t-s)s^{\alpha-2}H(s,u)ds) = -H(t,u). $$

By Theorem 1, we infer that u solves (1). Thus, the proof is completed.

□

Theorem 5. *Consider problem (1) with $g, h \in C[0,1], k \in C^1([0,1]\times \mathbb{R}), u \in C^2[0,1]$, and $g(t) \geq 0, t \in [0,1]$. If $h(t) + \lambda \frac{\partial k}{\partial u} \geq \eta > 0$, for some positive constant $\eta > (\alpha-1)t_0^{1-\alpha}$ with $t_0 \in (0,1)$, then the problem has no ordered solutions.*

Proof: Let $u_1 \leq u_2$ be two solutions of (1). We have

$$ u_1(t) = u_1(0) + u_1'(0)t - \int_0^t (t-s)s^{\alpha-2}H(s,u_1)\,ds, $$
$$ u_2(t) = u_2(0) + u_2'(0)t - \int_0^t (t-s)s^{\alpha-2}H(s,u_2)\,ds, $$

where $H(s,u(s)) = g(s)u'(s) + h(s)u(s) + \lambda k(s,u(s))$. Let $z(t) = u_2(t) - u_1(t) \geq 0 \in [0,1]$. We have

$$ \begin{aligned} z(t) =& u_2(0) - u_1(0) + \left(u_2'(0) - u_1'(0)\right)t \\ & - \int_0^t (t-s)s^{\alpha-2}\left(H(t,u_2(t)) - H(t,u_1(t))\right) \\ z(0) =& z'(0) \quad \text{and} \quad z(1) = -z'(1). \end{aligned} \tag{13} $$

Thus,

$$ \begin{aligned} z'(t) &= (u_2'(0) - u_1'(0) - \int_0^t s^{\alpha-2}\left(H(t,u_2) - H(t,u_1)\right) \\ &= z'(0) - \int_0^t s^{\alpha-2}\left(H(s,u_2) - H(s,u_1)\right)ds. \end{aligned} $$

Substituting $z' = u_2' - u_1'$ in the above equation and applying the mean value theorem yields

$$ \begin{aligned} z'(t) =& z'(0) - \int_0^t s^{\alpha-2}\left(g(s)\left(u_2' - u_1'\right)\right. \\ & \left. + h(s)\left(u_2 - u_1\right) + \lambda\left[k(s,u_2) - k(s,u_1)\right]\right)ds \\ =& z'(0) - \int_0^t s^{\alpha-2}\left(g(s)z'(s) + \left[h(s) + \lambda\frac{\partial k}{\partial u}(\xi)\right]z(s)\right)ds, \end{aligned} \tag{14} $$

for some ξ between u_1 and u_2. To reach a contradiction, we assume now $z(t) \neq 0$ for $t \in (0,1)$, then $z(t)$ will have a positive maximum in $[0,1]$. Let $t_0 \in [0,1]$ be such a positive maximum point. If $t_0 = 0$, then $z'\left(0^+\right) \leq 0$ by the classical calculus. But the boundary conditions in (13) imply that $z(0) \leq 0$ which is a contradiction. Using the similar argument, if $t_0 = 1$, then $z'\left(1^-\right) \geq 0$. Again the boundary conditions in (13) result in $z(1) \leq 0$ which is also a contradiction. consequently, $t_0 \in (0,1)$. We get $z(t_0) > 0$ and $z'(t_0) = 0$ and using the inequality $z'(0) \geq 0$, we arrive at $z'(t) \geq 0$ for all $t \in [0, t_0]$. Next, we examine two possibilities for $z(0)$.

Case 1. $z(0) = 0$; If $z(0) = 0$, then the boundary conditions (13) give $z'(0) = 0$. Using this and the above results in (14) with the facts that $g(s) \geq 0, s \in [0, t_0]$ and $h(s) + \lambda\frac{\partial k}{\partial u}(\xi) \geq \eta > 0$ yields $z'(t_0) < 0$, which contradicts to$z'(t_0) = 0$.
Case 2. $z(0) = \zeta > 0$; If $z(0) = \zeta > 0$, then we have

$$ \begin{aligned} \gamma &= \int_0^{t_0} s^{\alpha-2}\left(g(s)z'(s) + \left[h(s) + \lambda\frac{\partial k}{\partial u}(\xi)\right]z(s)\right)ds \\ &> \eta\zeta\int_0^{t_0} s^{\alpha-2}ds = \frac{\eta\zeta}{\alpha-1}t_0^{\alpha-1}. \end{aligned} \tag{15} $$

Plugging (15) into (14), we get

$$ z'(t_0) < z'(0) - \frac{\eta\zeta}{\alpha-1}t_0^{\alpha-1}. $$

Hence, since $\eta > (\alpha-1)t_0^{1-\alpha}$ it holds that

$$ z'(t_0) < \zeta - \frac{\eta\zeta}{\alpha-1}t_0^{\alpha-1} < 0 $$

which leads to a contradiction. Therefore, one must have $z = 0$. Thus, we complete the proof.

□

5 Examples

Consider the following fractional eigenvalue problem

$$\begin{aligned} {}_tT_\alpha u(x) + g(t)u' - u &= -\lambda e^t u, \quad \alpha \in (1,2], \quad t \in [0,1], \\ u(0) = u'(0), &\quad u(1) + u'(1) = 0. \end{aligned} \tag{16}$$

Here, we take

$$k(t,u) = e^t u \text{ and } h(t) = -1.$$

We observe that the partial derivative of $k(t,u)$ with respect to u $\dfrac{\partial k(t,u)}{\partial u} = e^t > 0$. Thus, there exist $\mu > 0$ such that $\dfrac{\partial k(t,u)}{\partial u} \geq \mu > 0$. Then by Corollary 1 (2) the eigenvalues of the problem (16) have the following lower bound

$$\lambda \geq \inf_{t\in[0,1]} \left(-\frac{-1}{e^t} \right) = \inf_{t\in[0,1]} \left(e^{-t} \right) = e^{-1}.$$

Consider now a more general eigenvalue problem

$$\begin{aligned} {}_tT_\alpha u(x) + h(t)u &= -\lambda r(t)u, \quad \alpha \in (1,2], \quad t \in [0,1], \\ u(0) = u'(0), &\quad u(1) + u'(1) = 0, \end{aligned} \tag{17}$$

where $h(t) < 0$ and $r(t) > 0$ or $h(t) < 0$ and $r(t) < 0$. Here we assume that $k(t,u) = r(t)u$.

We note that $\dfrac{\partial k(t,u)}{\partial u} = r(t)$. If $r(t) > 0$ and $h(t) < 0$, then by Corollary 1 (2) the eigenvalues of the problem (17) satisfy

$$\lambda \geq \inf_{t\in[0,1]} \left(-\frac{h(t)}{r(t)} \right) > 0.$$

Otherwise, if $r(t) < 0$ and $h(t) < 0$, then by Corollary 1 (1) the eigenvalues of the problem (17) satisfy

$$\lambda \leq \sup_{t\in[0,1]} \left(-\frac{h(t)}{r(t)} \right) < 0.$$

6 Conclusion

In this work, we presented necessary conditions for the existence of eigenfunctions of nonlinear fractional boundary value problems involving the conformable fractional derivative of order $\alpha \in (1,2)$ by applying the maximum principle theorem. We estimated lower and upper bounds for the eigenvlues associated with the eigenfunctions. We also provided a sufficient condition for the nonexistence of ordered solutions for the fractional boundary value problems by transferring the problem into an equivalent weakly singular integral equation. We gave two examples supporting our theoretical results.

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Fundamental Concepts in Variable Lebesgue Spaces Associated with Laplace-Bessel Differential Operator

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Abstract: In this study, we consider the concepts of convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. In variable Lebesgue spaces, there are three types of convergence: convergences with respect to modular, norm and measure. We investigate the relationship between these convergences. Furthermore, we prove that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ are Banach spaces.

Keywords: Cauchy sequence, completeness, variable Lebesgue spaces.

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1 Introduction

In this study, we investigate concepts of convergences in variable Lebesgue spaces connected with Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad 1 \le k \le n.$$

In variable Lebesgue spaces, there are three types of convergence: modular convergence, norm convergence and measure convergence. We will examine the relationship between norm, modular, and measure convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Also, we prove that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ are Banach spaces.

Variable Lebesgue spaces which have first been considered by Orlicz [2] and have a long history, play a key role in harmonic analysis theory. Indeed, these spaces are extensions of classical Lebesgue spaces by taking the exponent function $p(\cdot)$ instead of the constant exponent p. Therefore, they have many properties similar properties with $L_{p,\gamma}(\mathbb{R}^n_{k,+})$. Of course, they also differ in many ways and for this reason there is an increasing interest on variable Lebesgue spaces.

The motivation of this paper is to study fundamental concepts of analysis such as convergence, completeness in variable Lebesgue spaces. Then, we examine the relationship between norm, modular, and measure convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Also, we prove that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ are Banach spaces. Now, we are ready to recall important definitions and notations.

Let $x = (x', x'')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_k > 0,\ 1 \le k \le n\}$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$ and $|\gamma| = \gamma_1 + \ldots + \gamma_k$.

Let $\mathcal{P}(\mathbb{R}^n_{k,+}) = \Big\{p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty] : p(\cdot) \text{ is measurable}\Big\}$. Also let any element of $\mathcal{P}(\mathbb{R}^n_{k,+})$ is said to be a variable exponent function and also let

$$p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}} p(x), \qquad p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n_{k,+}} p(x),$$

satisfying the conditions for all $|x-y| \le \frac{1}{2}$, $x, y \in \mathbb{R}^n_{k,+}$,

$$|p(x) - p(y)| \le \frac{A_0}{-\log|x-y|},$$

and

$$|p(x) - p_\infty| \le \frac{A_\infty}{\log(e+|x|)}.$$

Here $p_\infty = \lim_{x\to\infty} p(x) > 1$. If the above inequalities hold for $p(\cdot)$, then we denote it by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$, and $p(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n_{k,+})$, respectively. Moreover, if $p(\cdot)$ provides both of inequalities, then it is denoted by $p(\cdot) \in LH(\mathbb{R}^n_{k,+})$. As in classical Lebesgue spaces, there exist three cases for $p(x)$, i.e., $p(x) = 1$, $p(x) = \infty$ or $1 < p(x) < \infty$. Therefore, three canonical subsets on $\mathbb{R}^n_{k,+}$ are introduced as follows:

$$(\mathbb{R}^n_{k,+})_\infty = \{x \in \mathbb{R}^n_{k,+} : p(x) = \infty\},$$
$$(\mathbb{R}^n_{k,+})_1 = \{x \in \mathbb{R}^n_{k,+} : p(x) = 1\},$$
$$(\mathbb{R}^n_{k,+})_0 = \{x \in \mathbb{R}^n_{k,+} : 1 < p(x) < \infty\}.$$

For $x \in \mathbb{R}^n_{k,+}$, conjugate exponent function is given by

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Then variable Lebesgue space is defined as follows:

$$L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+}) := \left\{ f : \|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})} = \inf\left\{\mu > 0 : \rho_{p(\cdot),\gamma}(f/\mu) \le 1\right\} < \infty \right\},$$

where f is a measurable function, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and

$$\rho_{p(\cdot),\gamma}(f) := \int_{\mathbb{R}^n_{k,+}\setminus(\mathbb{R}^n_{k,+})_\infty} |f(x)|^{p(x)} (x')^\gamma dx + \|f\|_{L_{\infty,\gamma}(\mathbb{R}^n_{k,+})_\infty} < \infty.$$

The next proposition follows easily from [1].

Proposition 1. *$f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ if and only if*

$$\rho_{p(\cdot),\gamma}(f) = \int_{\mathbb{R}^n_{k,+}} |f(x)|^{p(x)} (x')^\gamma dx < \infty,$$

where $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $p_+ < \infty$.

Lemma 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. If $\|f\|_{p(\cdot),\gamma} \le 1$, then $\rho_{p(\cdot),\gamma}(f) \le \|f\|_{p(\cdot),\gamma}$ and if $\|f\|_{p(\cdot),\gamma} > 1$, then $\rho_{p(\cdot),\gamma}(f) \ge \|f\|_{p(\cdot),\gamma}$.*

Proof: If $\|f\|_{p(\cdot),\gamma} = 0$, then $f \equiv 0$ and $\rho_{p(\cdot),\gamma}(f) = 0$. If $0 < \|f\|_{p(\cdot),\gamma} \le 1$, since the modular $\rho_{p(\cdot),\gamma}$ is convex, then we have

$$\begin{aligned}\rho_{p(\cdot),\gamma}(f) &= \rho_{p(\cdot),\gamma}\left(\|f\|_{p(\cdot),\gamma} \frac{f}{\|f\|_{p(\cdot),\gamma}}\right)\\ &\le \|f\|_{p(\cdot),\gamma}\rho_{p(\cdot),\gamma}\left(\frac{f}{\|f\|_{p(\cdot),\gamma}}\right) \le \|f\|_{p(\cdot),\gamma}.\end{aligned}$$

If $\|f\|_{p(\cdot),\gamma} > 1$, then one can write $\rho_{p(\cdot),\gamma}(f) > 1$. Also if $\rho_{p(\cdot),\gamma}(f) \le 1$, then $\|f\|_{p(\cdot),\gamma} \le 1$. But then we have

$$\begin{aligned}\rho_{p(\cdot),\gamma}\left(\frac{f}{\rho_{p(\cdot),\gamma}}\right) &= \int_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty} \left(\frac{|f(x)|}{\rho_{p(\cdot),\gamma}(f)}\right)^{p(x)} (x')^\gamma dx + \rho_{p(\cdot),\gamma}(f)^{-1}\|f\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)}\\ &\le \int_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty} |f(x)|^{p(x)}\rho_{p(\cdot),\gamma}(f)^{-1}(x')^\gamma dx + \rho_{p(\cdot),\gamma}(f)^{-1}\|f\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)} = 1.\end{aligned}$$

Consequently, we get $\|f\|_{p(\cdot),\gamma} \le \rho_{p(\cdot),\gamma}(f)$. This completes the proof. □

2 Convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$

In this section, we first give the relationship between convergences with respect to modular, norm and measure in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Given $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, it is said that $\{f_i\}$ converges with respect to norm to $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ if $\lim_{i\to\infty} \|f - f_i\|_{p(\cdot),\gamma} = 0$. If there exists $\lambda > 0$ such that $\rho_{p(\cdot),\gamma}(\lambda(f - f_i)) \to 0$ as $i \to \infty$, then it is said that f_i converges to f with respect to modular. Finally, given $\varepsilon > 0$, if

$$\lim_{i\to\infty} \nu\left\{ \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)| \ge \varepsilon\right\}\right|_\gamma < \varepsilon \right\} = 0$$

holds, then it is said that $\{f_i\}$ converges with respect to measure.

Theorem 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and for all $i \in \mathbb{N}$, $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ nonnegative functions such that f_i increases to a function f pointwise a.e. Then $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\|f_i\|_{p(\cdot),\gamma} \to \|f\|_{p(\cdot),\gamma}$ or $f \notin L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\|f_i\|_{p(\cdot),\gamma} \to \infty$.*

Proof: Since $\{f_i\}$ is an increasing sequence, $\{\|f_i\|_{p(\cdot),\gamma}\}$ is also increasing and so it either converges or diverges to ∞. If $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, then $\|f_i\|_{p(\cdot),\gamma} \leq \|f\|_{p(\cdot),\gamma}$ since $f_i \leq f$. Otherwise, since $f_i \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, we have $\|f_i\|_{p(\cdot),\gamma} < \infty = \|f\|_{p(\cdot),\gamma}$. In both cases it is sufficient to show that there holds $\mu < \|f_i\|_{p(\cdot),\gamma}$ for any $\mu < \|f\|_{p(\cdot),\gamma}$, and for all sufficiently large i.

Fix $\mu > 0$. Then it is obvious that $\rho_{p(\cdot),\gamma}(f/\mu) > 1$ and from monotone convergence theorem, we have

$$\begin{aligned}\rho_{p(\cdot),\gamma}(f/\mu) &= \int\limits_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty} \left(\frac{|f(x)|}{\mu}\right)^{p(x)} (x')^\gamma dx + \mu^{-1}\|f\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)} \\ &= \lim_{i\to\infty}\left(\int_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty} \left(\frac{|f_i(x)|}{\mu}\right)^{p(x)} (x')^\gamma dx + \mu^{-1}\|f_i\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)}\right) \\ &= \lim_{i\to\infty} \rho_{p(\cdot),\gamma}(f_i/\mu).\end{aligned}$$

Thus, $\rho_{p(\cdot),\gamma}(f_i/\mu) > 1$ and $\mu < \|f_i\|_{p(\cdot),\gamma}$ for all sufficiently large i. Therefore, we complete the proof. □

Now, we will give the analog of Fatou's Lemma.

Theorem 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. Assume that the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $f_i \to f$ pointwise a.e. If $\liminf\limits_{i\to\infty} \|f_i\|_{p(\cdot),\gamma} < \infty$, then $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\|f\|_{p(\cdot),\gamma} \leq \liminf\limits_{i\to\infty} \|f_i\|_{p(\cdot),\gamma}$.*

Proof: Firstly, we will define a sequence $g_i(x) = \inf_{i\leq m} |f_m(x)|$. Then $g_i(x) \leq |f_m(x)|$ for all $i \leq m$ and thus $g_i \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Furthermore, $\{g_i\}$ is an increasing sequence and

$$\lim_{i\to\infty} g_i(x) = \liminf_{m\to\infty} |f_m(x)| = |f(x)|$$

for $x \in \mathbb{R}^n_{k,+}$ a.e. by its definition. From Theorem 1, we get

$$\|f\|_{p(\cdot),\gamma} = \lim_{i\to\infty} \|g_i\|_{p(\cdot),\gamma} \leq \lim_{i\to\infty} \big(\inf_{i\leq m} \|f_m\|_{p(\cdot),\gamma}\big) = \liminf_{i\to\infty} \|f_i\|_{p(\cdot),\gamma} < \infty,$$

and $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Thus, this completes the proof. □

Notice that unlike the above theorems, to obtain the analog of dominated convergence theorem we have to suppose $p_+ < \infty$. The following theorem associated with convergence with respect to norm is required to convergence with respect to modular.

Theorem 3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $p_+ < \infty$. For $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, $\|f_i - f\|_{p(\cdot),\gamma} \to 0$ if and only if $\rho_{p(\cdot),\gamma}(f - f_i) \to 0$.*

Proof: Assume that $\{f_i\}$ converges with respect to norm. Then, from Lemma 1, we obtain

$$\rho_{p(\cdot),\gamma}(f - f_i) \leq \|f - f_i\|_{p(\cdot),\gamma} \leq 1,$$

for all sufficiently large i. Thus, $\rho_{p(\cdot),\gamma}(f - f_i) \to 0$.

To obtain the converse, let $\mu < 1$ be fixed. Then, we get

$$\rho_{p(\cdot),\gamma}\left(\frac{f - f_i}{\mu}\right) \leq \left(\frac{1}{\mu}\right)^{p_+} \rho_{p(\cdot),\gamma}(f - f_i).$$

This implies that $\rho_{p(\cdot),\gamma}\big(\frac{f-f_i}{\mu}\big) \leq 1$ for all i sufficiently large. Equivalently, $\|f - f_i\|_{p(\cdot),\gamma} \leq \mu$ for all sufficiently large i. Since μ is arbitrary, $f_i \to f$ with respect to norm. Therefore, we complete the proof. □

Theorem 4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $p_+ < \infty$. If there exists a sequence $\{f_i\}$ such that $f_i \to f$ pointwise a.e., and there exists $g \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $|f_i(x)| \leq g(x)$ a.e., then $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\lim\limits_{i\to\infty} \|f - f_i\|_{p(\cdot),\gamma} = 0$.*

Proof: From Proposition 1, we have

$$\begin{aligned}|f(x) - f_i(x)|^{p(x)} &\leq 2^{p(x)-1}\left(|f(x)|^{p(x)} + |f_i(x)|^{p(x)}\right) \\ &\leq 2^{p_+}|g(x)|^{p(x)} \in L_{1,\gamma}(\mathbb{R}^n_+).\end{aligned}$$

Therefore, from the dominated convergence theorem on $L_{1,\gamma}(\mathbb{R}^n_{k,+})$, $\rho_{p(\cdot),\gamma}(f - f_i) \to 0$ as $i \to 0$ and so $\|f - f_i\|_{p(\cdot),\gamma} \to 0$, by Theorem 3. Thus, we complete the proof. □

Theorem 5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. If there exists the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $\|f - f_i\|_{p(\cdot),\gamma} \to 0$, then the sequence $\{f_i\}$ converges to f with respect to measure.*

Proof: Assume that there exists a sequence $\{f_i\}$ converges to f with respect to norm but not with respect to measure. And we also assume that there exists $0 < \varepsilon < 1$ such that,

$$E_i := |\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)| \geq \varepsilon\}|_\gamma \geq \varepsilon,$$

for all i. Since there exists $|E_i \cap (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ or $|E_i \setminus (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ for each i, by taking another subsequence we suppose that one of these inequalities holds for all i.

If $|E_i \cap (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ for all i, then we find

$$\|f - f_i\|_{L_{p(\cdot),\gamma}} \geq \|(f - f_i)\chi_{(\mathbb{R}^n_{k,+})_\infty}\|_{L_{p(\cdot),\gamma}} = \|f - f_i\|_{L_{\infty,\gamma}((\mathbb{R}^n_{k,+})_\infty)} \geq \varepsilon.$$

Then, this is a contradiction. If $|E_k \setminus (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ for all k, then we get

$$\begin{aligned}\rho_{p(\cdot),\gamma}\left(\frac{f - f_k}{\varepsilon^2/2}\right) &\geq \int_{\mathbb{R}^n_{k,+}\setminus(\mathbb{R}^n_{k,+})_\infty} \left(\frac{|f(x) - f_k(x)|}{\varepsilon^2/2}\right)^{p(x)} (x')^\gamma dx \\ &\geq \int_{E_k\setminus(\mathbb{R}^n_{k,+})_\infty} \left(\frac{2}{\varepsilon}\right)^{p(x)} (x')^\gamma dx \geq \left(\frac{2}{\varepsilon}\right)^{p_-} |E_k \setminus (\mathbb{R}^n_{k,+})_\infty| \geq 1.\end{aligned}$$

Therefore, there exists $\|f - f_i\|_{L_{p(\cdot),\gamma}} \geq \varepsilon^2/2 > 0$. Again, it is a contradiction. Thus, if the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ converges to f with respect to norm, then the sequence $\{f_i\}$ converges to f in measure. □

Theorem 6. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. Assume that the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ converges with respect to norm to $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Then there exists a subsequence $\{f_{i_j}\}$ and $g \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $f_{i_j} \to f$ pointwise a.e. and $|f_{i_j}(x)| \leq g(x)$ for a.e. $x \in \mathbb{R}^n_{k,+}$.*

Proof: From Theorem 5, we have a subsequence $\{f_{i_j}\}$ such that $f_{i_j} \to f$ pointwise a.e. Furthermore, since a convergent sequence is also Cauchy, we can fix i_j large enough for each j, $\|f_{i_{j+1}} - f_{i_j}\|_{p(\cdot),\gamma} \leq 2^{-j}$.

Define for each j,

$$h_j(x) = \sum_{m=1}^{j-1} |f_{i_{m+1}}(x) - f_{i_m}(x)|,$$

which implies $\{h_j\}$ is an increasing and pointwise convergent to h. Therefore, we have

$$\|h_j\|_{p(\cdot),\gamma} \leq \sum_{m=1}^{j-1} 2^{-m} \leq 1.$$

By monotone convergence theorem, there exists $h \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. But, we get

$$|f_{i_j}(x) - f_1(x)| \leq \sum_{m=1}^{j-1} |f_{i_{m+1}}(x) - f_{i_m}(x)| = h_j(x) \leq h(x),$$

for every j and a.e. $x \in \mathbb{R}^n_{k,+}$. Hence, if we fix $g = h + |f_1|$, we have $g \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $|f_{i_j}(x)| \leq g(x)$ a.e. □

Now, we give the relationship between these convergence types.

Theorem 7. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$, $p_+ < \infty$, $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and a sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Then the followings are equivalent:*

(i) $f_i \to f$ with respect to norm,
(ii) $f_i \to f$ with respect to modular,
(iii) $f_i \to f$ with respect to measure and for some $\lambda > 0$, $\rho_{p(\cdot),\gamma}(\lambda f_i) \to \rho_{p(\cdot),\gamma}(\lambda f)$.

Proof: The equivalence of (i) and (ii) has been proved in Theorem 3. We will now prove the equivalence of (ii) and (iii).

To prove that (ii) implies (iii), by Theorem 5, notice that convergence with respect to norm implies convergence with respect to measure. To finish the proof of this argument we will obtain that convergence with respect to modular means $\rho_{p(\cdot),\gamma}(\lambda f_i) \to \rho_{p(\cdot),\gamma}(\lambda f)$ for $\lambda = 1$.

If $1 \le p < \infty$ and $u, v \ge 0$, by using the mean value theorem, then there holds

$$|u^p - v^p| \le p \max\{u^{p-1}, v^{p-1}\}|u - v| \le p(u^{p-1} + v^{p-1})|u - v|.$$

Therefore, we have

$$\begin{aligned}|\rho_{p(\cdot),\gamma}(f) - \rho_{p(\cdot),\gamma}(f_i)| &\le \int_{\mathbb{R}^n_{k,+}} \left| |f(x)|^{p(x)} - |f_i(x)|^{p(x)} \right| (x')^\gamma dx \\ &\le p_+ \int_{\mathbb{R}^n_{k,+}} \left(|f(x)|^{p(x)-1} - |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)| (x')^\gamma dx.\end{aligned}$$

To estimate the RHS, we write $\mathbb{R}^n_{k,+} = (\mathbb{R}^n_{k,+})_1 \cup (\mathbb{R}^n_{k,+})_0$. For the integral on $(\mathbb{R}^n_{k,+})_1$, we can write

$$\begin{aligned}&p_+ \int_{(\mathbb{R}^n_{k,+})_1} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)| (x')^\gamma dx \\ &\le 2p_+ \int_{(\mathbb{R}^n_{k,+})_1} |f(x) - f_i(x)|^{p(x)} (x')^\gamma dx \\ &\le 2p_+ \rho_{p(\cdot),\gamma}(f - f_i).\end{aligned}$$

Since convergences with respect to modular and norm are equivalent, RHS goes to 0 ($i \to \infty$).

To calculate the integral on $(\mathbb{R}^n_{k,+})_0$, let ε, $0 < \varepsilon < 1/4$ be fixed. Then by Young's inequality, we have

$$\begin{aligned}&p_+ \int_{(\mathbb{R}^n_{k,+})_0} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)| (x')^\gamma dx \\ &\le p_+ \int_{(\mathbb{R}^n_{k,+})_0} \frac{\varepsilon^{p'(x)}}{p'(x)} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right)^{p'(x)} (x')^\gamma dx \\ &\quad + p_+ \int_{(\mathbb{R}^n_{k,+})_0} \frac{\varepsilon^{-p(x)}}{p(x)} |f(x) - f_i(x)|^{p(x)} (x')^\gamma dx \\ &= I_1 + I_2.\end{aligned}$$

First, let us calculate I_2. Since $p(x) > 1$ for all $x \in (\mathbb{R}^n_{k,+})_0$, we get

$$I_2 \le p_+ \rho_{p(\cdot),\gamma}(\varepsilon^{-1}(f - f_i)).$$

To estimate I_1, we use the following inequalities:

$$\begin{array}{ll} u^p + v^p & \le \max\{1, 2^{1-p}\}(u + v)^p \\ (u + v)^p & \le \max\{1, 2^{1-p}\}(u^p + v^p) \end{array} \qquad p > 0, \quad u, v > 0.$$

Since $1 < p'(x) < \infty$ on $(\mathbb{R}^n_{k,+})_0$, we have

$$\begin{aligned}I_1 &\le p_+ \int_{(\mathbb{R}^n_{k,+})_0} \varepsilon^{p'(x)} \max\{1, 2^{2-p(x)}\}^{p'(x)} (|f(x)| + |f_i(x)|)^{p(x)} (x')^\gamma dx \\ &\le p_+ \int_{(\mathbb{R}^n_{k,+})_0} (4\varepsilon)^{p'(x)} (2|f(x)| + |f(x) - f_i(x)|)^{p(x)} (x')^\gamma dx \\ &\le 4\varepsilon p_+ \int_{(\mathbb{R}^n_{k,+})_0} 2^{p(x)-1} \left(2^{p(x)} |f(x)|^{p(x)} + |f(x) - f_i(x)|^{p(x)} \right)^{p(x)} (x')^\gamma dx \\ &\le \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f) + \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f - f_i).\end{aligned}$$

Thus, we can write

$$\begin{aligned}&p_+ \int_{(\mathbb{R}^n_{k,+})_0} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)| (x')^\gamma dx \\ &\le \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f) + \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f - f_i) + p_+ \rho_{p(\cdot),\gamma}(\varepsilon^{-1}(f - f_i)).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\limsup_{i \to \infty} p_+ \int_{(\mathbb{R}^n_{k,+})_0} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)| (x')^\gamma dx \\ &\le \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we follow from $|\rho_{p(\cdot),\gamma}(f) - \rho_{p(\cdot),\gamma}(f_i)| \to 0$.

Now assume that $f_i \to f$ with respect to measure and $\rho_{p(\cdot),\gamma}(\lambda f_i) \to \rho_{p(\cdot),\gamma}(\lambda f)$ for $\lambda > 0$. Since $\lambda f_i \to \lambda f$ with respect to measure, we may suppose that $\lambda = 1$. Then we have, for each ε, $0 < \varepsilon < 1$,

$$\left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)|^{p(x)} > \varepsilon\right\}\right|_\gamma \le \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)|^{p(x)} > \varepsilon^{1/p_-}\right\}\right|_\gamma$$
$$\le \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)|^{p(x)} > \varepsilon\right\}\right|_\gamma \le \varepsilon.$$

Hence, $|f(\cdot) - f_i(\cdot)|^{p(\cdot)} \to 0$ with respect to measure.

Furthermore, we get

$$\begin{aligned}
&\left||f(x)|^{p(x)} - |f_i(x)|^{p(x)}\right| \\
&\le p_+ \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1}\right) |f(x) - f_i(x)| \\
&\le p_+|f(x)|^{p(x)-1}|f(x) - f_i(x)| + p_+ \max\{1, 2^{p(x)-2}\}\times \\
&\quad \times \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1}\right) |f(x) - f_i(x)| \\
&\le p_+(2^{p_+ +1})|f(x)|^{p(x)-1}|f(x) - f_i(x)| + p_+2^{p_+}|f(x) - f_i(x)|^{p(x)}.
\end{aligned} \tag{1}$$

Now let ε, $0 < \varepsilon < 1$ be fixed. Since $|f(\cdot)|^{p(\cdot)} \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, there exists $K \ge 1$ such that

$$\left|\left\{x : |f(x)|^{p(x)-1} > K\right\}\right|_\gamma \le \left|\left\{x : |f(x)|^{p(x)} > K\right\}\right|_\gamma \le \varepsilon/2.$$

From inequality (1), since $f_i \to f$ and $|f(\cdot) - f_i(\cdot)|^{p(\cdot)} \to 0$ with respect to measure, we can write

$$\begin{aligned}
&\left|\left\{x : \left||f(x)|^{p(x)} - |f_i(x)|^{p(x)}\right| > \varepsilon\right\}\right|_\gamma \\
&\le \left|\left\{x : |f(x)|^{p(x)-1} > K\right\}\right|_\gamma + \left|\left\{x : p_+(2^{p_+ +1})K|f(x) - f_i(x)| > \varepsilon/2\right\}\right|_\gamma + \\
&\quad + \left|\left\{x : p_+2^{p_+}|f(x) - f_i(x)|^{p(x)} > \varepsilon/2\right\}\right|_\gamma \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2p_+(2^{p_+} + 1)K} + \frac{\varepsilon}{p_+2^{p_+ +1}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

for all sufficiently large i. Therefore, $|f_i(\cdot)|^{p(\cdot)} \to |f(\cdot)|^{p(\cdot)}$ with respect to measure. Define

$$h_i(x) = 2^{p_+ -1}|f_i(x)|^{p(x)} + 2^{p_+ -1}|f(x)|^{p(x)} - |f(x) - f_i(x)|^{p(x)} \ge 0,$$

then $h_i \to 2^{p_+}|f(\cdot)|^{p(\cdot)}$ with respect to measure. Hence, from Fatou's Lemma, we get

$$\begin{aligned}
&2^{p_+} \int_{\mathbb{R}^n_{k,+}} |f(x)|^{p(x)}(x')^\gamma dx \\
&\quad \le \liminf_{i\to\infty} \int_{\mathbb{R}^n_{k,+}} 2^{p_+ -1}|f_i(x)|^{p(x)} + 2^{p_+ -1}|f(x)|^{p(x)} - |f(x) - f_i(x)|^{p(x)}(x')^\gamma dx.
\end{aligned}$$

Since $\rho_{p(\cdot),\gamma}(f_i) \to \rho_{p(\cdot),\gamma}(f)$, we follow from

$$\limsup_{i\to\infty} \int_{\mathbb{R}^n_{k,+}} |f(x) - f_i(x)|^{p(x)}(x')^\gamma dx \le 0.$$

Thus, $f_i \to f$ with respect to modular. Therefore, we complete the proof. □

3 Completeness of $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$

Here, we are ready to obtain the completeness of $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, but we first need to prove the following theorem.

Theorem 8. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ be the sequence such that $\sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty$. Then there exists $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $\sum_{i=1}^{k} f_i \to f$ in norm as $k \to \infty$ and*

$$\|f\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma}.$$

Proof: Firstly, let us define F on $\mathbb{R}^n_{k,+}$ and $\{F_k\}$ as follows:

$$F(x) = \sum_{k=i}^{\infty} |f_i(x)|, \qquad F_k(x) = \sum_{i=1}^{k} |f_i(x)|.$$

Then the sequence $\{F_k\}$ is nonnegative and increasing pointwise a.e. to F. Furthermore, there exists $F_k \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, and its norm is uniformly bounded for each k, since

$$\|F_k\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{k} \|f_i\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty.$$

Therefore, from Theorem 1, we have $F \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Since F is finite a.e., $\{F_k\}$ converges pointwise a.e. Therefore, if we can define $\{G_k\}$ by

$$G_k(x) = \sum_{i=1}^{k} f_i(x),$$

then it is also pointwise convergent a.e., since absolute convergence means convergence. Let us denote this limit by f, i.e, $G_k \to f$.

Now fix $G_0 = 0$. Then $G_k - G_j \to f - G_j$ pointwise a.e. for $j \geq 0$. Moreover, we have

$$\liminf_{k\to\infty} \|G_k - G_j\|_{p(\cdot),\gamma} \leq \liminf_{k\to\infty} \sum_{i=j+1}^{k} \|f_i\|_{p(\cdot),\gamma} = \sum_{i=j+1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty.$$

From Theorem 2, if $j = 0$, then we get

$$\|f\|_{p(\cdot),\gamma} \leq \liminf_{k\to\infty} \|G_k\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty.$$

Also, we can write, for each j,

$$\|f - G_j\|_{p(\cdot),\gamma} \leq \liminf_{k\to\infty} \|G_k - G_j\|_{p(\cdot),\gamma} \leq \sum_{i=j+1}^{\infty} \|f_i\|_{p(\cdot),\gamma},$$

since the sum on the RHS goes to 0. Hence, $G_j \to f$ with respect to norm. Therefore, we complete the proof. □

Let us state the completeness of $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ which is a corollary of Theorem 8. This result also means that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is Banach space for $1 < p_- \leq p(x) \leq p_+ < \infty$.

Corollary 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is complete, i.e. every Cauchy sequence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is also convergent.*

Proof: Let $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ be a Cauchy sequence. Fix i_1 such that $\|f_k - f_j\|_{p(\cdot),\gamma} < 2^{-1}$ for $k, j \geq i_1$, fix i_2 such that $\|f_k - f_j\|_{p(\cdot),\gamma} < 2^{-2}$ for $k, j \geq i_2$ and so on. This gives a subsequence $\{f_{i_j}\}$, $i_j < i_{j+1}$, such that

$$\|f_{i_{j+1}} - f_{i_j}\|_{p(\cdot),\gamma} < 2^{-j}.$$

Let $\{g_j\}$ be defined by $g_1 = f_{i_1}$ and $g_j = f_{i_j} - f_{i_{j-1}}$ for $j > 1$. Then for all j, we have the telescoping sum $\sum_{k=1}^{j} g_k = f_{i_j}$. Furthermore, we obtain

$$\sum_{j=1}^{\infty} \|g_j\|_{p(\cdot),\gamma} \leq \|f_{i_1}\|_{p(\cdot),\gamma} + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Hence, from Theorem 8, there exists $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $f_{i_j} \to f$ in norm.

As a consequence of this, we have

$$\|f - f_i\|_{p(\cdot),\gamma} \leq \|f - f_{i_j}\|_{p(\cdot),\gamma} + \|f_{i_j} - f_i\|_{p(\cdot),\gamma}.$$

Since $\{f_i\}$ is a Cauchy sequence, we can get the RHS as small as desired. Therefore, $f_i \to f$ with respect to norm. This completes the proof. □

4 Conclusion

In this paper, the concepts of convergence in variable Lebesgue space has been investigated. In this space, there exists three types of convergence: convergences with respect to modular, norm, measure. The relationship between these convergences has been studied.

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Sequentially T_3 (regular Hausdorff) Spaces

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Abstract: The notion of sequentially open set is a generalization of the notion of open set in topological spaces since every open set is in fact a sequentially open set. According to this fact, in 2019, Akız and Koçak introduced the notion of sequentially Hausdorff spaces which also a generalization of Hausdorff spaces. That is, every Hausdorff space is a sequentially Hausdorff space. It is very interesting that a significant part of the properties provided by Hausdorff spaces are also provided for sequentially Hausdorff spaces. In this study, we introduce the notion of a sequentially T_3-space and investigate its properties. We also give the relation between sequentially T_3 property and other well-known separation properties.

Keywords: Regular Hausdorff (T_3) space, Sequentially open set, Sequentially space.

1 Introduction

In topology and other branches of mathematics, the Hausdorff space concept is the most used axiom among separation axioms. A Hausdorff space is a topological space in which any two distinct points have discrete open neighborhoods. The limit of a convergent sequence in a Hausdorff topological space is unique [1]-[5]. One of the two important spaces among Hausdorff spaces is regular Hausdorff (T_3) spaces and the other is normal Hausdorff (T_4) spaces.

With the help of the concept of convergence in sequences, sequentially open and sequentially closed sets, which are more general than open and closed sets in topological spaces, are defined [6]-[10]. Every open set is sequentially open, but the converse is not always true. In sequentially spaces, on the other hand, the concepts of openness and sequentially openness are equivalent to each other. These concepts were later extended to different subjects of topological spaces. One of them is the sequentially connected topological spaces, which have no other sequentially open and sequentially closed subsets than empty set. Every sequentially connected space is connected [11]-[14]. Later, under the title of $G-$method, $G-$sequentially openness, $G-$sequentially closure, $G-$sequentially continuity [15], $G-$sequentially compactness [16], $G-$continuity [17], $G-$convergence, $G-$sequentially connectivity concepts have been studied [18]-[20].

The concept of sequentially Hausdorff property, which is a broader construct than the concept of Hausdorff property, was recently introduced and exemplified by Akız and Koçak [21]. If any two distinct points in a space have discrete sequentially open neighbourhoods, this space is called a sequentially Hausdorff. Every Hausdorff space is a sequentially Hausdorff. In sequentially spaces, these two concepts are equivalent. After sequentially Hausdorff space concept defined by Akız and Koçak [21], the sequentially definition of other separation axioms and the examination of their properties became a matter of curiosity.

The most important goal in general point-set topology theory is to discover the topological properties, which is a very useful tool in determining whether any two spaces are homeomorphic according to the relation of being homeomorphic, which is called isomorphism of topological spaces. All separation axioms known in the literature are topological properties. As a result of obtaining some separation axioms using sequentially deficits, new topological properties have emerged in recent years.

In this study, it is aimed to define the characteristics of being a sequentially T_3-space and a sequentially T_4-space, to examine in detail, and to investigate the results and methods used in previous studies on T_3 and T_4-spaces. In addition, by using these methods, it is aimed to examine the properties of sequentially regular and sequentially normal spaces, which are more sensitive topological properties, in a sequentially sense, and to show whether these new properties are also topological properties.

2 Preliminaries

Let recall some preliminary definitions and properties.

Definition 1. *[8, 9] A subset U of a topological space X is called* ***sequentially open subset*** *if every convergent sequence with a limit point in U has finitely many terms in $X \backslash U$.*

Remark 1. *Here note that, according to the definition given above, every open subset of a topological space is a sequentially open subset.*

The converse of the fact given in the remark above is not true in general. That is, a sequentially open subset of a topological space need not to be an open subset. In fact, a topological space whose open subsets and sequentially open subsets are coincide is called a **sequentially space** [8, 9].

It is easy to see that in a topological space X the union of any collection of sequentially open subsets of X is again a sequentially open subset. However, this is not always true for intersection, even for finitely many sequentially open subsets.

Definition 2. *Let X be a topological space and let A be a subset of X. Then the union of sequentially open subsets which contained in A is called the **sequentially interior of** A and denoted by $\mathsf{s}-\mathsf{int}(A)$.*

Since the union of any collection of sequentially open subsets of a topological space is again a sequentially open subset then we can give a different description of the notion of sequentially interior as follows: Let $\mathcal{SO}_A$ be the collection of sequentially open subsets of a topological space X which contained in $A \subseteq X$. Thus

$$\mathsf{s}-\mathsf{int}(A) = \bigcup_{G \in \mathcal{SO}_A} G$$

For a subset A of a topological space X, one can see that $\mathsf{int}(A) \subseteq \mathsf{s}-\mathsf{int}(A) \subseteq A$ and $\mathsf{s}-\mathsf{int}(A)$ is sequentially open in X. Moreover, A is sequentially open if and only if $\mathsf{s}-\mathsf{int}(A) = A$, i.e. $A \subseteq \mathsf{s}-\mathsf{int}(A)$.

Definition 3. *[8, 9] A subset F of a topological space X is called **sequentially closed subset** if F contains every limit points of convergent sequences in F.*

Proposition 1. *[8, 9] A subset F of a topological space X is sequentially closed if and only if its complement in X, i.e. $X \backslash F$, is sequentially open.*

Corollary 1. *Every closed subset of a topological space is a sequentially closed subset. But the converse is not always true.*

We know that, in a topological space X, the intersection of any collection of sequentially closed subsets of X is again a sequentially closed subset. However, this is not always true for union of sequentially closed subsets, even for finitely many sequentially closed subsets.

Definition 4. *Let X be a topological space and let A be a subset of X. Then the intersection of sequentially closed subsets which contains A is called the **sequentially closure of** A and denoted by $\mathsf{s}-\mathsf{cl}(A)$.*

Since the intersection of any collection of sequentially closed subsets of a topological space is again a sequentially closed subset then we can give a different description of the notion of sequentially closure as follows: Let $\mathcal{SC}_A$ be the collection of sequentially closed subsets of a topological space X which contains $A \subseteq X$. Hence

$$\mathsf{s}-\mathsf{cl}(A) = \bigcap_{F \in \mathcal{SO}_A} F$$

It is easy to see that, for a subset A of a topological space X, $A \subseteq \mathsf{s}-\mathsf{cl}(A) \subseteq \mathsf{cl}(A)$ and $\mathsf{s}-\mathsf{cl}(A)$ is sequentially closed. Also, A is a sequentially closed subset of X if and only if $A = \mathsf{s}-\mathsf{cl}(A)$, i.e. $\mathsf{s}-\mathsf{cl}(A) \subseteq A$.

3 Sequentially T_0-Space

First we recall the definition of a T_0-space.

Definition 5. *A topological space X is called a T_0-**space** if for any two points $x, y \in X$ there exist an open set G such that $x \in G$ and $y \notin G$ or $x \notin G$ and $y \in G$.*

In this definition, if we take sequentially open set instead of open set then we get the definition of a sequentially T_0-space.

Definition 6. *A topological space X is called a sequentially T_0-space, or briefly an sT_0-space, if for any two points $x, y \in X$ there exist a sequentially open set G such that $x \in G$ and $y \notin G$ or $x \notin G$ and $y \in G$.*

Since every open set in a topological space is sequentially open set then we obtain the following proposition.

Proposition 2. *Any T_0-space is an sT_0-space.*

Corollary 2. *A discrete topological space is an sT_0-space.*

Theorem 1. *A topological space X is an sT_0-space if and only if $\mathsf{s}-\mathsf{cl}\{x\} \neq \mathsf{s}-\mathsf{cl}\{y\}$ for each $x, y \in X$ such that $x \neq y$.*

Proof: It is sufficient to prove the equivalent expression that "A topological space X is not an sT_0-space if and only if there exist $x, y \in X$ with $x \neq y$ such that $\mathsf{s}-\mathsf{cl}\{x\} = \mathsf{s}-\mathsf{cl}\{y\}$".

Assume that X is not an sT_0-space. Then there exist distinct elements $x, y \in X$ such that each sequentially open subset of X containing x also contains y and each sequentially open subset of X containing y also contains x. Hence y must be in $\mathsf{s}-\mathsf{cl}\{x\}$ since $X \backslash \mathsf{s}-\mathsf{cl}\{x\}$ is a sequentially open subset of X not containing x. Similarly, x must be in $\mathsf{s}-\mathsf{cl}\{y\}$. Thus, $\mathsf{s}-\mathsf{cl}\{x\} = \mathsf{s}-\mathsf{cl}\{y\}$.

Conversely, let there exist distinct points $x, y \in X$ such that $\mathsf{s}-\mathsf{cl}\{x\} = \mathsf{s}-\mathsf{cl}\{y\}$. Then each sequentially open subset of X containing x also contains y and each sequentially open subset of X containing y also contains x. Thus, X is not an sT_0-space. This completes the proof. □

Corollary 3. *A topological space X is an sT_0-space if each singleton in X is a sequentially closed subset of X.*

4 Sequentially T_1-Space

Now we remind the definition of a T_0-space.

Definition 7. *A topological space X is called a T_1-space if for any two points $x, y \in X$ there exist an open set G such that $x \in G$ and $y \notin G$, and $x \notin G$ and $y \in G$.*

In this definition, if we replace open sets with sequentially open set then we get the definition of a sequentially T_1-space.

Definition 8. *A topological space X is called a sequentially T_1-space, or briefly an sT_1-space, if for any two points $x, y \in X$ there exist sequentially open sets G and H such that $x \in G$ and $y \notin G$, and $x \notin H$ and $y \in H$.*

Since every open set in a topological space is sequentially open set then we obtain the following proposition.

Proposition 3. *Any T_1-space is an sT_1-space.*

Corollary 4. *A discrete topological space is an sT_1-space.*

From the definitions of an sT_0-space and of an sT_1-space one can easily figure the following proposition.

Proposition 4. *Any sT_1-space is an sT_0-space.*

Following theorem gives a criteria for a topological space to be an sT_1-space.

Theorem 2. *A topological space X is an sT_1-space if and only if each singleton in X is a sequentially closed in X.*

Proof: Let X be an sT_1-space and let $a \in X$. Take $x \in X \backslash \{a\}$. Since $a \neq x$ and X is an sT_1-space there exist sequentially open sets G and H such that $x \in G$ and $a \notin G$, and $x \notin H$ and $a \in H$. Here $x \in G \subseteq X \backslash \{a\}$ and hence $X \backslash \{a\}$ is sequentially open, i.e. $\{a\}$ is sequentially closed in X.

Conversely, assume that each singleton in X is a sequentially closed in X. Let $x, y \in X$ with $x \neq y$. If we take $G := X \backslash \{y\}$ and $H := X \backslash \{x\}$ then G and H become sequentially open sets from the assumption and it is obvious that $x \in G$ and $y \notin G$, and $x \notin H$ and $y \in H$. Thus X is an sT_1-space. This completes the proof. □

5 Sequentially T_2 (Hausdorff) Spaces

In this subsection, first we remind the definition of a T_2-space from [21].

Definition 9. *A topological space X is called a T_2 (Hausdorff) space if for any two points $x, y \in X$ there exist non-intersecting open sets G and H such that $x \in G$ and $y \in H$.*

In this definition, if we take sequentially open set instead of open set then we get the definition of a sequentially T_1-space.

Definition 10. *[21] A topological space X is called a sequentially T_2 (Hausdorff) space, or briefly an sT_2-space, if for any two points $x, y \in X$ there exist non-intersecting sequentially open sets G and H such that $x \in G$ and $y \in H$.*

Since every open set in a topological space is sequentially open set then we obtain the following proposition.

Proposition 5. *[21] Any T_2-space is an sT_2-space.*

Corollary 5. *A discrete topological space is an sT_2-space.*

From the definitions of a sT_1-space and of a sT_2-space one can easily figure the following proposition.

Proposition 6. *Any sT_2-space is an sT_1-space.*

Example 1. *[21] Let X be a non-empty set and let $\tau = \{U \subseteq X \mid X \backslash U \text{ is countable}\} \cup \{\emptyset\}$. In this case, any subset of X is sequentially open. Hence, for any distinct points $x, y \in X$ we can take non-intersecting sequentially open sets as $G := \{x\}$ and $H := \{y\}$. Thus (X, τ) is an sT_2-space.*

Example 2. *[21] Any first countable space is an sT_2-space.*

Proposition 7. *[21] Let X be an sT_2-space and let $x, y \in X$. If (x_n) be a convergent sequence in X such that $(x_n) \to x$ and $(x_n) \to y$ then $x = y$. In other words, every convergent sequence has a unique limit point in an sT_2-space.*

Proposition 8. *[21] Any subspace of an sT_2-space is also an sT_2-space.*

Proposition 9. *[21] Being an sT_2-space is a topological property.*

Theorem 3. *[21] A sequentially compact subset of an sT_2-space is closed.*

6 Sequentially Regular Spaces

Let remind the definition of a regular space in point-set topology theory.

Definition 11. *A topological space X is called a regular space if for any closed subset $F \subset X$ and for any point $x \in X \backslash F$ there exist non-intersecting open sets G and H such that $F \subseteq G$ and $x \in H$.*

In this definition, if we take sequentially open set and sequentially closed set instead of open set and closed set then we get the definition of a sequentially regular space.

Definition 12. *A topological space X is called a sequentially regular space if for any sequentially closed subset $F \subset X$ and for any point $x \in X \backslash F$ there exist non-intersecting sequentially open sets G and H such that $F \subseteq G$ and $x \in H$.*

Since every open (closed) set in a topological space is sequentially (closed) open set then we obtain the following proposition.

Proposition 10. *Any regular space is a sequentially regular space.*

Proposition 11. *Topological spaces in which every sequentially open set is also a sequentially closed set are sequentially regular.*

Proof: Let X be a topological space in which every sequentially open set is also a sequentially closed set and let $F \subset X$ be a sequentially closed set. Take an element $x \in X \backslash F$. according to the assumption F and $X \backslash F$ are both sequentially open and sequentially closed. Thus we can take $G := F$ and $H := X \backslash F$. This completes the proof. ☐

Corollary 6. *A discrete topological space is sequentially regular.*

7 Sequentially T_3 (regular Hausdorff) Spaces

First we recall the definition of a T_3-space.

Definition 13. *A topological space X is called a T_3-space if it is both a regular space and a T_1-space.*

Now we can give the generalized version of this separation axiom using sequentially open sets.

Definition 14. *A topological space X is called a sequentially T_3 (regular Hausdorff) space, or briefly an sT_3-space if it is both a sequentially regular space and an sT_1-space.*

Proposition 12. *Any sT_3-space is an sT_2-space.*

Proof: Let X be an sT_3-space and let $x, y \in X$ with $x \neq y$. Since X is an sT_3-space then it is both a sequentially regular space and an sT_1-space. We know from Theorem 2 that $F := \{x\}$ is sequentially closed subset of X and $y \notin F$. From the assumption that X being a sequentially regular space there exist non-intersecting sequentially open sets G and H such that $F \subseteq G$ and $y \in H$. That is there exist non-intersecting sequentially open sets G and H such that $x \in G$ and $y \in H$. Thus X is an sT_2-space. This completes the proof. ☐

Diagram given in Figure 1 shows the relation between separation axioms and generalized separation axioms.

$$\begin{array}{ccccccc} sT_3 & \Longrightarrow & sT_2 & \Longrightarrow & sT_1 & \Longrightarrow & sT_0 \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ T_3 & \Longrightarrow & T_2 & \Longrightarrow & T_1 & \Longrightarrow & T_0 \end{array}$$

Fig. 1: Relation between T_i and sT_i spaces for $i = 0, 1, 2, 3$.

8 Conclusion

In this proceeding, some basic concepts of sequentially separation axioms are obtained for the novel sequentially spaces theory, which is presented by further generalized point-set topology concepts.

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Some Basic Results on Groups up to Congruence Relations

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Abstract: It is a well-known fact that crossed modules and group-groupoids, i.e. strict categorical groups, are categorically equivalent algebraic structures. In 2020, Datuashvili et al. aim to obtain for categorical groups an analogous description in terms of certain crossed module type objects as strict categorical groups. Thus, they introduced the notion of a c-group which also known as a group up to congruence relation. In this study, we explore most significant properties of groups up to congruence relations. Moreover, we give different definitions for homomorphisms, kernels, and images up to congruence relations.

Keywords: Congruence relation, Group, Image, Kernel.

1 Introduction

We recall from [1] the definition of the category $\widetilde{\mathsf{Set}}$. $\widetilde{\mathsf{Set}}$ has non-empty sets with a congruence relation as objects and ordinary functions between sets which respect to congruence relations. Objects of the category $\widetilde{\mathsf{Set}}$ is denoted with X_R where X is a non-empty set and $R \subseteq X \times X$ is a congruence relation on X. See [1, Section 3] for details.

First we will introduce the concept of **c-group** which is a group up to a given congruence relation.

Definition 1. *A **c-semigroup** is an object G_R in $\widetilde{\mathsf{Set}}$ with a morphism*

$$\begin{array}{rcccl} m & : & G_R \times G_R & \longrightarrow & G_R \\ & & (a,b) & \longmapsto & m(a,b) = a+b \end{array}$$

in $\widetilde{\mathsf{Set}}$, i.e, $m \in \widetilde{\mathsf{Set}}((G \times G)_{R\times R}, G_R)$ such that

(i) $a + (b + c) \sim_R (a + b) + c$ *for all* $a, b, c \in G$*;*

*a **c-monoid** is a c-semigroup G_R which contains an element 0 such that*

(ii) $a + 0 \sim_R a$ *and* $0 + a \sim_R a$ *for all* $a \in G$*;*

*and a **c-group** [1] is a c-monoid G_R such that*

(iii) *there exists an element $-a$ such that $a + (-a) \sim_R 0$ and $-a + a \sim_R 0$ for each $a \in G$.*

Let G_R be a c-group. The element $0 \in G$ is called a **zero element** of G_R, and for any $a \in G$ the element $-a \in G$ is called an **inverse** of a [1].

Our main interest is in c-groups. So we will focus on the properties of c-groups. Other algebraic structures such as c-semigroups and c-monoids could be very interesting topics for other researchers to investigate.

Remark 1. *[1] Let G_R be a c-group. Then we have the following:*

1. *if $a \sim_R b$ and $c \sim_R d$ for $a, b, c, d \in G$, then $a + c \sim_R b + d$;*
2. *if 0 and $0'$ are both zero elements in G_R, then $0 \sim_R 0'$;*
3. *if $-a$ and a' are both inverses of $a \in G$, then $a' \sim_R -a$;*
4. *if $a \sim_R b$ then $-a \sim_R -b$.*

Following properties are group analogous properties such as cancellation, inverse of inverse and one variable equations.

Proposition 1. *Let G_R be a c-group, then*

1. *$a+b \sim_R a+c$ implies $b \sim_R c$, and $b+a \sim_R c+a$ implies $b \sim_R c$ for all $a,b,c \in G$;*
2. *$-(-a) \sim_R a$ for all $a \in G$;*
3. *$-(a+b) \sim_R -b+(-a)$ for all $a,b \in G$;*
4. *for all $a,b \in G$ there exist $x,y \in G$ such that $a+x \sim_R b$ and $y+a \sim_R b$ which are $x \sim_R -a+b$ and $y \sim_R b+(-a)$.*

Proof: These can be proven by easy calculations. So are omitted. □

Equality is an equivalence relation on any set. Thus any group can be considered as a c-group where the equivalence relation is equality "=". So the c-group notion is a generalization of the group notion.

Remark 2. *Let G_R be a c-group and R' is also a congruence relation on G such that $R \subseteq R'$. Then $G_{R'}$ is also a c-group with the same operation. Hence, since $= \subseteq R$ for any congruence relation R then any group G is a c-group with any of its congruence relation.*

Following is a geometric example of a c-group.

Example 1. *[1] Let X be a topological space and $x \in X$. The set $P(X,x)$ of all closed paths at x is a c-group with the composition of paths. Here the congruence relation $\simeq$ is the homotopy of the paths.*

Example 2. *[1] Let $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. Define an equivalence relation on $\mathbb{Z}^*$ by $x \sim_R y \Leftrightarrow xy > 0$. Then $\mathbb{Z}^*$ becomes a c-group with respect to the multiplication. The number 1 is a zero element and an inverse for any number could be taken itself this number.*

Following is the motivating example in [1].

Example 3. *[1] In a categorical group $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ the set C_1 of morphisms and the set C_0 of objects are both c-groups. The congruence relations are isomorphisms between arrows and between objects respectively. See [2] for details on categorical groups.*

Definition 2. *[1] Let G_R be a c-group (resp. c-semigroup, c-monoid). If $a+b \sim_R b+a$ for all $a,b \in G$, then G_R is called **c-abelian** (or **c-commutative**) c-group (resp. c-semigroup, c-monoid).*

In particular, if the operation is commutative (up to equality) then we use another term for that.

Definition 3. *Let G_R be a c-group (resp. c-semigroup, c-monoid). If $a+b = b+a$ for all $a,b \in G$, then G_R is called **abelian** c-group (resp. c-semigroup, c-monoid).*

Following is the extended version of Lemma 3.14 of [1].

Lemma 1. *Let G_R be a c-group (resp. c-semigroup, c-monoid). Then the quotient set $G/R = \{[a] \mid a \in G\}$ becomes a group (resp. semigroup, monoid) with the operation*

$$\begin{array}{rcccl} m^* & : & G/R \times G/R & \longrightarrow & G/R \\ & & ([a]\,,[b]) & \longmapsto & [a]+[b] = [a+b] \end{array}$$

induced by m where $[a]$ is the equivalence class of $a \in G$.

As a corollary of Lemma 1 we can also express the notion of a c-group in terms of diagrams as; an object G_R of $\widetilde{\mathsf{Set}}$ with a morphism $m\colon G_R \times G_R \to G_R$, $(a,b) \mapsto m(a,b) = a+b$ is a c-group if the diagrams given in Figure 1, Figure 2 and Figure 3 are commutative:

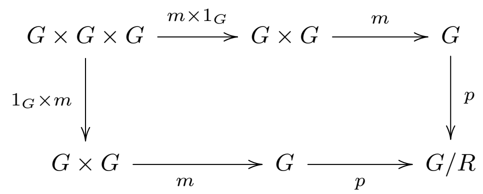


Fig. 1: Associativity diagram

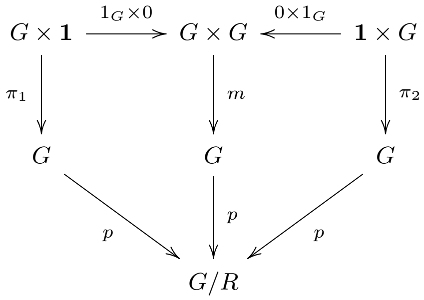


Fig. 2: Identity element diagram

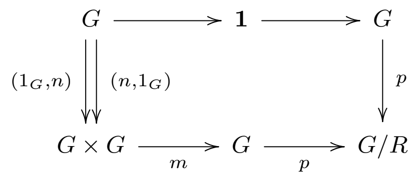


Fig. 3: Inverse element diagram

where $\mathbf{1}$ is a zero object in the category $\widetilde{\mathsf{Set}}$, $\mathbf{1} \to G$ is the unique initial morphism, $G \to \mathbf{1}$ is the unique terminal morphism, $n\colon G \to G$ is the inverse element morphism and $p\colon G \to G/R$ is the natural projection.

Remark 3. *Clearly, if G_R is a c-abelian c-group then G/R, with the induced operation, is an abelian group.*

Example 4. *Let $P(X,x)$ be the c-group in Example 1. Then the group $P(X,x)/\simeq$ is the fundamental group $\pi_1(X,x)$ of X at x.*

Example 5. *Let $\mathbb{Z}^*_R$ be the c-group given in Example 2. Then $\mathbb{Z}^*/R$ is isomorphic to the group $\mathbb{Z}_2$.*

Example 6. *[1] Let $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ be a categorical group. Then*

$$\mathsf{C}/{\approx} = (C_0/{\approx}, C_1/{\approx}, d_0^*, d_1^*, i^*, m^*)$$

becomes a strict categorical group which is also called by the name group-groupoid in [3].

Let G_R and H_S be two c-groups with zero elements 0_G and 0_H respectively. Then the product $(G \times H)_{R\times S}$ of G_R and H_S in $\widetilde{\mathsf{Set}}$ is a c-group with the zero element $(0_G, 0_H)$ and $(-g,-h)$ is the inverse element of $(g,h) \in G \times H$. This c-group is called the **direct product** of G_R and H_S. We will denote this c-group with $G_R \times H_S$.

2 Morphisms among c-groups

Definition 4. *Let G_R and H_S be c-groups. A* ***morphism of c-groups*** *$f\colon G_R \to H_S$ is a morphism in $\widetilde{\mathsf{Set}}$ such that $f(a+b) \sim_S f(a) + f(b)$ for any $a, b \in G$.*

Remark 4. *[1] Here note that, since $f\colon G_R \to H_S$ is a morphism in $\widetilde{\mathsf{Set}}$ then $a \sim_R b$ implies $f(a) \sim_S f(b)$. Furthermore, $f(0) \sim 0$ and $f(-a) = -f(a)$, for any $a \in G$.*

It is easy to see that if $f\colon G_R \to H_S$ is a morphism of c-groups then the induced morphism

$$\begin{array}{rccl} f^* & : & G/R & \longrightarrow & H/S \\ & & [a] & \longmapsto & f^*([a]) = [f(a)] \end{array}$$

becomes a group homomorphism.

Example 7. *Let X and Y be two topological spaces, $x \in X$ and $f\colon X \to Y$ a continuous function. Then*

$$\begin{array}{rccl} \bar{f} & : & P(X,x) & \longrightarrow & P(Y,f(x)) \\ & & \alpha & \longmapsto & \bar{f}(\alpha) = f \circ \alpha \end{array}$$

is a morphism of c-groups, where $P(X,x)$ and $P(Y,f(x))$ are c-groups and the congruence relation is homotopy of paths as in Example 1.

Example 8. *Let G_R and $G'_{R'}$ be two c-groups. Assume that 0 and $0'$ be two chosen zero elements of G_R and $G'_{R'}$, respectively. Then there are four morphisms of c-groups;*

$$G_R \underset{\imath_1}{\overset{\pi_1}{\leftrightarrows}} G_R \times G'_{R'} \underset{\imath_2}{\overset{\pi_2}{\rightleftarrows}} G'_{R'}$$

given by $\imath_1(g) = (g, 0')$, $\imath_2(g') = (0, g')$, $\pi_1(g, g') = g$ and $\pi_2(g, g') = g'$ for all $g \in G$ and $g' \in G'$ where $G_R \times G'_{R'}$ is the direct product c-group.

A **strict morphism of c-groups** $f\colon G_R \to H_S$ is a morphism of c-groups such that $f(a + b) = f(a) + f(b)$ for any $a, b \in G$. Clearly a strict morphism of c-groups is a morphism of c-groups. Strict morphisms of c-groups are given under the name *c-group morphisms* in [1].

It is easy to see that composition of two morphisms of c-groups is again a morphism of c-groups and the identity function 1_G on any c-group G_R is a morphism of c-groups. Hence c-groups and morphisms of c-groups forms a category which is denoted by cGr.

Definition 5. *[1] Let $f\colon G_R \to H_S$ be a morphism of c-groups. The subset $\mathrm{cKer}\, f = \{a \in G \mid f(a) \sim_S 0_H\}$ is said to be **c-kernel** of the c-group morphism f. The subset $\mathrm{cIm}\, f = \{b \in H \mid \exists a \in G, f(a) \sim_S b\}$ is said to be the **c-image** of the morphism f.*

We know from category theory a morphism in a category is called an isomorphism if it has inverse morphism. In the category cGr of c-groups there is a special morphism which is called c-isomorphism [1]. Before giving the definition of c-isomorphism we need to remind the equivalence relation between morphisms of c-groups with the same end points: Let $f, f'\colon G_R \to H_S$ be two morphisms of c-groups. Then we write $f \sim f'$ if $f(g) \sim_S f'(g)$ for all $g \in G$. This relation is clearly an equivalence relation on the set of morphisms of c-groups from G_R to H_S.

Definition 6. *[1] Let $f : G \to G'$ be a morphism in cGr. f is called an **isomorphism up to congruence relation** or **c-isomorphism** if there exist a morphism $f' : G' \to G$ in cGr, such that $ff' \sim 1_{G'}$ and $f'f \sim 1_G$.*

3 Subobjects and ideals

Definition 7. *[1] Let G_R be a c-group and H be a subset of the underlying set of G. H is called a c-subgroup in G_R if H_S is a c-group with the same operation on G_R and the congruence relation S induced from R, i.e. $S = R \cap (H \times H)$. We denote this situation with $H_S \lesssim G_R$.*

Following notations come from [1]. Let G_R be a c-group and H be a subset of G. If for an element $a \in G$ there exists an element $b \in H$ such that $a \sim_R b$ then we write $a \,\widetilde{\in}\, H$. Obviously $a \in H$ implies $a \,\widetilde{\in}\, H$ but converse is not true in general. In fact c-subgroups satisfying this condition are special (see Definition 9). If H and H' are two subsets of G_R, then we write $H \widetilde{\subset} H'$ if for any $h \in H$ we have $h \,\widetilde{\in}\, H'$. If $H \widetilde{\subset} H'$ and $H' \widetilde{\subset} H$, then we write $H \sim H'$.

Proposition 2. *Let G_R be a c-group and H be a non-empty subset of G which is closed under the binary operation given on G_R. Then $H_S \lesssim G_R$ if and only if $a + (-b) \,\widetilde{\in}\, H$ for all $a, b \in H$.*

Proof: First assume that $H_S \lesssim G_R$. Let $a, b \in H$. Since H_S is a c-subgroup of G_R then there exist an element $b' \in H$ such that $-b \sim_R b'$. By (i) of Remark 1, $a + (-b) \sim_R a + b'$ and from closedness of H_S under the operation we get $a + b' \in H$ and hence $a + (-b) \,\widetilde{\in}\, H$.

Conversely, we assume that $a + (-b) \,\widetilde{\in}\, H$ for all $a, b \in H$. Now we need to show that $H_S \lesssim G_R$, i.e. H_S is a c-group with the addition and congruence relation S induced from G_R. It is obvious that the operation is associative up to congruence relation. Since H is non-empty then there is an element a in H and by the assumption $a + (-a) \,\widetilde{\in}\, H$. Thus there exist an element $0' \in H$ such that $0 \sim_R 0'$ and consequently for any $b \in H$, $0' + (-b) \,\widetilde{\in}\, H$. Since $0' + (-b) \sim_R -b$ then there exist an element $b' \in H$ such that $b' \sim_S -b$ which acts as an inverse in H_S. □

As a consequence of this result we can give the following corollary.

Corollary 1. *Let G_R be a c-group and $\left\{(H_i)_{S_i} \mid i \in I\right\}$ a non-empty family of c-subgroups of G_R. Then the intersection of all these c-subgroups*

$$H_S = \left(\bigcap\nolimits_{i \in I} H_i\right)_{\left(\bigcap_{i \in I} S_i\right)}$$

is again a c-subgroup of G_R.

Definition 8. *Let G_R be a c-group, A be a subset of G, and $\left\{(H_i)_{S_i} \mid i \in I\right\}$ be the family of all c-subgroups of G_R containing A. Then the c-subgroup*

$$H_S = \left(\bigcap\nolimits_{i \in I} H_i\right)_{\left(\bigcap_{i \in I} S_i\right)}$$

*is called the **c-subgroup of** G_R **generated by** A. This c-subgroup is denoted by $\langle A \rangle$.*

A is called the **set of generators** of the c-subgroup $\langle A \rangle$. It should be noted that there could be another subset B of G such that $\langle A \rangle = \langle B \rangle$ even though $A \neq B$. If A is a finite set then $\langle A \rangle$ is called **finitely generated c-subgroup**. If A is a singleton then $\langle A \rangle$ is called **cyclic c-subgroup**.

Proposition 3. *Let G_R be a c-group and H be a subset of the underlying set of G. Then $H_S \lesssim G_R$ if and only if H/S is isomorphic to a subgroup of G/R.*

Proof: It is an obvious output of Lemma 1 and Proposition 2. □

Definition 9. *[1] Let G_R be a c-group. Then a c-subgroup H_S of G_R is called*

1. ***connected*** *if $H \times H \subseteq R$;*
2. ***perfect*** *if $g \,\widetilde{\in}\, H$ implies $g \in H$, for any $g \in G$.*

Example 9. *Let $\mathbb{Z}_R^*$ be the c-group given in Example 2. Then the c-subgroup $(2\mathbb{Z}+1)_S$ of $\mathbb{Z}_R^*$ is neither connected nor perfect. However the c-subgroup $\left(\mathbb{Z}^+\right)_S$ is both connected and perfect.*

Definition 10. *[1] Let G_R be a c-group and $H_S \lesssim G_R$. Then H_S is called a **normal c-subgroup** or an **ideal** of G_R if $g+h-g \,\widetilde{\in}\, H_S$ for any $h \in H$ and $g \in G$.*

If H_S is an ideal of G_R we will denote this situation with $H_S \,\widetilde{\triangleleft}\, G_R$.
Let G_R be a c-group, $A, B \subseteq G$ and $g \in G$. Then the sets $g+A$ and $A+B$ are defined in the following way as in the group case:

$$g + A = \{g + a \mid a \in A\}$$
$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}\,.$$

However we define new kind of subsets $g \widetilde{+} A$ and $A \widetilde{+} B$ of G_R with

$$g \widetilde{+} A = \{x \in G \mid \exists a \in A, x \sim_R g + a\}$$
$$A \widetilde{+} B = \{x \in G \mid \exists a \in A \text{ and } \exists b \in B, x \sim_R a + b\}\,.$$

It is easy to see that the set $A \widetilde{+} B$ is the union of sets $a \widetilde{+} B$ for all $a \in A$.

Remark 5. *In [1] it has been shown that the condition given in the definition of an ideal is equivalent to the condition $g + H_S - g \,\widetilde{\subset}\, H_S$, and hence to the condition $g + H_S \sim H_S + g$ for any $g \in G$.*

Proposition 4. *Let G_R be a c-group and N be a subset of G. Then $N_S \,\widetilde{\triangleleft}\, G_R$ if and only if N/S is isomorphic to a normal subgroup of G/R.*

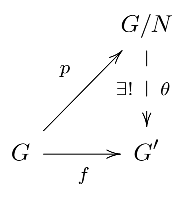
Proof: One can prove this using Lemma 1 and Proposition 2. □

Lemma 2. *[1] Let G_R and H_S be c-groups and let $f\colon G_R \to H_S$ be a morphism of c-groups. Then*

1. cKer f *is a perfect ideal of G_R, and*
2. cIm f *is perfect in H_S.*

Let G_R be a c-group and $N_S \,\widetilde{\triangleleft}\, G_R$. Now we remind the construction of the quotient object G_R/N_S as in [1] but using our notation. Consider the set $G/H = \left\{g \widetilde{+} N \mid g \in G\right\}$. The binary operation on this set is defined by $(g \widetilde{+} N) + (g' \widetilde{+} N) = (g + g') \widetilde{+} N$, for any $g, g' \in G$. This operation is well-defined, it is associative, $0 \widetilde{+} N$ is the zero element, and $(-g) \widetilde{+} N$ acts as an inverse for any $g \in G$. G/N becomes a c-group with this operation where the congruence relation is "=" (equality), i.e. G/N becomes a group. Moreover $p\colon G \to G/N$ is a morphism of c-groups which is called the canonical projection.

Lemma 3. *[1] Let G be a c-group and $N \,\widetilde{\triangleleft}\, G$. Then for any group G' and any morphism of c-groups $f : G \to G'$, if $f(n) = 0$ for any $n \in N$, there exists a unique morphism $\theta : G/N \to G'$ of c-groups such that $\theta p = f$. Moreover, if N is perfect in G, then $N =$ cKer p.*



4 Conclusion

The main purpose of defining c-groups is to generalize the categorical equivalence of crossed modules and group-groupoids given by Brown and Spencer [4] for categorical groups. In this paper, some basic concepts of algebra such as sub-objects, ideals, transformations between such objects are obtained for the new algebraic theory, which is presented by further generalized group theoretical concepts.

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Some Fixed-Point Theorems for Continuous Functions

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Abstract: In this study, the necessary and sufficient conditions for the convergence of some iteration methods created through any continuous mapping are examined and the convergence rates of these methods are compared. In addition, the result of the convergence rate was supported by numerical examples.

Keywords: Iteration method, Continuous mapping, Convergence rate.

1 Introduction

Fixed-point iteration methods for certain classes of mappings are an important tool in fixed-point theory. What is meant here by a certain class of mapping, is contraction, nonexpansive, Lipschitzian, etc. mapping types. All of the mappings mentioned here are continuous. However, in general, not every continuous mapping can be expressed in one of these types. Throughout this work, we will consider X as a closed interval on the real axis and g as a continuous self-function on X, and the set of all fixed points of g is shown as F_g. If X is bounded, we can say from the Intermediate Value Theorem that g has at least one fixed point. In this context, a lot of iteration methods have been defined and studied by numerous mathematicians. Now let's remind some of them:
The following methods are called Mann [1] and Ishikawa [2] iteration methods, respectively:

$$u_{n+1} = (1-\alpha_n)\,u_n + \alpha_n g(u_n),\ (n \in \mathbb{N}), \tag{1}$$

and

$$\begin{cases} x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n g(y_n), \\ y_n = (1-\beta_n)\,x_n + \beta_n g(x_n), \end{cases} \tag{2}$$

in which $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$.
Rhoades [3, 4] proved the strong convergence of the Mann iteration for the continuous and non-decreasing mapping classes defined in the closed unit interval and showed that the Ishikawa iteration is faster than the Mann iteration for such mappings.
Also, Borwein and Borwein [5] and Qing and Qihou [6] gave some convergence theorems for Mann and Ishikawa iterations, respectively, in an arbitrary interval by using continuous mappings.
Noor [7] defined the following iteration by generalizing the Ishikawa and thus the Mann iteration:

$$\begin{cases} x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n g(y_n), \\ y_n = (1-\beta_n)\,x_n + \beta_n g(z_n), \\ z_n = (1-\gamma_n)\,x_n + \gamma_n g(x_n), \end{cases} \tag{3}$$

in which $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$. It is obvious that Ishikawa and Mann iteration methods can be obtained by special selections of the control sequences here.
In addition, Phuengrattana and Suantai [8] proved that the sequence obtained from this iteration converges strongly to the fixed point of the continuous function g defined in an arbitrary interval and they defined the following iteration method called SP:

$$\begin{cases} u_{n+1} = (1-\alpha_n)\,v_n + \alpha_n g(v_n), \\ v_n = (1-\beta_n)\,w_n + \beta_n g(w_n), \\ w_n = (1-\gamma_n)\,u_n + \gamma_n g(u_n), \end{cases} \tag{4}$$

in which $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$. They proved that this method converges strongly to the fixed point of a continuous function g. Also, by comparing the convergence rates of Ishikawa, Mann, Noor, and SP-iterations, they proved that the SP-iteration is better (faster) than the others.

Gürsoy and Karakaya [9] introduced Picard-S iteration method as follows:

$$\begin{cases} u_1 \in E, \\ u_{n+1} = g(v_n) \\ v_n = (1-\alpha_n) g(u_n) + \alpha_n g(w_n) \\ w_n = (1-\beta_n) u_n + \beta_n g(u_n) \ (n \in \mathbb{N}), \end{cases} \tag{5}$$

in which $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in$ [0,1].

Now, we give PMP iteration method which is defined by Karakaya et al. [10] as follows:

$$\begin{cases} x_{n+1} = g(y_n) \\ y_n = (1-\alpha_n) z_n + \alpha_n g(z_n) \\ z_n = g(x_n) \ (n \in \mathbb{N}), \end{cases} \tag{6}$$

in which $\{\alpha_n\}_{n=1}^{\infty} \in$ [0,1].
Now, we will give some useful facts to proofs of our main results.

Lemma 1 ([11])**.** *Let E be a closed interval on the real line and $g : E \to E$ be a continuous and non-decreasing function. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence defined by iteration method (5) for $u_1 \in E$, with control sequence $\{\alpha_n\}_{n=1}^{\infty} \in$ [0,1]. Then the following hold:*

i. *If $g(u_1) < u_1$, then $g(u_n) \leq u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^{\infty}$ is non-increasing.*
ii. *If $g(u_1) > x_1$, then $g(u_n) \geq u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^{\infty}$ is non-decreasing.*

Proposition 1. *Let E be a closed interval on the real line and $g : E \to E$ be a continuous, non-decreasing function and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty} \in$ [0,1]. Also for the initial values $u_1 \in E$, let $\{u_n\}_{n=1}^{\infty}$ be defined by (5). Then, the following assertions are true:*

i. *F_g is nonempty and bounded with $u_1 < \inf \{q \in E : q = g(q)\}$. If $g(u_1) < u_1$, then the sequence $\{u_n\}_{n=1}^{\infty}$ defined by iteration method (5) does not converge to a fixed point of g.*
ii. *F_g is nonempty and bounded with $u_1 > \sup \{q \in E : q = g(q)\}$. If $g(u_1) > u_1$, then the sequence $\{u_n\}_{n=1}^{\infty}$ defined by iteration method (5) does not converge to a fixed point of g.*

Definition 1 ([8])**.** *Let E be a closed interval on the real line and $g : E \to E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ two iteration methods which converge to the fixed point q of g. Then, $\{x_n\}_{n=1}^{\infty}$ is said to converge faster than $\{s_n\}_{n=1}^{\infty}$ if*

$$|x_n - q| \leq |s_n - q| \quad \text{for all } n \in \mathbb{N}.$$

Theorem 1 ([12])**.** *Let E be a closed interval on the real line and $g : E \to E$ be a continuous, non-decreasing function such that F_g is nonempty and bounded and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty} \in$ [0,1]. Also for the initial values $x_1 = u_1 \in E$, let $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be defined by (4) and (6) respectively. If $\{u_n\}_{n=1}^{\infty}$ converges to fixed point $q \in F_g$, then $\{x_n\}_{n=1}^{\infty}$ converges to the same fixed point $q \in F_g$. Moreover, $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{u_n\}_{n=1}^{\infty}$.*

2 Main Results

Lemma 2. *Let E be a closed interval on the real line and $g : E \to E$ be a continuous and non-decreasing function. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by iteration method (6) for $x_1 \in E$, with control sequence $\{\alpha_n\}_{n=1}^{\infty} \in$ [0,1]. Then the following hold:*

i. *If $g(x_1) < x_1$, then $g(x_n) \leq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-increasing.*
ii. *If $g(x_1) > x_1$, then $g(x_n) \geq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-decreasing.*

Proof:

i. Let $g(x_1) < x_1$. Assume that $g(x_k) \leq x_k$ for $k > 1$. Then by (6) we have $g(x_k) = z_k \leq x_k$. Since g non-decreasing, we have $g(z_k) \leq g(x_k) = z_k \leq x_k$. Again using the same arguments, we obtain

$$g(z_k) \leq g(x_k) = z_k \leq x_k$$

$$g(y_k) \leq g(x_k) = z_k \leq x_k$$

and since

$$x_{k+1} = g(y_k),$$

we have

$$x_{k+1} = g(y_k) \leq g(x_k) = z_k \leq x_k .$$

Using the non-decreasing property of g, we obtain

$$g(x_{k+1}) \leq x_{k+1} = g(y_k) \leq g(x_k) = z_k \leq x_k. \tag{7}$$

By induction, we get

$$g(x_n) \leq x_n.$$

Hence

$$g(y_n) \leq g(x_n).$$

Considering (7), we can conclude that

$$x_{n+1} = g(y_n) \leq g(x_n) = z_n \leq x_n, \text{ for all } n \in \mathbb{N}.$$

Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is non-increasing.

ii. By using the same argument as in (i), we obtain the desired result.

□

Theorem 2. *Let E be a closed interval on the real line and $g : E \to E$ be a continuous and non-decreasing function. Let $\{x_n\}_{n=1}^{\infty}$ be defined by* (6) *for $x_1 \in E$, with control sequence $\{\alpha_n\}_{n=1}^{\infty} \in$ [0,1]. Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of g.*

Proof: Assume that $(x_n)_{n=1}^{\infty}$ is bounded. If $g(x_1) = x_1$, we have

$$z_1 = g(x_1) = x_1.$$
$$y_1 = (1-\alpha_1) z_1 + \alpha_1 g(z_1) = x_1$$
$$x_2 = g(y_1) = x_1$$

It is clear that $x_n = x_1$ and $\lim_{n\to\infty} x_n = x_1$, for all $n \geq 1$. If $g(x_1) < x_1$ or $g(x_1) > x_1$, then, by Lemma 2, we obtain that $(x_n)_{n=1}^{\infty}$ is non-increasing or non-decreasing. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, it implies that $\{x_n\}_{n=1}^{\infty}$ is convergent. Since $(x_n)_{n=1}^{\infty}$ is convergent, there is a $\lim_{n\to\infty} x_n = q \in E$. Using the continuity of g and $(x_n)_{n=1}^{\infty}$ is bounded, we obtain $\{g(x_n)\}_{n=1}^{\infty}$ is bounded. In addition, iteration method (6) can be edited as follows:

$$x_{n+1} = g(y_n)$$

$$y_n - z_n = \alpha_n [g(z_n) - z_n]$$

$$z_n = g(x_n).$$

We show in two steps that q is a fixed point of g.

Step 1. If $g(x_1) < x_1$, then $g(x_n) \leq x_n$ for all $n \geq 1$ and since $\lim_{n\to\infty} x_n = q \in E$, it is clear that $\lim_{n\to\infty} g(x_n) = g(q) \leq \lim_{n\to\infty} x_n = q \in E$. Also, the following inequalty was obtained by Lemma 2:

$$x_{n+1} = g(y_n) \leq g(x_n), \text{ for all } n \in \mathbb{N}.$$

Hence

$$q = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(y_n) \leq \lim_{n\to\infty} g(x_n) = g(q).$$

It contradicts our assumption. Therefore, $g(q) = q$.

Step 2. If $g(x_1) > x_1$, then $g(x_n) \geq x_n$ for all $n \geq 1$ and since $\lim_{n\to\infty} x_n = q \in E$, it is clear that $\lim_{n\to\infty} x_n = q \leq \lim_{n\to\infty} g(x_n) = g(q) \in E$. Also, the following inequalty was obtained by Lemma 2:

$$x_{n+1} = g(y_n) \geq g(x_n), \text{ for all } n \in \mathbb{N}.$$

Hence

$$q = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(y_n) \geq \lim_{n\to\infty} g(x_n) = g(q).$$

It contradicts our assumption. Therefore, $g(q) = q$. Hence q is a fixed point of g and $\{x_n\}_{n=1}^{\infty}$ converges to q.

□

Lemma 3. *Let E be a closed interval on the real line and $g : E \to E$ be a continuous and non-decreasing function. Let $\{x_n\}_{n=1}^{\infty}$ be defined by* (6) *for $x_1 \in E$, with control sequence $\{\alpha_n\}_{n=1}^{\infty} \in$ [0,1]. Then, the following assertions are true:*

i. *If $q \in F_g$ with $x_1 > q$, then $x_n \geq q$ for all $n \geq 1$.*
ii. *If $q \in F_g$ with $x_1 < q$, then $x_n \leq q$ for all $n \geq 1$.*

Proof:

i. From our claim, $q \in F_g$ with $x_1 > q$. Since g is non-decreasing, we obtain $g(x_1) \geq g(q)$. By iteration method (6), we have

$$z_1 = g(x_1) \geq g(q) = q$$

$$y_1 = (1-\alpha_1) z_1 + \alpha_1 g(z_1) \geq (1-\alpha_1) q + \alpha_1 g(q) = q.$$

The above inequalities imply that $g(z_1) \geq g(q)$ and $g(y_1) \geq g(q)$. Again we re handle iteration method (6), we obtain

$$x_2 = g(y_1) \geq g(q) = q.$$

Suppose that $x_k \geq q$ for $k > 2$. Then $g(x_k) \geq g(q) = q$. By using iteration method (6), we have

$$z_k = g(x_k) \geq g(q) = q$$

$$y_k = (1-\alpha_k) z_k + \alpha_k g(z_k) \geq (1-\alpha_k) q + \alpha_k g(q) = q.$$

Thus $g(z_k) \geq g(q) = q$ and $g(y_k) \geq g(q) = q$. Also, we obtain

$$x_{k+1} = g(y_k) \geq g(q) = q.$$

By induction, we have

$$x_n \geq q \text{ for all } n \geq 1.$$

ii. Using the same arguments in (i) one can easily show this assertion. For this reason, the proof will not be given.

□

Lemma 4. *Let E be a closed interval on the real line and $g : E \to E$ be a continuous, non-decreasing function and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ $\in$ [0,1]. Also for the initial values $x_1 = u_1 \in E$, let $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be defined by (5) and (6) respectively. Then, the following assertions are true:*

i. *If $g(u_1) < u_1$, then $x_n \leq u_n$ for all $n \geq 1$.*
ii. *If $g(u_1) > u_1$, then $x_n \geq u_n$ for all $n \geq 1$.*

Proof:

i. Since $x_1 = u_1$ we have $g(x_1) < x_1$. By (6) and g is non-decreasing, we obtain

$$g(z_1) \leq g(x_1) = z_1 \leq x_1$$

and

$$g(y_1) \leq g(x_1) = z_1 \leq x_1.$$

Also from [11], we get

$$g(v_n) \leq g(w_n) \leq g(u_n) \leq u_n.$$

Thus, by (6) and (5), we get

$$\begin{aligned} z_1 - u_1 &= g(x_1) - u_1 \\ &\leq 0 \end{aligned}$$

it implies that $z_1 \leq u_1$. That is, $g(z_1) \leq g(u_1)$,

$$\begin{aligned} z_1 - w_1 &= g(x_1) - (1-\beta_1) u_1 - \beta_1 g(u_1) \\ &= (1-\beta_1)(gx_1 - x_1) \\ &\leq 0 \end{aligned}$$

it implies that $z_1 \leq w_1$. That is, $g(z_1) \leq g(w_1)$,

$$\begin{aligned} y_1 - v_1 &= (1-\alpha_1) z_1 + \alpha_1 g(z_1) - (1-\alpha_1) gu_1 - \alpha_1 g(w_1) \\ &= (1-\alpha_1)(z_1 - gu_1) + \alpha_1 (gz_1 - gw_1) \\ &\leq 0 \end{aligned}$$

it implies that $y_1 \leq v_1$. That is, $g(y_1) \leq g(v_1)$ and

$$\begin{aligned} x_2 - u_2 &= g(y_1) - g(v_1) \\ &\leq 0 \end{aligned}$$

it implies that $x_2 \le u_2$. That is, $g(x_2) \le g(u_2)$. We suppose that $x_k \le u_k$, for $k \in \mathbb{N}$. Then $g(x_k) \le g(u_k)$. From Lemma 1, $g(u_k) \le u_k$ and from Lemma 2 $g(x_k) \le x_k$. This follows that

$$g(x_k) = z_k \le x_k \le u_k.$$

From properties of g, we get

$$g(z_k) \le g(u_k).$$

Also, by (6) and (5), we get

$$\begin{aligned} z_k - w_k &= g(x_k) - (1-\beta_k) u_k - \beta_k g(u_k) \\ &\le (1-\beta_k)(gx_k - u_k) + \beta_k(gx_k - gu_k) \\ &\le 0 \end{aligned}$$

it implies that $z_k \le w_k$. That is, $g(z_k) \le g(w_k)$ and

$$\begin{aligned} y_k - v_k &= (1-\alpha_k) z_k + \alpha_k g z_k - (1-\alpha_k) g u_k - \alpha_k g w_k \\ &= (1-\alpha_k)(z_k - gu_k) + \alpha_k(gz_k - gw_k) \\ &\le 0 \end{aligned}$$

it implies that $y_k \le v_k$. That is, $g(y_k) \le g(v_k)$ and

$$\begin{aligned} x_{k+1} - u_{k+1} &= g(y_k) - g(v_k) \\ &\le 0 \end{aligned}$$

we conclude that $x_{k+1} \le u_{k+1}$. That is, $g(x_{k+1}) \le g(u_{k+1})$. By induction, we obtain the desired result $x_n \le u_n$, for all $n \ge 1$.

ii. Using the same arguments in (i), one can easily show this assertion. For this reason, the proof of (ii) will not be given.

□

Proposition 2. *Let E be a closed interval on the real line and $g : E \to E$ be a continuous, non-decreasing function and $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$. Also for the initial values $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be defined by (6). Then, the following assertions are true:*

i. *F_g is nonempty and bounded with $x_1 < \inf\{q \in E : q = g(q)\}$. If $g(x_1) < x_1$, then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by iteration method (6) does not converge to a fixed point of g.*

ii. *F_g is nonempty and bounded with $x_1 > \sup\{q \in E : q = g(q)\}$. If $g(x_1) > x_1$, then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by iteration method (6) does not converge to a fixed point of g.*

Proof:

i. By Lemma 2 and by assertion of (i), since $\{x_n\}_{n=1}^{\infty}$ is non-decreasing and $x_1 < \inf\{q \in E : q = g(q)\}$, respectively. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by iteration method (6) does not converge to a fixed point of g.

ii. By Lemma 2 and by assertion of (i), since $\{x_n\}_{n=1}^{\infty}$ is non-increasing and $x_1 > \sup\{q \in E : q = g(q)\}$, respectively. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by iteration method (6) does not converge to a fixed point of g.

□

Theorem 3. *Let E be a closed interval on the real line and $g : E \to E$ be a continuous, non-decreasing function such that F_g is nonempty and bounded and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\in [0,1]$. Also for the initial values $x_1 = u_1 \in E$, let $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be defined by (4) and (6) respectively. If $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge to the same fixed point $q \in F_g$ then, the iteration method (6) converges faster than the iteration method (5).*

Proof: In [11], it was shown that Picard-S iteration method (5) is convergent to fixed point of g. Let $k = \inf\{q \in E : q = g(q)\}$ and $t = \sup\{q \in E : q = g(q)\}$. Our proof will be analyzed in three cases.

Case1. Let $t < x_1 = u_1$. From Proposition 1 and Proposition 2, we get $g(x_1) < x_1$ and $g(u_1) < u_1$. From Lemma 4 (i), we have $x_n \le u_n$ for all $n \ge 1$. By using Picard-S iteration method and mathematical induction, we can show that $t \le x_n$. Thus, we obtain

$$q \le x_n \le u_n,$$

so

$$|x_n - q| \le |u_n - q|$$

for all $n \ge 1$. That is, $\{x_n\}_{n=1}^{\infty}$ converges to $q \in F_g$ faster than $\{u_n\}_{n=1}^{\infty}$.

Case2. Let $k > x_1 = u_1$. From Proposition 1 and Proposition 2, we get $g(x_1) > x_1$ and $g(u_1) > u_1$. From Lemma 4 (ii), we have $x_n \geq u_n$ for all $n \geq 1$. By using iteration method (6) and mathematical induction, we can show that $x_n \leq k$. Thus, we obtain

$$u_n \leq x_n \leq q,$$

so

$$0 \leq |x_n - q| \leq |u_n - q|$$

for all $n \geq 1$. It follows that $\{x_n\}_{n=1}^{\infty}$ converges to $q \in F_g$ faster than $\{u_n\}_{n=1}^{\infty}$.

Case3. Let $t < x_1 = u_1 < k$. Suppose that $g(x_1) \neq x_1$. If $g(x_1) < x_1$, then by Lemma 1, we get $\{u_n\}_{n=1}^{\infty}$ iteration method is non-increasing. This implies that $q \leq u_n$ for all $n \geq 1$. From Lemma 3 and Lemma 4, we obtain $q \leq x_n \leq u_n$. That is,

$$0 \leq |x_n - q| \leq |u_n - q|$$

it follows that $\{x_n\}_{n=1}^{\infty}$ converges to $q \in F_g$ faster than $\{u_n\}_{n=1}^{\infty}$ iteration method.
Assume that $g(x_1) > x_1$, then by Lemma 1, we get that $\{u_n\}_{n=1}^{\infty}$ iteration method is non-decreasing. This implies that $u_n \leq q$ for all $n \geq 1$. From Lemma 3 and Lemma 4, we obtain $u_n \leq x_n \leq q$. That is,

$$0 \leq |x_n - q| \leq |u_n - q|$$

it follows that $\{x_n\}_{n=1}^{\infty}$ converges to $q \in F_g$ faster than $\{u_n\}_{n=1}^{\infty}$.

□

Example 1. *Let $g : [1,2] \to [1,2]$ defined by $g(x) = \frac{x^2+4\sqrt{x}+6}{9}$. It is easy to show that g is continuous and non-decrasing with fixed point $q = 1.4207$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{4}$, and an initial value $x_1 = 1$. The following table shows that the PMP iteration method (6) converges faster than all Mann (1), Ishikawa (2), Noor (3), SP (4) and, Picard-S (5) iteration methods.*

Table 1 Comparison rate of convergence among various iteration methods

Iter. No	Mann	Ishikawa	Noor	SP	Picard-S	PMP
1	1	1	1	1	1	1
2	1,0556	1,0618	1,0625	1,1448	1,3269	1,3361
⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	1,2617	1,2797	1,2817	1,3965	1,4206	1,4206
9	1,2820	1,2998	1,3018	1,4045	1,4207	1,4207
⋮	⋮	⋮	⋮	⋮	⋮	⋮
27	1,4083	1,4128	1,4132	1,4206	⋮	⋮
28	1,4098	1,4143	1,4132	1,4207	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮
72	1,4206	1,4206	1,4207	⋮	⋮	⋮
73	1,4206	1,4207	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮
82	1,4207	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮

3 Conclusion

In this work, we investigate the necessary and sufficient conditions for the convergence result of some iteration methods by using continuous mapping. We also compare the rate of convergence of two iteration methods and to support this, we give a numerical example.

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A Tauberian Theorem for the Weighted Mean Summability Method of Double Sequences in Ordered Spaces

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Abstract: In this paper, our aim is to extend a Tauberian theorem given for the Cesàro summability method and the weighted mean summability method of single sequences in ordered spaces to the weighted mean, or shortly, the $(\overline{N}, p, q)$ summability method of double sequences. Accordingly, we give some Tauberian conditions, controlling O_L-oscillatory behavior of a double sequence (s_{mn}) in certain senses, from the $(\overline{N}, p, q), (\overline{N}, p, *)$, and $(\overline{N}, *, q)$ summability to P-convergence with some restrictions on the weight sequences p and q in ordered spaces.

Keywords: Double sequences, Ordered linear spaces, Slowly decreasing sequences, Tauberian conditions, Tauberian theorems, Weighted mean summability method.

1 Introduction

A double sequence $s = (s_{mn})$ is a function s from $\mathbb{N} \times \mathbb{N}$ into the set $\mathbb{R}$ or $\mathbb{C}$. The real or complex number s_{mn} denotes the value of the function s at a point $(m, n) \in \mathbb{N} \times \mathbb{N}$ and is called the (m, n)-term of the double sequence.

A double sequence $s = (s_{mn})$ is said to be convergent in Pringsheim's sense, or shortly, P-convergent to L if for all $\epsilon > 0$ there exists a $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $|s_{mn} - L| < \epsilon$ whenever $m, n \geq n_0$ (see [1]). The number L is called the P-limit of s and we denote it by $P\text{-}\lim_{m,n\to\infty} s_{mn} = L$, where both m and n tend to ∞ independently of each other.

Let $p = (p_m), q = (q_n)$ be two sequences of nonnegative numbers such that $p_0,\ q_0 > 0$ and

$$P_m := \sum_{i=0}^{m} p_i \to \infty \quad \text{and} \quad Q_n := \sum_{j=0}^{n} q_j \to \infty \quad \text{as} \quad m, n \to \infty. \tag{1}$$

The weighted means of (s_{mn}) determined by the weight sequences (p_m) and (q_n) are defined by

$$\sigma_{mn}^{11} := \frac{1}{P_m Q_n} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j s_{ij}, \quad \sigma_{mn}^{10} := \frac{1}{P_m} \sum_{i=0}^{m} p_i s_{in}, \quad \sigma_{mn}^{01} := \frac{1}{Q_n} \sum_{j=0}^{n} q_j s_{mj}, \tag{2}$$

where $P_m Q_n > 0$ for all $m, n \in \mathbb{N}$ (see [5]).

A double sequence (s_{mn}) is called $(\overline{N}, p, q)$ summable to L if $P\text{-}\lim \sigma_{mn}^{11} = L$. Similarly, $(\overline{N}, p, *)$ and $(\overline{N}, *, q)$ summable sequences are defined via double sequences (σ_{mn}^{10}) and (σ_{mn}^{01}), respectively. If a bounded double sequence is P-convergent to L, then it is also $(\overline{N}, p, q)$ summable to same number under (1). However, the opposite of this implication is not true in general. The question of whether certain conditions imposed on the terms s_{mn} and $p_m,\ q_n$ under which its $(\overline{N}, p, q)$ summability implies its P-convergence exist comes to mind at this point. The condition $T\{s_{mn}\}$ making such a situation possible is called a *Tauberian condition.* The resulting theorem stating that P-convergence follows from its $(\overline{N}, p, q)$ summability and $T\{s_{mn}\}$ is called a *Tauberian Theorem*.

For a double sequence (u_{mn}), we define

$$\begin{aligned}
\Delta_{11} s_{mn} &:= \Delta_{10}\Delta_{01} s_{mn} = \Delta_{10} (\Delta_{01} s_{mn}) = \Delta_{01} (\Delta_{10} s_{mn}) \\
&= s_{mn} - s_{m,n-1} - s_{m-1,n} + s_{m-1,n-1}, \\
\Delta_{10} s_{mn} &:= s_{mn} - s_{m-1,n}, \\
\Delta_{01} s_{mn} &:= s_{mn} - s_{m,n-1}
\end{aligned}$$

for all $m, n \in \mathbb{N}$.

The double weighted Kronecker identity for a sequence (s_{mn}) are defined via $(V^{11}_{mn}(\Delta_{11}s))$ as follows:

$$s_{mn} - \sigma^{10}_{mn}(s) - \sigma^{01}_{mn}(s) + \sigma^{11}_{mn}(s) = V^{11}_{mn}(\Delta_{11}s),$$

where

$$V^{11}_{mn}(\Delta_{11}s) := \frac{1}{P_m Q_n} \sum_{i=1}^{m} \sum_{j=1}^{n} P_{i-1} Q_{j-1} \Delta_{11} s_{ij}$$

for all $m, n \in \mathbb{N}$ (see [8, 11]).
The double sequence $(V^{11}_{mn}(\Delta_{11}s))$ is the $(\overline{N}, p, q)$ mean of $\left(\frac{P_{m-1}Q_{n-1}}{p_m q_n} \Delta_{11} s_{mn}\right)$ and it is called the weighted generator sequence of (s_{mn}) in the sense $(1, 1)$.

Throughout this paper, we consider an ordered linear space $(X, \leq)$ over the real numbers, in which we denote by o the zero element and by τ a given nonnegative element. In addition, we assume that (s_{mn}) is a double sequence of elements in X.

Now, we give concepts of P-convergence and slow decrease in certain senses for double sequences in X (see [4]).

A double sequence (s_{mn}) in X is said to be P-convergent to $L \in X$, relative to $\tau \in X$, if for all $\epsilon > o$ there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$-\epsilon\tau \leq s_{mn} - L \leq \epsilon\tau \quad \text{whenever} \quad m, n > n_0.$$

A double sequence (s_{mn}) in X is said to be slowly decreasing in sense $(1, 1)$, relative to $\tau \in X$, if for all $\epsilon > o$ there exist $n_0 = n_0(\epsilon) \in \mathbb{N}$, and $\delta = \delta(\epsilon) > 0$ such that

$$s_{ij} - s_{in} - s_{mj} + s_{mn} \geq -\epsilon\tau \quad \text{whenever} \quad n_0 < m < i \leq m(1+\delta) \text{ and } n_0 < n < j \leq n(1+\delta),$$

and slowly decreasing in sense $(1, 0)$, relative to $\tau \in X$, if

$$s_{in} - s_{mn} \geq -\epsilon\tau \quad \text{whenever} \quad n_0 < m < i \leq m(1+\delta) \text{ and } n_0 < n,$$

and slowly decreasing in sense $(0, 1)$, relative to $\tau \in X$, if

$$s_{mj} - s_{mn} \geq -\epsilon\tau \quad \text{whenever} \quad n_0 < m \text{ and } n_0 < n < j \leq n(1+\delta).$$

Notice that when X is the real linear space $\mathbb{R}$ with its usual order, relative to 1, then these definitions reduce to the classical definitions of P-convergence and slow decrease of double sequences in certain senses.

In the remainder of this section, we mention the class of *SVA*, its characterization and two of its subclasses. Let $p = (p_m)$ be a sequence that satisfies $(p_m) = (P_m - P_{m-1})$, where $P_{-1} = 0$ and $P_m \neq 0$ for all $m \in \mathbb{N}$.
A sequence (P_m) of real or complex numbers is said to be *varying away from* 1 if

$$\liminf_{m\to\infty} \left| \frac{P_{\lambda_m}}{P_m} - 1 \right| > 0 \quad \text{for all} \quad \lambda > 0 \quad \text{with} \quad \lambda \neq 1, \tag{3}$$

i.e., for each $\lambda > 0$ with $\lambda \neq 1$ there exist $\delta_\lambda > 0$ and $m_\lambda \in \mathbb{N}$ such that

$$\left| \frac{P_{\lambda_m}}{P_m} - 1 \right| \geq \delta_\lambda \quad \text{whenever} \quad m > m_\lambda,$$

where λ_m denotes the integer part of the product λm. The set of all sequences (p_m) of real or complex numbers satisfying (3) is denoted by SVA_r or *SVA*, respectively.
The following lemma due to Chen and Hsu [3] gives another representation of the class of *SVA* (or SVA_r).

Lemma 1. *([3]) Let $p = (p_m)$ be a complex (or real) sequence with $P_m \neq 0$ for all $m \in \mathbb{N}$. Then, condition (3) is equivalent to condition*

$$\liminf_{m\to\infty} \left| \frac{P_m}{P_{\lambda_m}} - 1 \right| > 0 \quad \textit{for all} \quad \lambda > 0 \quad \textit{with} \quad \lambda \neq 1.$$

Analogously, the set of all sequences (p_m) of nonnegative numbers with $(p_m) \in SVA$ is denoted by SVA_+. It is clear that $(p_m) \in SVA_+$ if and only if (P_m) is a nondecreasing sequence of positive numbers with $p_0 > 0$ and any of the following conditions is satisfied:

$$\liminf_{m\to\infty} \frac{P_{\lambda_m}}{P_m} > 1 \quad \text{for all} \quad \lambda > 1, \tag{4}$$

$$\limsup_{m\to\infty} \frac{P_{\lambda_m}}{P_m} < 1 \quad \text{for all} \quad 0 < \lambda < 1. \tag{5}$$

The following lemma proved by Chen and Hsu [3] gives another representation of the class of SVA_+.

Lemma 2. *([3]) Let $p = (p_m)$ be a nonnegative sequence with $p_0 > 0$. Then, conditions (4) and (5) are equivalent to conditions*

$$\liminf_{m\to\infty} \frac{P_m}{P_{\lambda_m}} > 1 \quad \text{for all} \quad 0 < \lambda < 1$$

and

$$\limsup_{m\to\infty} \frac{P_m}{P_{\lambda_m}} < 1 \quad \text{for all} \quad \lambda > 1,$$

respectively.

A relation between the classes of SVA, SVA_r and SVA_+ is given as follows:

$$SVA_+ \subset SVA_r \subset SVA.$$

In the real case, one-sided Tauberian theorem for the $(\overline{N}, p, q)$ summability of double series states that if a double series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}$ is $(\overline{N}, p, q)$ summable to a finite number L with conditions $P_{m+1}/P_m \to 1$ as $m \to \infty$ and $Q_{n+1}/Q_n \to 1$ as $n \to \infty$, and if there exist N_0 and a constant $H > 0$ such that

$$\inf_{m\in N_0} \left\{\sum_{i=0}^{m} a_{in}\right\} \geq -H\frac{q_n}{Q_n} \quad \text{and} \quad \inf_{n\in N_0} \left\{\sum_{j=0}^{n} a_{mj}\right\} \geq -H\frac{p_m}{P_m},$$

then $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}$ is P-convergent to L (see [10]). More generally, it is also indicated in [10] that if $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}$ is $(\overline{N}, p, q)$ summable to a finite number L with conditions $P_{m+1}/P_m \to 1$ as $m \to \infty$ and $Q_{n+1}/Q_n \to 1$ as $n \to \infty$, and its partial sums sequence $(s_{mn}) = (\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij})$ with $a_{ij} = s_{ij} - s_{i,j-1} - s_{i-1,j} + s_{i-1,j-1}$ is slowly decreasing in certain senses, then $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}$ is P-convergent to L. In the literature, there are also other Tauberian theorems reflecting relation between the limits $\lim_{m,n\to\infty} \sigma_{mn}^{11} = L$ and $\lim_{m,n\to\infty} s_{mn} = L$. In particular, Baron and Stadtmüller [9], Chen and Hsu [3], Móricz and Stadtmüller [5] and Belen [2] have worked on Tauberian theorems for the $(\overline{N}, p, q)$ summability of double series or sequences.

In this paper, we aim to extend a Tauberian theorem for the Cesàro summability method due to Maddox [7] and the weighted mean summability method due to Çanak [6] in ordered spaces to the $(\overline{N}, p, q)$, summability method of double sequences. These researchers formulate the related results as follows, respectively:

Theorem 1. *([7]) Let $(X, \leq)$ be an ordered linear space over the real numbers and suppose that a sequence (s_n) is Cesàro summable to $L \in X$, relative to $\tau \in X$. If (s_n) is slowly decreasing, relative to $\tau \in X$, then (s_n) is convergent to L, relative to $\tau \in X$.*

Theorem 2. *([6]) Let $(X, \leq)$ be an ordered linear space over the real numbers, and let (4) be satisfied. Suppose that a sequence (s_n) is summable by the weighted mean method to $L \in X$, relative to $\tau \in X$. If (s_n) is slowly decreasing, relative to $\tau \in X$, then (s_n) is convergent to L, relative to $\tau \in X$.*

Accordingly, we give some Tauberian conditions, controlling O_L-oscillatory behavior of a double sequence (s_{mn}) in certain senses, from the $(\overline{N}, p, q)$, $(\overline{N}, p, *)$, and $(\overline{N}, *, q)$ summability to P-convergence with some restrictions on the weight sequences p and q in ordered spaces.

In a general ordered linear space $(X, \leq)$, we consider a given series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ with its double sequence of partial sums (s_{mn}). The weighted means of (s_{mn}) are defined by (2).

2 Main Results

In this section, we formulate our main result for the $(\overline{N}, p, q)$ summable double sequences in $(X, \leq)$ as follows:

Theorem 3. *Let $(X, \leq)$ be an ordered linear space over the real numbers, and let $p = (p_m), q = (q_n) \in SVA_+$, i.e.,*

$$\alpha := \liminf_{m\to\infty} \frac{P_{\lambda_m}}{P_m} > 1 \quad \text{and} \quad \beta := \liminf_{n\to\infty} \frac{Q_{\lambda_n}}{Q_n} > 1 \quad \text{for all} \ \lambda > 1 \tag{6}$$

be satisfied. Suppose that a sequence (s_{mn}) is $(\overline{N}, p, q)$, $(\overline{N}, p, *)$ and $(\overline{N}, *, q)$ summable to $L \in X$, relative to $\tau \in X$. If (s_{mn}) is slowly decreasing in senses $(1,1)$, $(1,0)$, and $(0,1)$, relative to $\tau \in X$, then (s_{mn}) is P-convergent to L, relative to $\tau \in X$.

Proof: Without loss of generality, we suppose that $L = o$. Otherwise, we consider the series

$$(a_{00} - x) + \sum_{m=1}^{\infty} a_{m0} + \sum_{n=1}^{\infty} a_{0n} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}.$$

Set the double sequence (t_{mn}^{11}) as

$$t_{mn}^{11} := \sum_{i=1}^{m}\sum_{j=1}^{n} P_{i-1}Q_{j-1}\Delta_{11}s_{ij} \tag{7}$$

for all $m, n \geq 1$. Since we have

$$\Delta_{11}\sigma_{mn} = \sigma_{mn} - \sigma_{m-1,n} - \sigma_{m,n-1} + \sigma_{m-1,n-1} = \frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} t_{mn}^{11} \tag{8}$$

for $m, n \geq 1$, it follows from (8) that

$$\sigma_{m+p,n+r}^{11} - \sigma_{m+p,n}^{11} - \sigma_{m,n+r}^{11} + \sigma_{mn}^{11} = \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \Delta_{11}\sigma_{ij}^{11} = \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} t_{ij}^{11} \tag{9}$$

for $p, r \geq 1$. Let $\epsilon > 0$ be given. Define $\epsilon' = \epsilon/\zeta$, where $\zeta = 4\alpha^2\beta^2/(\alpha-1)(\beta-1)$ for $\alpha, \beta > 1$. It is known that (s_{mn}) is slowly decreasing in senses $(1,1)$, $(1,0)$, and $(0,1)$, relative to $\tau \in X$, there exist $n_1 = n_1(\epsilon')$, $n_2 = n_2(\epsilon') \in \mathbb{N}$ and $0 < \delta < 1$ such that

$$s_{in} - s_{mn} \geq -\left(\frac{\epsilon'}{40}\right)\tau \quad \text{whenever } n_1 < m < i \leq m + [m\delta] \text{ and } n_1 \leq n,$$

$$s_{mj} - s_{mn} \geq -\left(\frac{\epsilon'}{40}\right)\tau \quad \text{whenever } n_2 < n < j \leq n + [n\delta] \text{ and } n_2 \leq m,$$

and additionally there exist $n_0 = n_0(\epsilon') = \min\{n_1, n_2\} \in \mathbb{N}$ and $0 < \delta < 1$ such that

$$s_{ij} - s_{in} - s_{mj} + s_{mn} \geq -\left(\frac{\epsilon'}{40}\right)\tau \quad \text{whenever } n_0 < m < i \leq m + [m\delta] \text{ and } n_0 < n < j \leq n + [n\delta].$$

Since (σ_{mn}^{11}) is P-convergent to o, relative to $\tau \in X$, it follows from (9) that, writing $p = [m\delta]$ and $r = [n\delta]$,

$$-\left(\frac{\epsilon'\delta}{40}\right)\tau \leq \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} t_{ij}^{11} \leq \left(\frac{\epsilon'\delta}{40}\right)\tau \tag{10}$$

for sufficiently large m, n. Define the double sequence (γ_{mn}) as

$$\gamma_{mn} := \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}}$$

for sufficiently large m, n. Then, we obtain

$$\begin{aligned}
\gamma_{mn} t_{mn}^{11} &= \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} t_{mn}^{11} \\
&= \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} t_{ij}^{11} - \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{mj}^{11} - t_{mn}^{11}) \\
&\quad - \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{in}^{11} - t_{mn}^{11}) - \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11}) \\
&\leq \left(\frac{\epsilon'\delta}{20}\right)\tau - \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{mj}^{11} - t_{mn}^{11}) - \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{in}^{11} - t_{mn}^{11}) \\
&\quad - \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11})
\end{aligned} \tag{11}$$

for sufficiently large m, n. It is clear that

$$\begin{aligned}
t_{mn}^{11} = \sum_{i=1}^{m} \sum_{j=1}^{n} P_{i-1} Q_{j-1} \Delta_{11} s_{ij} &= \sum_{i=1}^{m} P_{i-1} \sum_{j=1}^{n} Q_{j-1} \Delta_{01}(\Delta_{10} s_{ij}) = \sum_{i=1}^{m} P_{i-1} \left(Q_n \Delta_{10} s_{in} - \sum_{j=0}^{n} q_j \Delta_{10} s_{ij} \right) \\
&= Q_n \sum_{i=1}^{m} P_{i-1} \Delta_{10} s_{in} - \sum_{j=0}^{n} q_j \sum_{i=1}^{m} P_{i-1} \Delta_{10} s_{ij} \\
&= Q_n \left(P_m s_{mn} - \sum_{i=0}^{m} p_i s_{in} \right) - \sum_{j=0}^{n} q_j \left(P_m s_{mj} - \sum_{i=0}^{m} p_i s_{ij} \right) \\
&= P_m Q_n s_{mn} - Q_n \sum_{i=0}^{m} p_i s_{in} - P_m \sum_{j=0}^{n} q_j s_{mj} + \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j s_{ij}
\end{aligned} \tag{12}$$

for all $m, n \geq 1$. From this point of view, we find

$$t_{mj}^{11} - t_{mn}^{11} = P_m Q_n (s_{mj} - s_{mn}) + P_m \sum_{k=n+1}^{j-1} q_k (s_{mj} - s_{mk}) + \sum_{r=0}^{m} \sum_{k=n+1}^{j} p_r q_k (s_{rk} - s_{rn}) + P_m Q_j (\sigma_{mn}^{10} - \sigma_{mj}^{10}), \tag{13}$$

$$t_{in}^{11} - t_{mn}^{11} = P_m Q_n (s_{in} - s_{mn}) + Q_n \sum_{r=m+1}^{i-1} p_r (s_{in} - s_{rn}) + \sum_{k=0}^{n} \sum_{r=m+1}^{i} p_r q_k (s_{rk} - s_{mk}) + P_i Q_n (\sigma_{mn}^{01} - \sigma_{in}^{01}), \tag{14}$$

and

$$\begin{aligned} t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11} = {} & P_m Q_n (s_{ij} - s_{in} - s_{mj} + s_{mn}) + \sum_{r=m+1}^{i-1} \sum_{k=n+1}^{j-1} p_r q_k (s_{ij} - s_{rj} - s_{ik} + s_{rk}) \\ & + Q_n \sum_{r=m+1}^{i-1} p_r (s_{ij} - s_{rj} - s_{in} + s_{rn}) + P_m \sum_{k=n+1}^{j-1} q_k (s_{ij} - s_{ik} - s_{mj} + s_{mk}) \end{aligned} \tag{15}$$

for sufficiently large m, n. From slow decrease of (s_{mn}) in senses $(1,1)$, $(1,0)$, and $(0,1)$, relative to $\tau \in X$, we have

$$\begin{aligned} s_{in} - s_{\mu n} &\geq -\left(\frac{\epsilon'}{40}\right)\tau \quad \text{whenever } n_1 < m < i \leq m+p,\ m < \mu < i \text{ and } n_1 \leq n, \\ s_{mj} - s_{m\nu} &\geq -\left(\frac{\epsilon'}{40}\right)\tau \quad \text{whenever } n_2 < n < j \leq n+r,\ n < \nu < j \text{ and } n_2 \leq m, \\ s_{ij} - s_{i\nu} - s_{\mu j} + s_{\mu\nu} &\geq -\left(\frac{\epsilon'}{40}\right)\tau \quad \text{whenever } n_0 < m < i \leq m+p,\ m < \mu < i \text{ and } n_0 < n < j \leq n+r,\ n < \nu < j. \end{aligned}$$

Since (s_{mn}) is $(\overline{N}, p, *)$ and $(\overline{N}, *, q)$ summable to $o \in X$, relative to $\tau \in X$, the differences $(\sigma_{mn}^{10} - \sigma_{mj}^{10})$ and $(\sigma_{mn}^{01} - \sigma_{in}^{01})$ are P-convergent to o, relative to $\tau \in X$, as $m, n \to \infty$. Hence, considering these situations, we attain from (13)-(15) that

$$\begin{aligned} t_{mj}^{11} - t_{mn}^{11} &\geq -\left(\frac{\epsilon'}{40}\right)\tau P_m Q_n - \left(\frac{\epsilon'}{40}\right)\tau P_m (Q_{j-1} - Q_n) - \left(\frac{\epsilon'}{40}\right)\tau P_m (Q_j - Q_n) - \left(\frac{\epsilon'}{40}\right)\tau P_m Q_j \\ &\geq -\left(\frac{\epsilon'}{40}\right)\tau P_m Q_n - \left(\frac{\epsilon'}{40}\right)\tau P_m (Q_j - Q_n) - \left(\frac{\epsilon'}{40}\right)\tau P_m (Q_j - Q_n) - \left(\frac{\epsilon'}{40}\right)\tau P_m Q_j \\ &= -\left(\frac{\epsilon'}{40}\right)\tau P_m Q_n - 2\left(\frac{\epsilon'}{40}\right)\tau P_m (Q_j - Q_n) - \left(\frac{\epsilon'}{40}\right)\tau P_m Q_j \\ &= -\left(\frac{\epsilon'}{40}\right)\tau \left(3 P_m Q_j - P_m Q_n\right), \end{aligned} \tag{16}$$

$$t_{in}^{11} - t_{mn}^{11} \geq -\left(\frac{\epsilon'}{40}\right)\tau \left(3 P_i Q_n - P_m Q_n\right), \tag{17}$$

$$\begin{aligned} t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11} &\geq -\left(\frac{\epsilon'}{40}\right)\tau P_m Q_n - \left(\frac{\epsilon'}{40}\right)\tau (P_{i-1} - P_m)(Q_{j-1} - Q_n) \\ &\quad - \left(\frac{\epsilon'}{40}\right)\tau Q_n (P_{i-1} - P_m) - \left(\frac{\epsilon'}{40}\right)\tau P_m (Q_{j-1} - Q_n) \\ &= -\left(\frac{\epsilon'}{40}\right)\tau P_{i-1} Q_{j-1}, \end{aligned} \tag{18}$$

respectively, and by (16)-(18) with conditions (6)

$$\begin{aligned} -\sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{mj}^{11} - t_{mn}^{11}) &\leq \left(\frac{\epsilon'}{40}\right)\tau \left(\sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{3 P_m p_i q_j}{P_i P_{i-1} Q_{j-1}} - \sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{P_m Q_n p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} \right) \\ &\leq \left(\frac{\epsilon'}{40}\right)\tau \left(3\left(\frac{P_{m+p}}{P_m} - 1\right)\left(\frac{Q_{n+r}}{Q_n} - 1\right) - \frac{P_m Q_n (P_{m+p} - P_m)(Q_{n+r} - Q_n)}{P_{m+p} P_{m+p-1} Q_{n+r} Q_{n+r-1}} \right) \\ &\leq \left(\frac{\epsilon'}{40}\right) 3\tau \left(\frac{P_{m+p}}{P_m} - 1\right)\left(\frac{Q_{n+r}}{Q_n} - 1\right), \end{aligned} \tag{19}$$

$$-\sum_{i=m+1}^{m+p} \sum_{j=n+1}^{n+r} \frac{p_i q_j}{P_i P_{i-1} Q_j Q_{j-1}} (t_{in}^{11} - t_{mn}^{11}) \leq \left(\frac{\epsilon'}{40}\right) 3\tau \left(\frac{P_{m+p}}{P_m} - 1\right)\left(\frac{Q_{n+r}}{Q_n} - 1\right), \tag{20}$$

and

$$-\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+r}\frac{p_iq_j}{P_iP_{i-1}Q_jQ_{j-1}}(t_{ij}^{11}-t_{in}^{11}-t_{mj}^{11}+t_{mn}^{11})\le\left(\frac{\epsilon'}{40}\right)\tau\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+r}\frac{p_iq_j}{P_iQ_j}\le\left(\frac{\epsilon'}{40}\right)\tau\left(\frac{P_{m+p}}{P_m}-1\right)\left(\frac{Q_{n+r}}{Q_n}-1\right) \tag{21}$$

for sufficiently large m, n, respectively. From (11) together with (19)-(21), we get

$$\begin{aligned}\gamma_{mn}t_{mn}^{11}&\le\left(\frac{\epsilon'\delta}{40}\right)\tau-\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+r}\frac{p_iq_j}{P_iP_{i-1}Q_jQ_{j-1}}(t_{mj}^{11}-t_{mn}^{11})-\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+r}\frac{p_iq_j}{P_iP_{i-1}Q_jQ_{j-1}}(t_{in}^{11}-t_{mn}^{11})\\&-\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+r}\frac{p_iq_j}{P_iP_{i-1}Q_jQ_{j-1}}(t_{ij}^{11}-t_{in}^{11}-t_{mj}^{11}+t_{mn}^{11})\\&\le\left(\frac{\epsilon'\delta}{40}\right)\tau+\left(\frac{\epsilon'}{40}\right)7\tau\left(\frac{P_{m+p}}{P_m}-1\right)\left(\frac{Q_{n+r}}{Q_n}-1\right)\\&\le\left(\frac{\epsilon'\delta}{40}\right)\tau+\left(\frac{\epsilon'}{40}\right)28\tau\alpha\beta\\&\le\frac{3\epsilon'\tau\alpha\beta}{4}\end{aligned} \tag{22}$$

for sufficiently large m, n. When we simplify γ_{mn}, we obtain that

$$\begin{aligned}\gamma_{mn}=\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+r}\frac{p_iq_j}{P_iP_{i-1}Q_jQ_{j-1}}&=\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+r}\left(\frac{1}{P_{i-1}}-\frac{1}{P_i}\right)\left(\frac{1}{Q_{j-1}}-\frac{1}{Q_j}\right)\\&=\left(\frac{1}{P_m}-\frac{1}{P_{m+p}}\right)\left(\frac{1}{Q_n}-\frac{1}{Q_{n+r}}\right)=\left(\frac{P_{m+p}-P_m}{P_mP_{m+p}}\right)\left(\frac{Q_{n+r}-Q_n}{Q_nQ_{n+r}}\right),\end{aligned}$$

and so,

$$\gamma_{mn}P_mQ_n=\left(1-\frac{P_m}{P_{m+p}}\right)\left(1-\frac{Q_n}{Q_{n+r}}\right)\to\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\beta-1}{\beta}\right)\quad\text{as } m,n\to\infty.$$

Therefore, we reach

$$\frac{t_{mn}^{11}}{P_mQ_n}=\frac{\gamma_{mn}t_{mn}^{11}}{P_mQ_n\gamma_{mn}}\le\frac{3\epsilon'\tau\alpha\beta}{4}\left(\frac{2\alpha}{\alpha-1}\right)\left(\frac{2\beta}{\beta-1}\right)\le\frac{3\epsilon'\tau\zeta}{4}\le\frac{3\epsilon\tau}{4} \tag{23}$$

for sufficiently large m, n. If we consider the double weighted Kronecker identity for (s_{mn}), we find

$$s_{mn}=V_{mn}^{11}(\Delta_{11}s)+\sigma_{mn}^{10}+\sigma_{mn}^{01}-\sigma_{mn}^{11}=\frac{t_{mn}^{11}}{P_mQ_n}+\sigma_{mn}^{10}+\sigma_{mn}^{01}-\sigma_{mn}^{11} \tag{24}$$

for sufficiently large m, n. Since (s_{mn}) is $(\overline{N}, p, q)$, $(\overline{N}, p, *)$ and $(\overline{N}, *, q)$ summable to $o \in X$, relative to $\tau \in X$, the sequences (σ_{mn}^{11}), (σ_{mn}^{10}), and (σ_{mn}^{01}) are P-convergent to o, relative to $\tau \in X$. As a result, we conclude by (23) and (24) that

$$s_{mn}=\frac{t_{mn}^{11}}{P_mQ_n}+\sigma_{mn}^{10}+\sigma_{mn}^{01}-\sigma_{mn}^{11}\le\frac{3\epsilon\tau}{4}+\frac{\epsilon\tau}{12}+\frac{\epsilon\tau}{12}+\frac{\epsilon\tau}{12}=\epsilon\tau$$

for sufficiently large m, n. To indicate that $s_{mn} \ge -\epsilon\tau$ ultimately in m, n we define $p' = [m(1-\delta)]$ and $r' = [n(1-\delta)]$ for $0 < \delta < 1$, and consider

$$\sum_{i=p'}^{m}\sum_{j=r'}^{n}\frac{p_iq_j}{P_iP_{i-1}Q_jQ_{j-1}}t_{ij}^{11}$$

for all $m, n \ge 1$. As a result of the calculations made in parallel with that made in the first part of the proof, we complete the second part of the proof via Lemma 2 and we find $s_{mn} \ge -\epsilon\tau$ for sufficiently large m, n. Therefore, (s_{mn}) is P-convergent to L, relative to $\tau \in X$. ☐

3 Conclusion

In this paper, we extended a Tauberian theorem for the Cesàro summability method due to Maddox [7] and the weighted mean summability method due to Çanak [6] in ordered spaces to the $(\overline{N}, p, q)$, summability method of double sequences. In an ordered linear space $(X, \le)$ over the real numbers, we proved that under $p, q \in SVA_+$, if a double sequence (s_{mn}) is $(\overline{N}, p, q)$, $(\overline{N}, p, *)$ and $(\overline{N}, *, q)$ summable to $L \in X$, relative to a $\tau \in X$ and slowly decreasing in senses $(1, 1)$, $(1, 0)$, and $(0, 1)$, relative to $\tau \in X$, then it is P-convergent to L, relative to $\tau \in X$.

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