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# Preface

It is my great pleasure and honor to welcome you at the 11th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2022). I am pleased to acknowledge the official sponsorship of the conference by Yildiz Technical university, Society of Geometers, and Turkish World Mathematical Society.

Established since 2012, the series of IECMSA features the latest developments in the field of mathematics and applications. The previous conferences were held as follows: IECMSA-2012, Prishtine, Kosovo, IECMSA-2013, Sarajevo, Bosnia and Herzegovina, IECMSA-2014, Vienna, Austria, IECMSA-2015, Athens, Greece, IECMSA-2016, Belgrade, Serbia, IECMSA-2017, Budapest, Hungary, IECMSA-2018, Kyiv, Ukraine, IECMSA-2019, Baku, Azerbaijan, IECMSA-2020 (online), Skopje, North Macedonia, IECMSA-2021, Sakarya. These conferences gathered a large number of international world-renowned participants.

I would like to thank the members of the scientific committees. They have worked very hard in reviewing process and making valuable suggestions for the authors to improve their work. I also would like to express our gratitude to the external reviewers, for providing extra help in the review process, and the authors for contributing their research result to the conference. At IECMSA-2022, the scientific committee members and the external reviewers accepted 132 virtual presentations. Despite the effects of coronavirus, 167 participants are attending the conference from 25 different countries. The scientific program of the conference features 6 keynote talks, followed by 171 contributed presentations in three parallel sessions.

The conference program represents the efforts of many people. I would like to express my gratitude to all members of the organizing committee, sponsors and, honorary committee for their continued support to the IECMSA. I also thank the invited speakers for presenting their talks on current researches.

Also, the success of IECMSA depends on the effort and talent of researchers that have shared their studies on a variety of topics in mathematics and its applications. So, I would like to sincerely thank all participants of IECMSA-2022 for contributing to this great meeting

Wish you all health and safety during this difficult time
Prof. Dr. Murat TOSUN
Chairman
On behalf of the Organizing Committee

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# On Gould-Hopper Based Degenerate Truncated Frobenius-Euler Polynomials

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**Abstract:** In this study, we consider the truncated degenerate Frobenius-Euler polynomials based on the Gould-Hopper polynomials and examine diverse properties and formulas covering addition formulas, correlations and derivation property. Then, we derive some interesting implicit summation formulas and symmetric identities. Moreover, we define Gould-Hopper based truncated degenerate Frobenius-Euler polynomials of order $r$ and give some of their properties and relations.

**Keywords:** Degenerate exponential function, Frobenius-Euler polynomials, Gould-Hopper polynomials, Truncated exponential function.

## 1 Introduction

Along this paper, the usual notations $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$, are referred to the set of all natural numbers, the set of all non-negative integers, the set of all real numbers and the set of all complex numbers, respectively.

The truncated form of the exponential polynomials $e_n(z)$ are the first $(n+1)$ terms of the Taylor series for $e^z$ at $z=0$ [2], namely

$$e_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}. \tag{1}$$

One can see [2] to get the detailed information about $e_n(z)$.

For $\lambda \in \mathbb{C}$, the $\lambda$-falling factorial $(z)_{n,\lambda}$ is defined by $(z)_{n,\lambda} = z(z-\lambda)(z-2\lambda)\cdots(z-(n-1)\lambda)$ for $n \in \mathbb{N}$ with $(z)_{0,\lambda} = 1$, [1], [3]-[5], [8], [10]. In the case $\lambda = 1$, the $\lambda$-falling factorial becomes to the usual falling factorial given by $(z)_{n,1} := (z)_n = z(z-1)\cdots(z-n+1)$ with $(z)_{0,1} = 1$.

Let $\lambda \in \mathbb{R}/\{0\}$. The degenerate form of the exponential function $e_\lambda^z(z)$ is defined by [1], [3]-[5], [8], [10]

$$e_\lambda^\omega(z) = (1+\lambda z)^{\frac{\omega}{\lambda}} \text{ and } e_\lambda^1(z) := e_\lambda(z). \tag{2}$$

We note that $\lim_{\lambda\to 0} e_\lambda^\omega(z) = e^{\omega z}$. From (2), we attain

$$e_\lambda^\omega(z) = \sum_{n=0}^{\infty} (\omega)_{n,\lambda} \frac{z^n}{n!}. \tag{3}$$

The degenerate truncated form of the exponential polynomials (also called the $Detr$-exponential polynomials) are considereed as the first $(n+1)$ terms of the Mac Laurin series expansion of $e_\lambda(z)$ in (3) [3]:

$$e_{n,\lambda}(z) = \sum_{k=0}^{n} (1)_{k,\lambda} \frac{z^k}{k!}. \tag{4}$$

Also, when $\lambda \to 0$, the polynomials $e_{n,\lambda}(z)\, in$ (4) become the polynomials $e_n(z)$ in (1). To get more detailed information about the $Detr$-exponential polynomials and their properties, see [3].

The Stirling numbers $S_2(n,k)$ and polynomials $S_2(n,k:\omega)$ of the second kind are provided as follows [1], [3]-[5], [9]:

$$\sum_{n=0}^{\infty} S_2(n,k)\frac{z^n}{n!}=\frac{(e^z-1)^k}{k!} \text{ and } \sum_{n=0}^{\infty} S_2(n,k:\omega)\frac{z^n}{n!}=\frac{(e^z-1)^k}{k!}e^{z\omega}. \tag{5}$$

The degenerate form of the Stirling polynomials of the second kind are given below [3]-[5], [9]:

$$\sum_{n=0}^{\infty} S_{2,\lambda}(n,k:\omega)\frac{z^n}{n!}=\frac{(e_\lambda(z)-1)^k}{k!}e_\lambda^{\omega}(z). \tag{6}$$

The degenerate truncated form of the Stirling polynomials of the second kind are considered as follows [3]:

$$\sum_{n=0}^{\infty} S_{2,m;\lambda}(n,k:\omega)\frac{z^n}{n!}=\frac{\left(e_\lambda(z)-1-e_{m-1,\lambda}(z)\right)^k}{k!}e_\lambda^{\omega}(z). \tag{7}$$

The Gould-Hopper polynomials $H_n^{(j)}(\omega,\theta)$ are defined by (see [4], [11]):

$$\sum_{n=0}^{\infty} H_n^{(j)}(\omega,\theta)\frac{z^n}{n!}=e^{\omega z+\theta z^j}, \tag{8}$$

where $j\in\mathbb{N}$ with $j\geq 2$. Choosing $j=1$ in (8), the polynomials $H_n^{(j)}(\omega,\theta)$ reduce to the Newton binomial formula. Moreover, taking $j=2$ in (8), the polynomials $H_n^{(j)}(\omega,\theta)$ become the Hermite polynomials $H_n(\omega,\theta)$ [11]. The two polynomials $H_n^{(j)}(\omega,\theta)$ and $H_n(\omega,\theta)$ have been utilized to generalize multifarious special polynomials including Bell, Bernoulli, Genocchi and Euler polynomials (see [4], [11]).

Let $j\in\mathbb{N}$ and $\lambda\in\mathbb{R}\backslash\{0\}$. The degenerate Gould-Hopper polynomials $H_{n,\lambda}^{(j)}(\omega,\theta)$ are defined below [4]:

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(j)}(\omega,\theta)\frac{z^n}{n!}=e_\lambda^{x}(z)\,e_\lambda^{y}\left(z^j\right). \tag{9}$$

Several applications and properties of the polynomials $H_{n,\lambda}^{(j)}(\omega,\theta)$ are investigated in [4], [11].

## 2 The Gould-Hopper based degenerate truncated Frobenius-Euler polynomials

In this chapter, we consider the Gould-Hopper based degenerate truncated Frobenius-Euler polynomials and examine diverse formulas and correlations such as implicit summation formulas, derivation rule and symmetric identities.

Let $u(\neq 1)\in\mathbb{C}$ is an algebraic number. The classical Frobenius-Euler $\Omega_n(u,x)$ polynomials are given as follows [5], [7], [8], [13].

$$\sum_{n=0}^{\infty} \Omega_n(u:x)\frac{t^n}{n!}=\frac{1-u}{e^t-u}e^{xt}.$$

The usual degenerate Frobenius-Euler $\Omega_{n,\lambda}(u,x)$ polynomials are defined as follows [5], [8]:

$$\sum_{n=0}^{\infty} \Omega_{n,\lambda}(u,x)\frac{t^n}{n!}=\frac{1-u}{e_\lambda(t)-u}e_\lambda^{x}(t).$$

Several degenerate forms of the Frobenius-Euler polynomials have been recently studied and investigated by many mathematicians, [4], [7], [8], [13] and also cited references therein.

Let $x$ be an independent variable. The degenerate truncated Frobenius-Euler polynomials are defined by the following exponential generating function [5]:

$$\sum_{n=0}^{\infty} \Omega_{m,n,\lambda}(u,x)\frac{t^n}{n!}=\frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}e_\lambda^{x}(t). \tag{10}$$

When $x=0$ in (10), the $Detr$-Frobenius-Euler polynomials $\Omega_{m,n,\lambda}(u,x)$ reduce to the corresponding numbers called the $Detr$-Frobenius-Euler numbers denoted by $\Omega_{m,n,\lambda}(u)$:

$$\sum_{n=0}^{\infty} \Omega_{m,n,\lambda}(u)\frac{t^n}{n!}=\frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}. \tag{11}$$

The polynomials $\Omega_{m,n,\lambda}(x)$ in conjuction with the several identities and formulas are analyzed in [5] with details.

We now introduce the Gould-Hopper based degenerate truncated Frobenius-Euler polynomials as follows.

 

**Definition 1.** *Let $x$ and $y$ be two independent variables and $j \in \mathbb{N}_0$. The Gould-Hopper based degenerate truncated Frobenius-Euler polynomials are defined below:*

$$\sum_{n=0}^{\infty} {}_H\Omega_{m,n,\lambda}^{(j)}(x,y)\frac{z^n}{n!} = \frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)} e_\lambda^x(z)\, e_\lambda^y\left(z^j\right). \tag{12}$$

We choose to call the Gould-Hopper based $Detr$-Frobenius-Euler polynomials instead of the Gould-Hopper based degenerate truncated Frobenius-Euler polynomials.

**Remark 1.** *When $x = 0$ in Definition 1, the Gould-Hopper based $Detr$-Frobenius-Euler polynomials $\Omega_{m,n,\lambda}(x)$ reduce to the following polynomials which is also new extension of the $Detr$-Frobenius-Euler polynomials:*

$$\sum_{n=0}^{\infty} \Omega_{m,n,\lambda}^{(j)}(y)\frac{z^n}{n!} = \frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)} e_\lambda^y\left(z^j\right). \tag{13}$$

**Remark 2.** *Taking $x = y = 0$ in Definition 1, the polynomials ${}_H\Omega_{m,n,\lambda}^{(j)}(x,y)$ reduce to the degenerate truncated Frobenius-Euler numbers in (11).*

**Theorem 1.** *The following summation formulae holds for $n \in \mathbb{N}_0$:*

$${}_H\Omega_{m,n,\lambda}^{(j)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}\ {}_H\Omega_{m,n-k,\lambda}^{(j)}(y) \tag{14}$$

*and*

$${}_H\Omega_{m,n,\lambda}^{(j)}(x,y) = \sum_{k=0}^{n}\binom{n}{k} H_{n-k,\lambda}^{(j)}(x,y)\,\Omega_{m,k,\lambda}.$$

*Proof:* By Definition 1 and utilizing the (13) and (11), we attain

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_H\Omega_{m,n,\lambda}^{(j)}(x,y)\frac{z^n}{n!} &= \frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)} e_\lambda^x(z)\, e_\lambda^y\left(z^j\right) \\
&= \sum_{n=0}^{\infty} {}_H\Omega_{m,n,\lambda}^{(j)}(y)\frac{z^n}{n!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}\ {}_H\Omega_{m,n-k,\lambda}^{(j)}(y)\right)\frac{z^n}{n!}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_H\Omega_{m,n,\lambda}^{(j)}(x,y)\frac{z^n}{n!} &= \frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)} e_\lambda^x(z)\, e_\lambda^y\left(z^j\right) \\
&= \sum_{n=0}^{\infty} H_{n,\lambda}^{(j)}(x,y)\frac{z^n}{n!}\sum_{n=0}^{\infty}\Omega_{m,n,\lambda}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} H_{n-k,\lambda}^{(j)}(x,y)\,\Omega_{m,k,\lambda}\right)\frac{z^n}{n!}
\end{aligned}$$

which complete the proof. □

We give the following lemma.

**Lemma 1.** *[11] The following series manipulation is valid:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/j\rfloor} A(k,n-jk), \tag{15}$$

*where $\lfloor\cdot\rfloor$ is the Gauss symbol, and shows the maximum integer that does not exceed the number in the square brackets.*

We give the following theorem.

**Theorem 2.** *We have*

$$ {}_H\Omega^{(j)}_{m,n,\lambda}(x,y) = n! \sum_{k=0}^{\lfloor n/j \rfloor} \frac{(y)_{n-jk,\lambda}}{k!\,(n-jk)!} \Omega_{m,k,\lambda}(x). \tag{16} $$

*Proof:* By applying (15) and using the following equality

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\Omega^{(j)}_{m,n,\lambda}(x,y) \frac{z^n}{n!} &= \frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)} e^x_\lambda(z)\, e^y_\lambda\left(z^j\right) \\ &= \left(\sum_{n=0}^{\infty} \Omega_{m,n,\lambda}(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y)_{n,\lambda}\frac{z^{jn}}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(n!\sum_{k=0}^{\lfloor n/j\rfloor}\frac{(y)_{n-jk,\lambda}}{k!\,(n-jk)!}\Omega_{m,k,\lambda}(x)\right)\frac{z^n}{n!}, \end{aligned} $$

which is the claimed result (16). □

**Theorem 3.** *We have*

$$ {}_H\Omega^{(j)}_{m,n,\lambda}(x_1+x_2,y_1+y_2) = \sum_{k=0}^{n}\binom{n}{k} {}_H\Omega^{(j)}_{m,k,\lambda}(x_1,y_1)\, H^{(j)}_{n-k,\lambda}(x_2,y_2). \tag{17} $$

*Proof:* Using the following equality

$$ \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)} e^{x_1+x_2}_\lambda(z)\, e^{y_1+y_2}_\lambda\left(z^j\right) = \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)} e^{x_1}_\lambda(z)\, e^{y_1}_\lambda\left(z^j\right) e^{x_2}_\lambda(z)\, e^{y_2}_\lambda\left(z^j\right), $$

the proof is similar to Theorem 1. We, therefore, choose to omit details involved. □

**Theorem 4.** *We have*

$$ \frac{\partial}{\partial x}\, {}_H\Omega^{(j)}_{m,n,\lambda}(x,y) = n! \sum_{s=1}^{\infty} {}_H\Omega^{(j)}_{m,n-s,\lambda}(x,y)\frac{(-1)^{s+1}}{(n-s)!s}\lambda^{s-1}. \tag{18} $$

*Proof:* By applying the operator $\frac{\partial}{\partial x}$ to both sides of (12), we then derive

$$ \begin{aligned} \sum_{n=0}^{\infty}\frac{\partial}{\partial x}\, {}_H\Omega^{(j)}_{m,n,\lambda}(x,y)\frac{z^n}{n!} &= \frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)} e^y_\lambda\left(z^j\right)\frac{\partial}{\partial x}(1+\lambda z)^{\frac{x}{\lambda}} \\ &= \frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)} e^y_\lambda\left(z^j\right)(1+\lambda z)^{\frac{x}{\lambda}}\ln(1+\lambda z)^{\frac{1}{\lambda}} \\ &= \sum_{n=0}^{\infty} {}_H\Omega^{(j)}_{m,n,\lambda}(x,y)\frac{z^n}{n!}\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}\lambda^{s-1}z^s \\ &= \sum_{n=0}^{\infty}\sum_{s=1}^{\infty} {}_H\Omega^{(j)}_{m,n,\lambda}(x,y)\frac{(-1)^{s+1}}{s}\lambda^{s-1}\frac{z^{n+s}}{n!} \end{aligned} $$

which means the assertion in (18). □

**Theorem 5.** *For $n, m \in \mathbb{N}_0$, we have*

$$ \begin{aligned} {}_H\Omega^{(j)}_{m+1,n,\lambda}(x,y) &= n\frac{1-m\lambda}{m+1}\, {}_H\Omega^{(j)}_{m,n-1,\lambda}(x,y) \\ &\quad + \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\Omega_{m,k;\lambda}\, {}_H\Omega^{(j)}_{m+1,n-k,\lambda}(x,y). \end{aligned} \tag{19} $$

*Proof:* Utilizing the following equality

$$\begin{aligned}(1)_{m+1,\lambda}\, 2\frac{z^{m+1}}{(m+1)!}e_{\lambda}^{x}(z)\, e_{\lambda}^{y}\left(z^{j}\right) &= \left(e_{\lambda}(z)+1-e_{m,\lambda}(z)\right)\sum_{n=0}^{\infty}{}_{H}\Omega_{m+1,n,\lambda}^{(j)}(x,y)\frac{z^{n}}{n!}\\ &= \left(e_{\lambda}(z)+1-e_{m-1,\lambda}(z)\right)\sum_{n=0}^{\infty}{}_{H}\Omega_{m+1,n,\lambda}^{(j)}(x,y)\frac{z^{n}}{n!}\\ &\quad -(1)_{m,\lambda}\frac{z^{m}}{m!}\sum_{n=0}^{\infty}{}_{H}\Omega_{m+1,n,\lambda}^{(j)}(x,y)\frac{z^{n}}{n!},\end{aligned}$$

the proof is similar to Theorem 1. We, therefore, choose to omit details involved. □

**Theorem 6.** *For $n, m \in \mathbb{N}_0$, we have*

$$2\frac{(1)_{m,\lambda}}{(m)!}H_{n,\lambda}^{(j)}(x,y)=\sum_{k=0}^{n}n!\,(1)_{k+m,\lambda}\frac{{}_{H}\Omega_{m,n-k,\lambda}^{(j)}(x,y)}{(k+m)!\,(n-k)!}-n!\frac{{}_{H}\Omega_{m,n+m,\lambda}^{(j)}(x,y)}{(n+m)!}. \tag{20}$$

*Proof:* By Definition 1, we have

$$\begin{aligned}2\frac{z^{m}}{m!}(1)_{m,\lambda}\, e_{\lambda}^{x}(z)\, e_{\lambda}^{y}\left(z^{j}\right) &= \left(e_{\lambda}(z)+1-e_{m-1,\lambda}(z)\right)\sum_{n=0}^{\infty}{}_{H}\Omega_{m,n,\lambda}^{(j)}(x,y)\frac{z^{n}}{n!}\\ &= \sum_{n=m}^{\infty}(1)_{n,\lambda}\frac{z^{n}}{n!}\sum_{n=0}^{\infty}{}_{H}\Omega_{m,n,\lambda}^{(j)}(x,y)\frac{z^{n}}{n!}+\sum_{n=0}^{\infty}{}_{H}\Omega_{m,n,\lambda}^{(j)}(x,y)\frac{z^{n}}{n!},\end{aligned}$$

which yields the asserted result (20). □

In recent years, many mathematicians have been studied special polynomials to acquire some of their symmetric identities and implicit summation formulas, *cf.* [5,15] and see also each of the references cited therein. We now derive some the mentioned formulas and identities for the polynomials ${}_{H}\Omega_{m,n,\lambda}^{(j)}(x,y)$.

**Theorem 7.** *For $n, m \in \mathbb{N}_0$, we have*

$${}_{H}\Omega_{m,n,\lambda}^{(j)}(x,y)=\sum_{l=0}^{n}\sum_{k=0}^{n}\binom{n}{k}\Omega_{m,n-k,\lambda}^{(j)}(y)\, S_{2;\lambda}(k,l:-l)\,(x)^{(l)}, \tag{21}$$

*where $(x)^{(l)}=x(x+1)(x+2)\cdots(x+(l-1))$ for $l\in\mathbb{N}$ with $(x)^{(l)}:=1$.*

*Proof:* From Definition 1 and utilizing (6) and (13), we acquire

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{H}\Omega_{m,n,\lambda}^{(j)}(x,y)\frac{z^{n}}{n!} &= \frac{2\frac{z^{m}}{m!}(1)_{m,\lambda}}{e_{\lambda}(z)+1-e_{m-1,\lambda}(z)}e_{\lambda}^{y}\left(z^{j}\right)\left(e_{\lambda}^{-1}(z)-1+1\right)^{x}\\ &= \frac{2\frac{z^{m}}{m!}(1)_{m,\lambda}}{e_{\lambda}(z)+1-e_{m-1,\lambda}(z)}e_{\lambda}^{y}\left(z^{j}\right)\sum_{l=0}^{\infty}\binom{x+l-1}{l}\left(1-e_{\lambda}^{-1}(z)\right)^{l}\\ &= \frac{2\frac{z^{m}}{m!}(1)_{m,\lambda}}{e_{\lambda}(z)+1-e_{m-1,\lambda}(z)}e_{\lambda}^{y}\left(z^{j}\right)\sum_{l=0}^{\infty}\binom{x+l-1}{l}\frac{\left(e_{\lambda}(z)-1\right)^{l}}{l!}e_{\lambda}^{-l}(z)\,l!\\ &= \sum_{l=0}^{\infty}(x)^{(l)}\sum_{n=0}^{\infty}\Omega_{m,n,\lambda}^{(j)}(y)\frac{z^{n}}{n!}\sum_{n=0}^{\infty}S_{2;\lambda}(n,l:-l)\frac{z^{n}}{n!}\\ &= \sum_{l=0}^{\infty}(x)^{(l)}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\Omega_{m,n-k,\lambda}^{(j)}(y)\, S_{2;\lambda}(k,l:-l)\right)\frac{z^{n}}{n!},\end{aligned}$$

which means the assertion (21). □

Note that [11]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^{N}}{N!}=\sum_{n,s=0}^{\infty}f(n+s)\frac{x^{n}}{n!}\frac{y^{s}}{s!}. \tag{22}$$

We give the following theorem.

**Theorem 8.** *We have*

$$ {}_H\Omega^{(j)}_{m,k+l,\lambda}(x,y) = \sum_{n,s=0}^{k,l} \binom{k}{n}\binom{l}{s} (\mu - x)_{n+s,\lambda} \; {}_H\Omega^{(j)}_{m,k+l-n-s,\lambda}(\mu,y). \tag{23} $$

*Proof:* Taking $z$ by $z + \omega$ in (12), it yields

$$ \frac{2\frac{(z+\omega)^m}{m!}(1)_{m,\lambda}}{e_\lambda(z+\omega)+1-e_{m-1,\lambda}(z+\omega)} e^y_\lambda\left((z+\omega)^j\right) = e^\mu_\lambda(z+\omega)\sum_{k,l=0}^{\infty} {}_H\Omega^{(j)}_{m,k+l,\lambda}(\mu,y)\frac{z^k}{k!}\frac{\omega^l}{l!} $$

and similarly we acquire

$$ \frac{2\frac{(z+\omega)^m}{m!}(1)_{m,\lambda}}{e_\lambda(z+\omega)+1-e_{m-1,\lambda}(z+\omega)} e^y_\lambda\left((z+\omega)^j\right) = e^x_\lambda(z+\omega)\sum_{k,l=0}^{\infty} {}_H\Omega^{(j)}_{m,k+l,\lambda}(x,y)\frac{z^k}{k!}\frac{\omega^l}{l!}. $$

By the last two equalities, we write

$$ \begin{aligned} \sum_{k,l=0}^{\infty} {}_H\Omega^{(j)}_{m,k+l,\lambda}(x,y)\frac{z^k}{k!}\frac{\omega^l}{l!} &= e^{\mu-x}_\lambda(z+\omega)\sum_{k,l=0}^{\infty} {}_H\Omega^{(j)}_{m,k+l,\lambda}(\mu,y)\frac{z^k}{k!}\frac{\omega^l}{l!} \\ &= \sum_{n,s=0}^{\infty}(\mu-x)_{n+s,\lambda}\frac{z^n}{n!}\frac{\omega^m}{s!}\sum_{k,l=0}^{\infty} {}_H\Omega^{(j)}_{m,k+l,\lambda}(\mu,y)\frac{z^k}{k!}\frac{\omega^l}{l!}. \end{aligned} $$

By using (22), we acquire

$$ \sum_{k,l=0}^{\infty} {}_H\Omega^{(j)}_{m,k+l,\lambda}(x,y)\frac{z^k}{k!}\frac{\omega^l}{l!} = \sum_{k,l=0}^{\infty}\sum_{n,s=0}^{k,l}\frac{(\mu-x)_{n+s,\lambda}\; {}_H\Omega^{(j)}_{m,k+l-n-s,\lambda}(\mu,y)}{n!s!(k-l)!(l-s)!} z^k\omega^l, $$

which means the assertion (23). □

**Theorem 9.** *The following symmetric identity holds for $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$:*

$$ \sum_{k=0}^{n}\binom{n}{k} {}_H\Omega^{(j)}_{m,n-k,\lambda}(bx,y)\; {}_H\Omega^{(j)}_{m,k,\lambda}(ax,y)\,a^{n-k}b^k = \sum_{k=0}^{n}\binom{n}{k} {}_H\Omega^{(j)}_{m,n-k,\lambda}(ax,y)\; {}_H\Omega^{(j)}_{m,k,\lambda}(bx,y)\,b^{n-k}a^k. \tag{24} $$

*Proof:* Let

$$ \Upsilon = \frac{(az)^m(bz)^m\left(2\frac{(1)_{m,\lambda}}{m!}\right)^2 e^{bx}_\lambda(az)\,e^{ax}_\lambda(bz)\,e^y_\lambda\left(a^jz^j\right)e^y_\lambda\left(b^jz^j\right)}{\left(e_\lambda(az)+1-e_{m-1,\lambda}(az)\right)\left(e_\lambda(bz)+1-e_{m-1,\lambda}(bz)\right)} $$

Then, thanks to $\Upsilon$ being symmetric in $a$ and $b$, we have two expansions of $\Upsilon$ as follows:

$$ \begin{aligned} \Upsilon &= \sum_{n=0}^{\infty} {}_H\Omega^{(j)}_{m,n,\lambda}(bx,y)\frac{(az)^n}{n!}\sum_{n=0}^{\infty} {}_H\Omega^{(j)}_{m,n,\lambda}(ax,y)\frac{(bz)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_H\Omega^{(j)}_{m,n-k,\lambda}(bx,y)\; {}_H\Omega^{(j)}_{m,k,\lambda}(ax,y)\,a^{n-k}b^k\frac{z^n}{n!} \end{aligned} $$

and similarly

$$ \Upsilon = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_H\Omega^{(j)}_{m,n-k,\lambda}(ax,y)\; {}_H\Omega^{(j)}_{m,k,\lambda}(bx,y)\,b^{n-k}a^k\frac{z^n}{n!}, $$

which means the assertion (24).

□

## 3 Further remarks

Now, we introduce the Gould-Hopper based $Detr$-Frobenius-Euler polynomials $\Omega_{m,n,\lambda}^{(r)}(x)$ of order $r$ as follows:

$$\sum_{n=0}^{\infty} {}_H\Omega_{m,n,\lambda}^{(j,r)}(x,y)\frac{z^n}{n!} = \left(\frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)}\right)^r e_\lambda^x(z)\, e_\lambda^y\left(z^j\right). \tag{25}$$

We note that ${}_HE_{m,n,\lambda}^{(j,1)}(x,y) := {}_HE_{m,n,\lambda}^{(j)}(x,y)$. Also, upon letting $x=y=0$, the polynomials in (25) reduces to the Gould-Hopper based $Detr$-Frobenius-Euler numbers of order $r$ below:

$$\sum_{n=0}^{\infty} \Omega_{m,n,\lambda}^{(r)}\frac{z^n}{n!} = \left(\frac{2\frac{z^m}{m!}(1)_{m,\lambda}}{e_\lambda(z)+1-e_{m-1,\lambda}(z)}\right)^r.$$

We first give the following summation formula.

**Theorem 10.** *We have*

$${}_H\Omega_{m,n,\lambda}^{(j,r)}(x,y) = \sum_{l=0}^{n}\binom{n}{l}\Omega_{m,n-l,\lambda}^{(r)}H_{l,\lambda}^{(j)}(x,y).$$

*Proof:* By using (9) and (25), the proof is similar to Theorem 1. We, therefore, choose to omit details involved. □

Addition property of the Gould-Hopper based $Detr$-Frobenius-Euler polynomials of order $r$ is given below.

**Theorem 11.** *We have*

$${}_H\Omega_{m,n,\lambda}^{(j,r_1+r_2)}(x_1+x_2,y_1+y_2) = \sum_{u=0}^{n}\binom{n}{u}\, {}_H\Omega_{m,u,\lambda}^{(j,r_1)}(x_1,y_1)\ {}_H\Omega_{m,n-u,\lambda}^{(j,r_2)}(x_2,y_2).$$

*Proof:* By utlizing (9) and (25), the proof is similar to Theorem 1. We, thus, choose to omit details involved. □

**Theorem 12.** *We have*

$$\frac{\partial}{\partial x}\, {}_H\Omega_{m,n,\lambda}^{(j,r)}(x,y) = n!\sum_{s=1}^{\infty} {}_H\Omega_{m,n-s,\lambda}^{(j,r)}(x,y)\frac{(-1)^{s+1}}{(n-s)!s}\lambda^{s-1}. \tag{26}$$

*Proof:* By (25), the proof is similar to Theorem 4. We, hence, choose to omit details involved. □

## 4 Conclusion

In this study, we have introduced the Gould-Hopper based truncated degenerate Frobenius-Euler polynomials and have examined diverse properties and formulas covering addition formulas, derivation rule and relationships with the Gould-Hopper polynomials and the degenerate Stirling numbers of the second. Then, we have derived some interesting symmetric relations and implicit summation identities. Moreover, we have defined Gould-Hopper based truncated degenerate Frobenius-Euler polynomials of order $r$ and have given some of their properties and relations. For future directions, we will consider that the polynomials introduced in this paper can be examined within the context of monomiality principle and umbral calculus to have alternative ways of deriving our results.

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# Applications of Infinite Series and Trigonometric Fourier Series

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**Abstract:** In the present paper, a general absolute matrix summability method is studied. Two general theorems on $\varphi - \mid A, \beta; \delta \mid_k$ summability of infinite series and Fourier series are obtained. Also, two known results on absolute Riesz summability are deduced.
**Keywords:** Absolute matrix summability, Fourier series, Infinite series, Summability factors.

## 1 Introduction

Let $\sum a_n$ be an infinite series with partial sums $(s_n)$. Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where $A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$ Further, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad \text{and} \quad \hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ...$$

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{1}$$

The series $\sum a_n$ is said to be summable $\varphi - \mid A, \beta; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$ and $\beta$ is a real number, if (see [1])

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k + k - 1)} \mid A_n(s) - A_{n-1}(s) \mid^k < \infty. \tag{2}$$

Let $(p_n)$ be a sequence of positive numbers such that $P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-k} = p_{-k} = 0, \quad k \geq 1)$.
By taking $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in (2), we get $|\bar{N}, p_n|_k$ summability method (see [2]). For any sequence $(\lambda_n)$, it should be noted that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, $\Delta^0 \lambda_n = \lambda_n$ and $\Delta^k \lambda_n = \Delta\Delta^{k-1}\lambda_n$ for $k = 1, 2, ...$ (see [3]) also $(t_n)$ is the $n$-th $(C, 1)$ mean of the sequence $(na_n)$, i.e., $t_n = \frac{1}{n+1} \sum_{v=1}^{n} va_v$. If we write $X_n = \sum_{v=0}^{n} \frac{p_v}{P_v}$, then $(X_n)$ is a positive increasing sequence tending to infinity as $n \to \infty$. Additionally, a function $f$ defined on an interval $[a, b]$ is said to be of bounded variation if $sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\} < \infty$ for every partition of $[a, b]$ and the set of functions of bounded variation on $(0, \pi)$ is denoted by $\mathcal{BV}(0, \pi)$.

## 2 Main result

The main object of this paper is to prove following theorem on matrix summability of the series $\sum a_n \lambda_n$ . Further different applications on infinite series, we can refer the papers [4]-[12].

**Theorem 1.** *Let $\varphi_n p_n = O(P_n)$ and let the conditions*

$$\lambda_m = O(1) \quad as \quad m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} nX_n \left| \Delta^2 \lambda_n \right| = O(1) \quad as \quad m \to \infty \tag{4}$$

 

*be satisfied. Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1,\ n = 0, 1, ..., \qquad a_{n-1,v} \geq a_{nv},\ for\ n \geq v+1, \tag{5}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{6}$$

$$\mid \hat{a}_{n,v+1} \mid = O(v \mid \Delta_v(\hat{a}_{nv}) \mid). \tag{7}$$

*If the conditions*

$$\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v \hat{a}_{nv}| = O\left(\varphi_v^{\beta(\delta k+k-1)-k}\right) \quad as \quad m \to \infty$$

*are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - \mid A, \beta; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$ and $-\beta(\delta k + k - 1) + k > 0$.*

We need the following lemma for proof.

**Lemma 1.** ([13]) *Under the conditions of Theorem* 1*, we have*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty, \tag{9}$$

$$X_n|\lambda_n| = O(1) \quad as \quad n \to \infty.$$

## 3 Proof of Theorem 1

Let $(I_n)$ denotes $A$-transform of the series $\sum a_n \lambda_n$. Then, from (1), we have

$$\bar{\Delta} I_n \quad = \quad \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v.$$

Applying Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v \left(\hat{a}_{nv}\right) \lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{t_v}{v} + \frac{n+1}{n} a_{nn} \lambda_n t_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 1, we need to prove $\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k+k-1)} \mid I_{n,r} \mid^k < \infty$ for $r = 1, 2, 3, 4$. First, applying Hölder's inequality with indices $k$ and $k'$, where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \\
&\times \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\right)^{k-1}.
\end{aligned}$$

Then, by using $\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1}(a_{n-1,v} - a_{nv}) \leq a_{nn}$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} \, |\lambda_v| \, \frac{|t_v|^k}{X_v^{k-1}}.
\end{aligned}$$

Also, by applying Abel's transformation, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid I_{n,1} \mid^k &= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \varphi_r^{\beta(\delta k+k-1)-k} \frac{|t_r|^k}{X_r^{k-1}} \\
&\quad + O(1)|\lambda_m| \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} \frac{|t_v|^k}{X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)|\lambda_m| X_m = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

By using (7) and Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid I_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left( \sum_{v=1}^{n-1} v \, |\Delta_v(\hat{a}_{nv})| \, |\Delta\lambda_v| |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} (v|\Delta\lambda_v|)^k \, |\Delta_v(\hat{a}_{nv})| \, |t_v|^k \\
&\quad \times \left( \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} (v|\Delta\lambda_v|)^{k-1} \, (v|\Delta\lambda_v|) \, |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} v|\Delta\lambda_v| \frac{|t_v|^k}{X_v^{k-1}}.
\end{aligned}$$

Then, applying Abel's transformation and using the conditions (4), (9) and (8), we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid I_{n,2} \mid^k &= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \varphi_r^{\beta(\delta k+k-1)-k} \frac{|t_r|^k}{X_r^{k-1}} \\
&\quad + O(1)m|\Delta\lambda_m| \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} \frac{|t_v|^k}{X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) X_v + O(1)m|\Delta\lambda_m| X_m \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta^2\lambda_v| X_v + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)m|\Delta\lambda_m| X_m = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Again by using (7), we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |I_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_{v+1}|^k |t_v|^k \\
&\quad \times \left( \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} |\lambda_{v+1}| \frac{|t_v|^k}{X_v^{k-1}} = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

as in $I_{n,1}$.

Finally, we have

$$\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \mid I_{n,4} \mid^k = O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k} |\lambda_n| \frac{|t_n|^k}{X_n^{k-1}} = O(1) \quad as \quad m \to \infty$$

as in $I_{n,1}$. This completes the proof of Theorem 1.

## 4 An application to Fourier series

There are some papers on the field of summability of Fourier series [14]-[21]. Let $f$ be a periodic function with period $2\pi$ and Lebesgue integrable over $(-\pi, \pi)$. The trigonometric Fourier series of $f$ is defined as

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad \texttt{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

Write

$$\phi(t) = \frac{1}{2} \{ f(x+t) + f(x-t) \} \quad \texttt{and} \quad \phi_1(t) = \frac{1}{t} \int_0^t \phi(u) du.$$

**Theorem 2.** *Let $A = (a_{nv})$ be a positive normal matrix which satisfies the conditions (5)-(7). If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, and the sequences $(p_n)$, $(\lambda_n)$, $(\varphi_n)$ and $(X_n)$ satisfy the all conditions of Theorem 1, then the series $\sum A_n(x) \lambda_n$ is summable $\varphi - \mid A, \beta; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$ and $-\beta(\delta k + k - 1) + k > 0$.*

*Proof:* If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $n$-th $(C, 1)$ mean of the sequence $(nA_n(x))$ (see [22]). By using this, Theorem 2 can be easily proved as in the proof of Theorem 1. □

## 5 Corollaries

If we take $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 1 and Theorem 2, then we get the following known results which obtained in [23].

**Corollary 1.** *Let $(p_n)$ be a sequence of positive numbers such that $P_n = O(np_n) \quad as \quad n \to \infty$. If the conditions (3), (4) and*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty$$

*are satisfied, then the series $\sum a_n \lambda_n$ is summable $\left| \bar{N}, p_n \right|_k$, $k \geq 1$.*

**Corollary 2.** *If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, and the sequences $(p_n)$, $(\lambda_n)$ and $(X_n)$ satisfy the conditions of Corollary 1, then the series $\sum A_n(x) \lambda_n$ is summable $\left| \bar{N}, p_n \right|_k$, $k \geq 1$.*

## 6 Conclusion

In this paper, two general theorems on absolute matrix summability of infinite series and Fourier series are proved. Also, two known results on $\left|\bar{N}, p_n\right|_k$ summability are deduced.

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# Korovkin Type Theorems for Lebesgue Spaces via Statistical Convergence with Respect to A Power Series Method

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**Abstract:** In the present paper we consider the problem of approximating a function in the spaces $L_q[a,b]$ and $L_{q,\omega}(\mathbb{R})$. In order to solve this problem our main tool is $P$-statistical convergence which is recently introduced in [18] and effective to use since it makes a nonconvergent sequence to converge. It is already shown that $P$-statistical convergence is not included by previously known methods. Here it is noteworthy to mention that the approximation on the whole real axis differs radically from the one on bounded intervals.

**Keywords:** Korovkin type approximation, Lebesgue spaces, Power series method, Statistical convergence.

## 1 Introduction

The simplicity and efficiency of Korovkin type theorems make the Korovkin theory lie in the central themes in the approximation theory of functions. These types of theorems state the uniform convergence of a sequence of positive linear operators in $C[a,b]$, the space of continuous real valued functions defined on $[a,b]$, by checking the convergence only on three test functions $\{1,x,x^2\}$ [12]. Korovkin theorem has been considered from different perspectives in [2, 4, 11, 14]. Later than, Gadjiev and Orhan have combined the summability and approximation theories by using statistical convergence and in this direction many generalizations of Korovkin theorem have been stated [3, 6, 7, 15, 17, 18]. In the present paper some Korovkin theorems are extended on the spaces $L_q[a,b]$ and $L_{q,\omega}(\mathbb{R})$ by considering $P$-statistical convergence which is recently introduced in [18]. We also present an example showing that our Theorem 2 is stronger than Theorem 1 since $P$-statistical convergence provides a nonconvergent sequence to converge.

Now we pause to collect some notions and definitions. Let $(p_j)$ be real sequence such that $p_0>0, p_1,p_2,...\geq 0$, and $p(t):=\sum\limits_{j=0}^{\infty} p_j t^j$ has radius of convergence $R$ with $0<R\leq\infty$. If the limit

$$\lim_{t\to R^-}\frac{1}{p(t)}\sum_{j=0}^{\infty} x_j p_j t^j = l$$

exists then it is said that $x=(x_j)$ is convergent in the sense of power series method [13, 16]. The regularity of a power series method $P$ is characterized in [5] by the following condition:

$$\lim_{t\to R^-}\frac{p_j t^j}{p(t)}=0,\ for\ each\ j\in\mathbb{N}_0.$$

Let the method $P$ be regular and $E\subset\mathbb{N}_0$. If the limit

$$\delta_P(E):=\lim_{t\to R^-}\frac{1}{p(t)}\sum_{j\in E} p_j t^j$$

exists then $\delta_P(E)$ is called the $P$-density of $E$. Also, if for every $\varepsilon>0$, $\delta_P(E_\varepsilon)=0$ that is for every $\varepsilon>0$

$$\lim_{t\to R^-}\frac{1}{p(t)}\sum_{j\in E_\varepsilon} p_j t^j = 0,$$

where $E_\varepsilon=\{j\in\mathbb{N}_0:\ \ |x_j-l|\geq\varepsilon\}$ then it is said that the sequence $x=(x_j)$ of real numbers is $P$-statistically convergent to $l$ and it is denoted by $st_P-\lim x=l$. From [18], we have known that statistical convergence and $P$-statistical convergence do not imply each other. Let us introduce the concepts of $P$-statistical boundedness and uniform $P$-boundedness.

**Definition 1.** *The sequence $x = (x_j)$ of real numbers is said to be $P$-statistically bounded if there exists a positive constant $M$ such that the set $\{j \in \mathbb{N}_0 : \ |x_j| > M\}$ has $P$-density 0.*

**Definition 2.** *The sequence of linear operators $(T_j)$ from $L_{q,\omega}$ into itself is said to be uniformly $P$-bounded if there exists a positive constant $M$ and a subset $K \subset \mathbb{N}_0$ having $P$-density 1 such that for every $j \in K$,*

$$\|T_j\| := \|T_j\|_{L_{q,\omega} \to L_{q,\omega}} \leq M.$$

Note that the method $P$ is assumed to be regular throughout the paper. Therefore $P$-statistical convergence is regular due to Theorem 1 in [18].

## 2 Korovkin Theorem for Lebesgue spaces on bounded intervals

In this section, our main goal is to obtain a Korovkin theorem for Lebesgue spaces on bounded intervals by using $P$-statistical convergence. Eventhough the space $L_q[a,b]$ is well known, it would be nice to recall it. Let $L_q[a,b], 1 \leq q < \infty$ be the space of measurable, real valued, $q$th power absolutely Lebesgue integrable functions $f$ on $[a,b]$ with $\|f\|_q := \|f\|_{L_q[a,b]} := (\int_a^b |f|^q d\mu)^{\frac{1}{q}} < \infty$.

**Theorem 1.** *[8] Let $(T_j)$ be a sequence of uniformly bounded positive linear operators from $L_q[a,b]$ into $L_q[a,b]$ such that*

$$\lim_{j\to\infty} \|T_j(F_v) - F_v\|_{L_q} = 0,$$

*holds, where $F_v(x) = x^v$, $(v = 0,1,2)$. Then for every $f \in L_q[a,b]$, we have*

$$\lim_{j\to\infty} \|T_j(f) - f\|_{L_q} = 0.$$

An extension of Dzjadyk's theorem is given via statistical convergence in [[9], Theorem 7]. Our next result, Theorem 2, is an analog of this result for $P$-statistical convergence.

**Theorem 2.** *Let $(T_j)$ be a sequence of uniformly $P$-bounded, positive linear operators from $L_q[a,b]$ into itself such that*

$$st_P - \lim \|T_j(F_v) - F_v\|_{L_q} = 0,$$

*holds, where $F_v(x) = x^v$, $(v = 0,1,2)$. Then for every $f \in L_q[a,b]$, we have*

$$st_P - \lim \|T_j(f) - f\|_{L_q} = 0.$$

*Proof:* According to the hypotheses, for every $\varepsilon > 0$ there exist the subsets $K_v \subset \mathbb{N}$ for $v = 0,1,2$ such that $\delta_P(K_v) = 1$ and for each $j \in K_v$, $\|T_j(F_v) - F_v\|_{L_q} < \varepsilon$. Moreover since $(T_j)$ is uniformly $P$-bounded, one can find a subset $K_3 \subset \mathbb{N}$ such that for any $j \in K_3$, $\|T_j\| \leq M$ where $M$ is a positive real constant. Since $\delta_P(K_0 \cap K_1 \cap K_2) = 1$, for each $j \in K := K_0 \cap K_1 \cap K_2$, we obtain

$$\|T_j(F_v) - F_v\|_{L_q} < \varepsilon.$$

On the other hand by using Lusin's Theorem, for any given function $f \in L_q[a,b]$ there exists a function $g \in C[a,b]$ such that $\|f - g\|_{L_q} < \varepsilon$. By using the following inequality

$$|T_j(f(y);x) - f(x)| \leq |T_j(f(y) - g(y);x)| + |T_j(g(y);x) - g(x)| + |f(x) - g(x)|$$

for every $j \in K_3$, we have

$$\begin{aligned} \|T_j(f) - f\|_{L_q} &\leq \|T_j(f-g)\|_{L_q} + \|T_j(g) - g\|_{L_q} + \|f - g\|_{L_q} \\ &\leq \varepsilon(1+M) + \|T_j(g) - g\|_{L_q}. \end{aligned} \tag{1}$$

Since $g$ is continuous on $[a,b]$, for every $x \in [a,b]$ there exists $C > 0$ such that $|g(x)| < C$ and also $g$ is uniformly continuous on $[a,b]$. Therefore for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|g(y) - g(x)| < \varepsilon + \frac{2C}{\delta^2}(y-x)^2$$

holds. Then the following

$$\begin{aligned} |T_j(g(y);x) - g(x)| &\leq |T_j(|g(y) - g(x)|;x)| + C|T_j(F_0(y);x) - F_0(x)| \\ &\leq |T_j(\varepsilon;x)| + \frac{2C}{\delta^2}|T_j((y-x)^2;x)| + C|T_j(F_0(y);x) - F_0(x)| \\ &\leq \varepsilon\Big(|T_j(F_0(y);x) - F_0(x)| + F_0(x)\Big) + \frac{2C}{\delta^2}|T_j(F_2(y);x) - F_2(x)| \\ &+ \frac{2C}{\delta^2}2\beta|T_j(F_1(y);x) - F_1(x)| + \frac{2C}{\delta^2}\beta^2|T_j(F_0(y);x) - F_0(x)| + C|T_j(F_0(y);x) - F_0(x)| \end{aligned}$$

holds, where $\beta := \max\{|a|,|b|\}$. We also get that

 

$$\|T_j(g)-g\|_{L_q} \le \varepsilon\Big(\|T_j(F_0)-F_0\|_{L_q}+b-a\Big)+\frac{2C}{\delta^2}\|T_j(F_2)-F_2\|_{L_q}$$
$$+\frac{2C}{\delta^2}2\beta\|T_j(F_1)-F_1\|_{L_q}+\frac{2C}{\delta^2}\beta^2\|T_j(F_0)-F_0\|_{L_q}+C\|T_j(F_0)-F_0\|_{L_q}. \tag{2}$$

Combining (1) and (2), we obtain

$$\|T_j(f)-f\|_{L_q} \le \varepsilon(1+M)+\varepsilon\Big(\|T_j(F_0)-F_0\|_{L_q}+b-a\Big)+\frac{2C}{\delta^2}\|T_j(F_2)-F_2\|_{L_q}$$
$$+\frac{2C}{\delta^2}2\beta\|T_j(F_1)-F_1\|_{L_q}+\frac{2C}{\delta^2}\beta^2\|T_j(F_0)-F_0\|_{L_q}+C\|T_j(F_0)-F_0\|_{L_q}.$$

By using the hypotheses, for $j \in K \cap K_3$ and every $f \in L_q[a,b]$, we have

$$st_P - \lim \|T_j(f)-f\|_{L_q} = 0.$$

This completes the proof. □

Now consider the following example of a sequence of positive linear operators from $L_q[-1,1]$ into $L_q[-1,1]$ showing that our Theorem 2 is stronger than Theorem 1.

**Example 1.** *Define $(p_j)$ and $(s_j)$ as follows*

$$p_j = \begin{cases} 1 & ,j=2k \\ 0 & ,j=2k+1 \end{cases}$$

*and*

$$s_j = \begin{array}{ll} 0 & ,j=2k \\ 1 & ,j=2k+1. \end{array}$$

*We can immediately show that the method $P$ is regular. For every $\varepsilon>0$, since $E_\varepsilon=\{j\in\mathbb{N}_0:\ |s_j-0|\ge\varepsilon\}=\{j\in\mathbb{N}_0:\ j=2k+1\}$ we get $\delta_P(E_\varepsilon)=0$. This means that $(s_j)$ is $P$-statistically convergent to 0. Let us consider a sequence of positive linear operators $W=(W_j)$ on $L_q[-1,1]$ as follows*

$$W_j(f;x)=\frac{1}{1+2^{-j}}\begin{cases} f(x) & ,\frac{1}{2^j}\le|x|\le 1 \\ \frac{1}{2}\int_{-1}^{1} f(t)dt & ,|x|<\frac{1}{2^j}. \end{cases} \tag{3}$$

*By using (3), we can construct the sequence of operators*

$$V_j(f;x)=(1+s_j)W_j(f;x).$$

*Since*

$$\begin{aligned}
\|V_j(F_0)-F_0\|_{L_q}^q &= \int_{-1}^{1}|V_j(F_0;x)-F_0(x)|^q dx \\
&= \int_{-1}^{1}\Big|(1+s_j)W_j(F_0;x)-F_0\Big|^q dx \\
&= \int_{-1}^{1}\Big|(1+s_j)\frac{1}{2^{-j}+1}-1\Big|^q dx \\
&= \int_{-1}^{1}\Big|\frac{1}{2^{-j}+1}+\frac{s_j}{2^{-j}+1}-1\Big|^q dx \\
&= \int_{-1}^{1}\Big|\frac{s_j-2^{-j}}{2^{-j}+1}\Big|^q dx \\
&= 2\Big|\frac{s_j-2^{-j}}{2^{-j}+1}\Big|^q.
\end{aligned}$$

*The sequence of operators $(V_j)$ does not satisfy conditions of Theorem 1 in the present paper and Theorem 7 in [9]. According to [8] the sequence of operators $(W_j)$ satisfies conditions of Theorem 1 and the sequence $(s_j)$ is $P$-statistical convergent, therefore, we obtain that the sequence of operators $(V_j)$ satisfies conditions of Theorem 2.*

## 3 Korovkin Theorem for weighted Lebesgue spaces on real axis

Now we deal with the Korovkin type approximation theorem for weighted Lebesgue spaces on whole real axis. Similar type theorems have been studied in [1, 10]. The approximation on the whole real axis differs radically from the one on bounded intervals. Hence we need some

assumptions. Let $\omega$ be a positive continuous function on the whole real axis such that

$$\int_{\mathbb{R}} x^{2q}\omega(x)dx < \infty$$

holds for a fixed $q \in [1, \infty)$. The space of all measurable, $q$-absolutely integrable functions on $\mathbb{R}$ with respect to the weight function $\omega$ is denoted by $L_{q,\omega}(\mathbb{R})(1 \leq q < \infty)$. That is

$$L_{q,\omega}(\mathbb{R}) = \{f : \ \|f\|_{q,\omega} := \Big(\int_{\mathbb{R}} |f(x)|^q \omega(x)dx\Big)^{\frac{1}{q}} < \infty\}.$$

By $\omega_{\min}$ and $\omega_{\max}$, we denote the minimum and maximum values of $w$, respectively.

**Theorem 3.** *Let $(T_j)$ be a sequence of uniformly $P$-bounded positive linear operators from $L_{q,\omega}(\mathbb{R})$ into itself such that*

$$st_P - \lim \|T_j(F_v) - F_v\|_{q,\omega} = 0,$$

*holds, where $F_v(x) = x^v$, $(v = 0, 1, 2)$. Then*

$$st_P - \lim \|T_j(f) - f\|_{q,\omega} = 0$$

*holds for every $f \in L_{q,\omega}(\mathbb{R})$.*

*Proof:* According to the hypotheses, for every $\varepsilon > 0$ there exist the subsets $K_v \subset \mathbb{N}$ for $v = 0, 1, 2$ such that $\delta_P(K_v) = 1$ and for each $j \in K_v$, $\|T_j(F_v) - F_v\|_{q,\omega} < \varepsilon$. Let $\chi_1^A$ be the characteristic function of the interval $[-A, A]$ and $\chi_2^A(x) = 1 - \chi_1^A(x)$ for any positive number $A$. For a sufficiently large $A$ and every $\varepsilon > 0$ we can write

$$\|f\chi_2^A\|_{q,\omega} < \varepsilon. \tag{4}$$

By using the linearity of the operators $T_j$, we have

$$\begin{aligned} \|T_j f - f\|_{q,\omega} &\leq \|T_j(\chi_1^A f) - \chi_1^A f\|_{q,\omega} + \|T_j(\chi_2^A f) - \chi_2^A f\|_{q,\omega} \\ &= I_j^{(1)} + I_j^{(2)}. \end{aligned} \tag{5}$$

Let us begin with $I_j^{(2)}$. Since $(T_j)$ is uniformly $P$-bounded, there exists a constant $M$ and a subset $K_3 \subset \mathbb{N}$ having $P$-density one such that for any $j \in K_3$ and $f \in L_{q,\omega}(\mathbb{R})$

$$\|T_j f\|_{q,\omega} \leq M\|f\|_{q,\omega}. \tag{6}$$

Considering (4) and (6), for any $j \in K_3$ and $f \in L_{q,\omega}(\mathbb{R})$ we get

$$\begin{aligned} I_j^{(2)} &\leq \|T_j(\chi_2^A f)\|_{q,\omega} + \|\chi_2^A f\|_{q,\omega} \\ &\leq (M+1)\|\chi_2^A f\|_{q,\omega} \\ &\leq (M+1)\varepsilon. \end{aligned} \tag{7}$$

On the other hand since for every $f \in L_{q,\omega}(\mathbb{R})$

$$\|\chi_1^A f\|_{L_q} \leq \omega_{\min}^{-\frac{1}{q}} \|f\|_{q,\omega}$$

holds, we obtain $L_{q,\omega}(\mathbb{R}) \subset L_q[-A, A]$. By using Lusin's Theorem, for every $\varepsilon' > 0$ there exists a function $g \in C[-A, A]$ satisfying the condition $g(x) = 0$ for $|x| > A$ such that

$$\|(f-g)\chi_1^A\|_{L_q} < \frac{\varepsilon'}{(M+1)\omega_{\max}^{\frac{1}{q}}}. \tag{8}$$

From inequality (8), we have for every $j \in K_3$

$$\begin{aligned} I_j^{(1)} &\leq \|T_j(f-g)\chi_1^A\|_{q,\omega} + \|T_j(g\chi_1^A) - g\chi_1^A\|_{q,\omega} + \|(f-g)\chi_1^A\|_{q,\omega} \\ &\leq \|T_j(g\chi_1^A) - g\chi_1^A\|_{q,\omega} + \varepsilon'. \end{aligned} \tag{9}$$

Since $\chi_2^{A_1}\chi_1^A g = 0$ for some $A_1 > A$, we obtain

$$\|T_j(g\chi_1^A) - g\chi_1^A\|_{q,\omega} \leq \|[T_j(g\chi_1^A) - g\chi_1^A]\chi_1^{A_1}\|_{q,\omega} + \|\chi_2^{A_1} T_j(g\chi_1^A)\|_{q,\omega}. \tag{10}$$

Let us consider the second term in the last inequality, we have

$$\begin{aligned} \|\chi_2^{A_1} T_j(g\chi_1^A)\|_{q,\omega} &= \Big(\int_{|x|>A_1} |T_j(g\chi_1^A; x)|^q \omega(x)dx\Big)^{\frac{1}{q}} \\ &\leq M_g\Big(\int_{|x|>A_1} |T_j(F_0(t); x) - F_0(x)|^q \omega(x)dx\Big)^{\frac{1}{q}} + M_g\Big(\int_{\mathbb{R}} \chi_2^{A_1}(x)\omega(x)dx\Big)^{\frac{1}{q}}, \end{aligned}$$

 

where $M_g = \max_{x\in\mathbb{R}} |g(x)|\chi_1^A(x)$. Since $\omega \in L_1(\mathbb{R})$, we can choose a constant $A_1$ such that

$$\Big(\int_{\mathbb{R}} \chi_2^{A_1}(x)\omega(x)dx\Big)^{\frac{1}{q}} < \frac{\varepsilon'}{M_g}.$$

Using the last inequality, we have

$$\|\chi_2^{A_1} T_j(g\chi_1^A)\|_{q,\omega} \leq M_g\|T_j(F_0) - F_0\|_{q,\omega} + \varepsilon'. \tag{11}$$

If (10) is combined with (11), we obtain

$$\|T_j(g\chi_1^A) - g\chi_1^A\|_{q,\omega} \leq \|[T_j(g\chi_1^A) - g\chi_1^A]\chi_1^{A_1}\|_{q,\omega} + M_g\|T_j(F_0) - F_0\|_{q,\omega} + \varepsilon'.$$

And also by using (9), we have

$$I_j^{(1)} \leq 2\varepsilon' + M_g\|T_j(F_0) - F_0\|_{q,\omega} + \|[T_j(g\chi_1^A) - g\chi_1^A]\chi_1^{A_1}\|_{q,\omega}. \tag{12}$$

Since the function $g\chi_1^A$ is continuous on $[-A, A]$, for every $\varepsilon' > 0$ there exists a $\delta > 0$ such that

$$|g(y)\chi_1^A(y) - g(x)\chi_1^A(x)| < \varepsilon' + 2M_g\frac{(y-x)^2}{\delta^2}.$$

Then we have that

$$\begin{aligned}\Big|[T_j(g(y)\chi_1^A(y); x) - g(x)\chi_1^A(x)]\chi_1^{A_1}(x)\Big| &\leq \Big|[T_j(|g(y)\chi_1^A(y) - g(x)\chi_1^A(x)|; x)]\chi_1^{A_1}(x)\Big| + \Big|g(x)\chi_1^A(x)(T_j(F_0(y); x) - F_0(x))\Big| \\ &\leq \varepsilon'|T_j(F_0(y); x)| + M_g|T_j(F_0(y); x) - F_0(x)| \\ &+ \frac{2M_g}{\delta^2}\Big(\|T_j(F_2(y); x) - F_2(x)| + 2A|T_j(F_1(y); x) - F_1(x)| + A^2|T_j(F_0(y); x) - F_0(x)|\Big).\end{aligned}$$

We therefore get

$$\|[T_j(g\chi_1^A) - g\chi_1^A]\chi_1^{A_1}\|_{q,\omega} \leq \varepsilon' M\|\omega\|_1^{\frac{1}{q}} + \frac{2M_g}{\delta^2}(1+A)^2 \max_{v=0,1,2}\|T_j(F_v) - F_v\|_{q,\omega} + \varepsilon'.$$

For each $j \in K_0 \cap K_1 \cap K_2$, we can write

$$\max_{v=0,1,2}\|T_j(F_v) - F_v\|_{q,\omega} \leq \frac{\varepsilon'\delta^2}{2M_g(1+A)^2}.$$

Hence using (12), we get for each $j \in K_0 \cap K_1 \cap K_2 \cap K_3$

$$I_j^{(1)} < (5 + M\|\omega\|_1^{\frac{1}{q}})\varepsilon'. \tag{13}$$

Finally if we take into account (7) and (13) with (5), we obtain

$$\|T_j f - f\|_{q,\omega} \leq (5 + M\|\omega\|_1^{\frac{1}{q}})\varepsilon' + (M+1)\varepsilon$$

which completes the proof. □

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# Common Fixed Points of Generalized Meir-Keeler Contraction in Non-Archimedean Modular Metric Spaces

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**Abstract:** In this study, we establish the existence and uniqueness of the common fixed points of $(\alpha,\beta)-\psi-$Meir-Keeler contraction on non-Archimedean modular metric spaces. Our results generalize and extend various comparable results in the literature.
**Keywords:** Cyclic$(\alpha,\beta)$ admissible mapping, Meir-Keeler contraction, Non-Archimedean modular metric space.

## 1 Introduction

Banach contraction principle is one of the most famous and useful results in mathematics. In the last 100 years, it has been extended in many directions. The substitution of the metric space by other generalized metric spaces is one normal way to strengthen the Banach contraction principle. A self-mapping $V$ on a metric space $(M,d)$ is called a contraction if there exists $k\in[0,1)$ such that

$$d\left(V\xi,V\eta\right)\leq kd\left(\xi,\eta\right)$$

for all $\xi,\eta\in M$. The contraction principle simply states that if $(M,d)$ is complete, such a mapping has a unique fixed point. Based on the simplicity, usefulness, and applications of the Banach fixed point theorem, it has become a very popular tool in solving the existing problems in many branches of mathematical analysis. So, many authors have improved, extended and generalized Banach fixed point theorem in many directions.

In 2010, Chistyakov [1, 2] established a modular metric space. It is an extension of metric space and modular linear space. Let $M$ be a nonempty set and $\kappa:(0,\infty)\times M\times M\to[0,\infty]$ be a function; for simplicity, we will write:

$$\kappa_\lambda\left(\xi,\eta\right)=\kappa\left(\lambda,\xi,\eta\right) \tag{1}$$

for all $\lambda>0$ and $\xi,\eta\in M$.

**Definition 1.1.** Let $M$ be nonempty set and $\kappa:(0,\infty)\times M\times M\to[0,\infty]$ be a function satisfying the following conditions:

$\kappa_1$. $\xi=\eta$ if and only if $\kappa_\lambda\left(\xi,\eta\right)=0$ for all $\lambda>0$ and $\xi,\eta\in M$;
$\kappa_2$. $\kappa_\lambda\left(\xi,\eta\right)=\kappa_\lambda\left(\eta,\xi\right)$ for all $\lambda>0$ and $\xi,\eta\in M$;
$\kappa_3$. $\kappa_{\lambda+\mu}\left(\xi,\eta\right)\leq\kappa_\lambda\left(\xi,\nu\right)+\kappa_\mu\left(\nu,\eta\right)$ for all $\lambda,\ \mu>0$ and $\xi,\eta,\nu\in M$.

Then, $\kappa$ is called modular metric in $M$, so $M_\kappa$ is modular metric space. If we replace $(\kappa_1)$ with

$\kappa_4$. $\kappa_\lambda\left(\xi,\xi\right)=0$ for all $\lambda>0$ and $\xi\in M$,

then $\kappa$ is said to be a pseudomodular metric on $M$. A modular metric $\kappa$ on $M$ is called regular if the following weaker version of $(\kappa_1)$ is satisfied:

$\kappa_5$. $\xi=\eta$ if and only if $\kappa_\lambda\left(\xi,\eta\right)=0$ for some $\lambda>0$.

Moreover, $\kappa$ is called convex if for $\lambda,\ \mu>0$ and $\xi,\eta,\nu\in M$, the inequality holds:

$\kappa_6$. $\kappa_{\lambda+\mu}\left(\xi,\eta\right)\leq\frac{\lambda}{\lambda+\mu}\kappa_\lambda\left(\xi,\nu\right)+\frac{\mu}{\lambda+\mu}\kappa_\mu\left(\nu,\eta\right).$

**Definition 1.2.** If we replace $(\kappa_3)$ by

 

$\kappa 7.$ $\kappa_{\max\{\lambda,\mu\}}(\xi,\eta) \leq \kappa_\lambda(\xi,\nu) + \kappa_\mu(\nu,\eta)$

for all $\lambda,\ \mu > 0$ and $\xi,\eta,\nu \in M_\kappa$. Then, $M_\kappa$ is called the non-Archimedean modular metric space.

Suppose that $\kappa$ is a pseudomodular on $M$ and $\xi_0 \in M$ and fixed. Therefore, the two sets:

$$M_\kappa = M_\kappa(\xi_0) = \{\xi \in M : \kappa_\lambda(\xi,\xi_0) \ \text{ as } \ \lambda \to \infty\}$$

and

$$M_\kappa^* = M_\kappa^*(\xi_0) = \{\xi \in M : \exists\lambda = \lambda(\xi) > 0 \quad \text{such that } \ \kappa_\lambda(\xi,\xi_0) < \infty\}.$$

$M_\kappa$ and $M_\kappa^*$ are called modular spaces (around $\xi_0$).

It is clear that $M_\kappa \subset M_\kappa^*$, but this inclusion may be proper in general. Suppose that $\kappa$ is a modular on $M$; from [1, 2], we derive that the modular space $M_\kappa$ can be equipped with a (nontrivial) metric, induced by $\kappa$ and given by:

$$d_\kappa(\xi,\eta) = \inf\{\lambda > 0 : \kappa_\lambda(\xi,\eta) < \lambda\}$$

for all $\xi,\eta \in M_\kappa$.

Note that if $\kappa$ is a convex modular on $M$, then according to [1, 2], the two modular spaces coincide, i.e., $M_\kappa = M_\kappa^*$, and this common set can be endowed with the metric $d_\kappa^*$ given by:

$$d_\kappa^*(\xi,\eta) = \inf\{\lambda > 0 : \kappa_\lambda(\xi,\eta) < 1\}$$

for all $\xi,\eta \in M_\kappa$. Such distances are called Luxemburg distances.

**Definition 1.3.** Let $M_\kappa$ be a modular metric space, $S$ be a subset of $M_\kappa$ and $(\xi_n)_{n\in N}$ be a sequence in $M_\kappa$. Then,

i. A sequence $(\xi_n)_{n\in N}$ is called $\kappa-$convergent to $\xi \in M_\kappa$ if and only if $\kappa_\lambda(\xi_n,\xi) \to 0$ as $n \to \infty$ for all $\lambda > 0$, $\xi$ is said to be the $\kappa-$limit of $(\xi_n)$.
ii. A sequence $(\xi_n)_{n\in N}$ is called $\kappa-$Cauchy if $\kappa_\lambda(\xi_n,\xi_m) \to 0$, as $m,n \to \infty$ for all $\lambda > 0$.
iii. A subset $S$ is called $\kappa-$closed if the $\kappa-$limit of $\kappa-$convergent sequence of $S$ always belongs to $S$.
iv. A subset $S$ is called $\kappa-$complete if any $\kappa-$Cauchy sequence in $S$ is $\kappa-$convergent to a point of $S$.

**Definition 1.4.** Let $M_\kappa$ be a modular metric space and $V : M_\kappa \to M_\kappa$ be a mapping. We say that $V$ is a $\kappa-$continuous when if $\kappa_\lambda(\xi_n,\xi) \to 0$, then $\kappa_\lambda(V\xi_n,V\xi) \to 0$ as $n \to \infty$.

**Definition 1.5.** Let $M_\kappa$ be a modular metric space and $V : M_\kappa \to M_\kappa$ be a mapping. A mapping $V$ is called a $\kappa$-contraction if for each $\xi,\eta \in M_\kappa$ and for all $\lambda > 0$, there exists $k \in [0,1)$ such that

$$\kappa_\lambda(V\xi,V\eta) \leq k\,\kappa_\lambda(\xi,\eta).$$

**Theorem 1.6.** Let $M_\kappa$ be a $\kappa-$complete modular metric space and $V : M_\kappa \to M_\kappa$ be a $\kappa$-contraction. Then $V$ has a unique fixed point.

After that, many mathematicians improved some new fixed point results in modular metric spaces [3]-[8].

**Definition 1.7.** [9] Let $V : M \to M$ and $\alpha : M \times M \to [0,\infty)$ be two mappings. $V$ is called $\alpha-$admissible if

$$\alpha(\xi,\eta) \geq 1 \quad \Rightarrow \quad \alpha(V\xi,V\eta) \geq 1,$$

for all $\xi,\eta \in M$.

**Definition 1.8.** [10] Let $V : M \to M$ and $\alpha : M \times M \to [0,\infty)$ be two mappings. $V$ is called triangular $\alpha$-admissible if

$i.$ $\alpha(\xi,\eta) \geq 1 \quad \Rightarrow \quad \alpha(V\xi,V\eta) \geq 1$, for all $\xi,\eta \in M$;
$ii.$ $\alpha(\xi,\nu) \geq 1$ and $\alpha(\nu,\eta) \geq 1$, then $\alpha(\xi,\eta) \geq 1$, for all $\xi,\eta,\nu \in M$.

Recently, Alizadeh et al. [11] defined the concept of cyclic $(\alpha,\beta)-$admissible mapping as follows:

**Definition 1.9.** Let $M$ be a nonempty set, $V$ be a self-mapping on $M$ and $\alpha,\beta : M \to [0,\infty)$ be two mappings. We say that $V$ is a cyclic $(\alpha,\beta)-$admissible mapping if

$(i)$ $\alpha(\xi) \geq 1$ for some $x \in M$ implies $\beta(V\xi) \geq 1$;
$(ii)$ $\beta(\xi) \geq 1$ for some $\xi \in M$ implies $\alpha(V\xi) \geq 1$.

They also established the existence of fixed point theorems using the concept of cyclic $(\alpha,\beta)-$admissible mapping.

**Example 1.10.** Let $V : R \to R$ be defined by $V\xi = -\left(\xi + \xi^3\right)$. Suppose that $\alpha,\beta : R \to R_+$ are given mapping for all $\xi,\eta \in R_+$ such that

$$\alpha(\xi) = e^\xi \ \text{ and } \ \beta(\eta) = e^{-\eta}.$$

Then $V$ is a cyclic $(\alpha,\beta)-$admissible mapping.

Latif and Ansari [12] generalized cyclic $(\alpha,\beta)-$admissible mapping as follows:

**Definition 1.11.** Let $M$ be a nonempty set, $V,Z$ be two self-mappings on $M$, and $\alpha,\beta : M \to [0,\infty)$ be two mappings. $(V,Z)$ is called a cyclic $(\alpha,\beta)-$admissible pair if the following two statements hold;

$(i)$ $\alpha(\xi) \geq 1$ for some $\xi \in M$ implies $\beta(V\xi) \geq 1$;
$(ii)$ $\beta(\xi) \geq 1$ for some $\xi \in M$ implies $\alpha(Z\xi) \geq 1$.

In the above definition, if we take $V = Z$, we get $V$ is a cyclic $(\alpha,\beta)$-admissible mapping.

**Definition 1.12.** [13] A mapping $V$ on a metric space $(M,d)$ is said to be a Meir-Keeler contraction if given $\epsilon > 0$ there exists $\delta > 0$ such that for all $\xi, \eta \in M$,

$$\varepsilon \leq d(\xi,\eta) < \varepsilon + \delta \quad \Rightarrow \quad d(V\xi, V\eta) < \varepsilon.$$

In the sequel, $N$ denotes the set of positive integers. Let $\Psi$ the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where $\psi^n$ is the nth iterate of $\psi$.

**Remark 1.13.** Every function $\psi \in \Psi$ is called a $(c)-$comparison function. It is easy to show that if $\psi$ is a comparison function, then $\psi(t) < t$ for any $t > 0$ and $\psi(0) = 0$.

**Definition 1.14.** [14] Let $(M,d)$ be a metric space and $\psi \in \Psi$. Suppose that $V : M \to M$ is a triangular $\alpha-$admissible mapping satisfying the following condition: for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \psi(d(\xi,\eta)) < \varepsilon + \delta \quad \Rightarrow \quad \alpha(\xi,\eta)\, d(V\xi, V\eta) < \varepsilon$$

for all $\xi, \eta \in M$. Then $V$ is called an $\alpha - \psi-$Meir-Keeler contractive mapping.

**Remark 1.15.** [14] Let $V$ be an $\alpha - \psi-$Meir-Keeler contractive mapping. Then

$$\alpha(\xi,\eta)\, d(V\xi, V\eta) < \psi(d(\xi,\eta)),$$

for all $\xi, \eta \in M$ when $\xi \neq \eta$. Also, if $\xi = \eta$ then $d(V\xi, V\eta) = 0$. i.e.,

$$\alpha(\xi,\eta)\, d(V\xi, V\eta) < \psi(d(\xi,\eta))$$

for all $\xi, \eta \in M$.

Meir-Keeler's fixed point theorems have been extended in many directions [15]-[18].

## 2 Main results

In the sequel, the function $\kappa$ is convex and regular.

Using the above, we introduce generalized $(\alpha,\beta) - \psi-$Meir-Keeler contraction and establish common fixed point theorems in non-Archimedean modular metric spaces.

**Definition 2.1** Let $M_\kappa$ be a non-Archimedean modular metric space and $V, Z : M_\kappa \to M_\kappa$ and $\alpha, \beta : M_\kappa \to [0,\infty)$ be four mappings. We say that $V$ and $Z$ are generalized $(\alpha,\beta) - \psi-$Meir-Keeler contraction if there exist $\psi \in \Psi$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{array}{l} \alpha(\xi)\,\beta(\eta) \geq 1 \quad \Rightarrow \\ \varepsilon \leq \psi(M(\xi,\eta)) < \varepsilon + \delta \quad \Rightarrow \quad \kappa_\lambda(V\xi, Z\eta) < \varepsilon, \end{array} \tag{2}$$

where

$$M(\xi,\eta) = \max\{\kappa_\lambda(\xi,\eta), \kappa_\lambda(\xi, V\xi), \kappa_\lambda(\eta, Z\eta)\}$$

for all $\xi, \eta \in M_\kappa$.

**Theorem 2.2** Let $M_\kappa$ be a $\kappa-$complete non-Archimedean modular metric space and $V$ and $Z$ are generalized $(\alpha,\beta) - \psi-$Meir-Keeler contraction. Suppose that the following conditions hold:

$i.$ $(V,Z)$ is a cyclic $(\alpha,\beta)-$admissible pair,
$ii.$ there exists $\xi_0 \in M_\kappa$ such that $\alpha(\xi_0) \geq 1$,
$iii.$ $V$ or $Z$ is $\kappa-$continuous,
$iv.$ if $\{\xi_n\}$ is a sequence in $M_\kappa$ such that $\xi_n \to \xi$ and $\alpha(\xi_{2n}) \geq 1$ and $\beta(\xi_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(\xi) \geq 1$ and $\beta(\xi) \geq 1$.

Then, $V$ and $Z$ admit a common fixed point. Moreover, if $\alpha(\xi)\,\beta(\eta) \geq 1$ for all $\xi, \eta \in Fix(V,Z)$, then $V$ and $Z$ admit a unique common fixed point.

*Proof:* Let $\xi_0 \in M_\kappa$ be such that $\alpha(\xi_0) \geq 1$. We will construct a sequence $\{\xi_n\}$ in $M_\kappa$ by

$$\begin{array}{l} \xi_{2n+2} = Z\xi_{2n+1}, \\ \xi_{2n+1} = V\xi_{2n}, \end{array} \tag{3}$$

for all $n \in \mathbb{N}$. Also, as $(V,Z)$ is a cyclic $(\alpha,\beta)-$admissible pair and $\alpha(\xi_0) \geq 1$, then

$$\beta(\xi_1) = \beta(V\xi_0) \geq 1$$

which implies

$$\alpha(\xi_2) = \alpha(Z\xi_1) \geq 1.$$

By proceeding with this process, we get $\alpha(\xi_{2n}) \geq 1$ and $\beta(\xi_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$. Thus, $\alpha(\xi_{2n})\,\beta(\xi_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$. Since $V$ and $Z$ are generalized $(\alpha,\beta) - \psi-$Meir-Keeler contraction, we get

$$\kappa_\lambda(\xi_{2n+1}, \xi_{2n+2}) = \kappa_\lambda(V\xi_{2n}, Z\xi_{2n+1}) \leq \psi(M(\xi_{2n}, \xi_{2n+1})), \tag{4}$$

 

where

$$\begin{aligned} M(\xi_{2n},\xi_{2n+1}) &= \max\{\kappa_\lambda(\xi_{2n},\xi_{2n+1}),\kappa_\lambda(\xi_{2n},V\xi_{2n}),\kappa_\lambda(\xi_{2n+1},Z\xi_{2n+1})\} \\ &= \max\{\kappa_\lambda(\xi_{2n},\xi_{2n+1}),\kappa_\lambda(\xi_{2n},\xi_{2n+1}),\kappa_\lambda(\xi_{2n+1},\xi_{2n+2})\} \\ &= \max\{\kappa_\lambda(\xi_{2n},\xi_{2n+1}),\kappa_\lambda(\xi_{2n+1},\xi_{2n+2})\}. \end{aligned}$$

If $\max\{\kappa_\lambda(\xi_{2n},\xi_{2n+1}),\kappa_\lambda(\xi_{2n+1},\xi_{2n+2})\} = \kappa_\lambda(\xi_{2n+1},\xi_{2n+2})$ for each $n\in\mathbb{N}$, then from (4) and $\psi\in\Psi$ we have

$$\kappa_\lambda(\xi_{2n+1},\xi_{2n+2}) \le \psi(\kappa_\lambda(\xi_{2n+1},\xi_{2n+2})) < \kappa_\lambda(\xi_{2n+1},\xi_{2n+2}),$$

which is a contradiction. So, $M(\xi_{2n},\xi_{2n+1}) = \kappa_\lambda(\xi_{2n},\xi_{2n+1})$ for each $n\in\mathbb{N}$. Consequently, (4) gives

$$\kappa_\lambda(\xi_{2n+1},\xi_{2n+2}) \le \psi(\kappa_\lambda(\xi_{2n},\xi_{2n+1})). \tag{5}$$

Hence for all $n\in\mathbb{N}$ continuing this way we get

$$\kappa_\lambda(\xi_n,\xi_{n+1}) \le \psi^n(\kappa_\lambda(\xi_0,\xi_1)). \tag{6}$$

Now, we prove that $\{\xi_n\}$ is a Cauchy sequence. Regarding the properties of the function $\psi$, for any $t>0$ $\sum\limits_{n=1}^{\infty}\psi^n(t)<\infty$. From (5) and using $(\kappa_7)$, for all $k\ge 1$ we have

$$\begin{aligned} \kappa_\lambda(\xi_n,\xi_{n+k}) &\le \kappa_\lambda(\xi_n,\xi_{n+1}) + \kappa_\lambda(\xi_{n+1},\xi_{n+2}) + \ldots + \kappa_\lambda(\xi_{n+k-1},\xi_{n+k}) \\ &\le \sum_{p=n}^{n+k-1}\psi^p(\kappa_\lambda(\xi_0,\xi_1)) \\ &\le \sum_{p=n}^{\infty}\psi^p(\kappa_\lambda(\xi_0,\xi_1)). \end{aligned} \tag{7}$$

Letting $p\to\infty$ in (7), we obtain $\{\xi_n\}$ is a Cauchy sequence.

As $M_\kappa$ is a $\kappa-$complete non-Archimedean modular metric space, there exists $u\in M_\kappa$ such that $\kappa_\lambda(\xi_n,u)\to 0$ as $n\to\infty$.

Now we present $u$ is a common fixed point of $V$ and $Z$. From $(iii)$, we will presume $V$ is a $\kappa-$continuous mapping. Since $\kappa_\lambda(\xi_{2n},u)\to 0$ as $n\to\infty$ and $V$ is a $\kappa-$continuous mapping, we get $\kappa_\lambda(V\xi_{2n},Vu) = \kappa_\lambda(\xi_{2n+1},Vu)\to 0$ as $n\to\infty$. But by the uniqueness of the limit, we achieve $Vu=u$.

Next, we will demonstrate that $u$ is a fixed point of $Z$. We presume that $u\ne Zu$, that is $\kappa_\lambda(u,Zu)>0$. From $(iv)$, we have $\beta(u)\ge 1$. This implies that $\alpha(\xi_{2n})\beta(u)\ge 1$, for all $n\in\mathbb{N}$. By $V$ and $Z$ are generalized $(\alpha,\beta)-\psi-$Meir-Keeler contraction, we attain

$\kappa_\lambda(V\xi_{2n},Zu)\le\psi(M(\xi_{2n},u))$, where

$$\begin{aligned} M(\xi_{2n},u) &= \max\{\kappa_\lambda(\xi_{2n},u),\kappa_\lambda(\xi_{2n},V\xi_{2n}),\kappa_\lambda(u,Zu)\} \\ &= \max\{\kappa_\lambda(\xi_{2n},u),\kappa_\lambda(\xi_{2n},\xi_{2n+1}),\kappa_\lambda(u,Zu)\}. \end{aligned}$$

Letting $n\to\infty$, we have

$$\kappa_\lambda(u,Zu)\le\psi(\kappa_\lambda(u,Zu)) < \kappa_\lambda(u,Zu)$$

which is a contradiction. Thus $u=Zu$ and hence $u$ is a common fixed point of $V$ and $Z$.

Ultimately, we will demonstrate that the uniqueness of a common fixed point of $V$ and $Z$. We consume that $s$ is another common fixed point of $V$ and $Z$, that is, $\kappa_\lambda(u,s)\ne 0$. From the hypothesis, we gain $\alpha(u)\beta(s)\ge 1$. Since $V$ and $Z$ are generalized $(\alpha,\beta)-\psi-$Meir-Keeler contraction, we attain

$$\kappa_\lambda(u,s) = \kappa_\lambda(Vu,Zs)\le\psi(M(u,s)),$$

where

$$\begin{aligned} M(u,s) &= \max\{\kappa_\lambda(u,s),\kappa_\lambda(u,Vu),\kappa_\lambda(s,Zs)\} \\ &= \max\{\kappa_\lambda(u,s),0\} \\ &= \kappa_\lambda(u,s). \end{aligned}$$

That is

$$\kappa_\lambda(u,s)\le\psi(\kappa_\lambda(u,s)) < \kappa_\lambda(u,s)$$

which is a contradiction. Thus $u=s$. □

**Corollary 2.3** Let $M_\kappa$ be a $\kappa-$complete non-Archimedean modular metric space and $V,Z: M_\kappa\to M_\kappa$ be mappings. Suppose that the following conditions hold:

$i.$ $(V,Z)$ is a cyclic $(\alpha,\beta)-$admissible pair,
$ii.$ there exists $\xi_0\in M_\kappa$ such that $\alpha(\xi_0)\ge 1$,
$iii.$ $V$ or $Z$ is $\kappa-$continuous,
$iv.$ there exist $\psi\in\Psi$, and for each $\epsilon>0$ there exists $\delta>0$ such that

$$\begin{aligned} &\alpha(\xi)\beta(\eta)\ge 1 \quad\Rightarrow \\ &\varepsilon\le\psi(\kappa_\lambda(\xi,\eta)) < \varepsilon+\delta \quad\Rightarrow\quad \kappa_\lambda(V\xi,Z\eta)<\varepsilon, \end{aligned}$$

$v.$ if $\{\xi_n\}$ is a sequence in $M_\kappa$ such that $\xi_n\to\xi$ and $\alpha(\xi_{2n})\ge 1$ and $\beta(\xi_{2n-1})\ge 1$ for all $n\in N$, then $\alpha(\xi)\ge 1$ and $\beta(\xi)\ge 1$.

Then, $V$ and $Z$ admit a common fixed point. Moreover, if $\alpha(\xi)\beta(\eta)\ge 1$ for all $\xi,\eta\in Fix(V,Z)$, then $V$ and $Z$ admit a unique common fixed point.

**Corollary 2.4** Let $M_\kappa$ be a $\kappa-$complete non-Archimedean modular metric space and $V : M_\kappa \to M_\kappa$ be mapping. Suppose that the following conditions hold:

$i.$ $V$ is a cyclic $(\alpha, \beta)$ $-$admissible mapping,
$ii.$ there exists $\xi_0 \in M_\kappa$ such that $\alpha(\xi_0) \geq 1$ and $\beta(\xi_0) \geq 1$,
$iii.$ $V$ is $\kappa-$continuous
$iv.$ there exist $\psi \in \Psi$, and for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{array}{l} \alpha(\xi)\beta(\eta) \geq 1 \quad \Rightarrow \\ \varepsilon \leq \psi(M(\xi,\eta)) < \varepsilon + \delta \quad \Rightarrow \quad \kappa_\lambda(V\xi, V\eta) < \varepsilon, \end{array} \tag{8}$$

where

$$M(\xi,\eta) = \max\{\kappa_\lambda(\xi,\eta), \kappa_\lambda(\xi, V\xi), \kappa_\lambda(\eta, V\eta)\}$$

for all $\xi, \eta \in M_\kappa$.

Then, $V$ admits a fixed point. Moreover, if $\alpha(\xi)\beta(\eta) \geq 1$ for all $\xi, \eta \in Fix(V)$, then $V$ admits a unique fixed point.
**Corollary 2.5** Let $M_\kappa$ be a $\kappa-$complete non-Archimedean modular metric space and $V : M_\kappa \to M_\kappa$ be mapping. Suppose that the following conditions hold:

$i.$ $V$ is a cyclic $(\alpha, \beta)$ $-$admissible mapping,
$ii.$ there exists $\xi_0 \in M_\kappa$ such that $\alpha(\xi_0) \geq 1$ and $\beta(\xi_0) \geq 1$,
$iii.$ $V$ is $\kappa-$continuous,
$iv.$ there exist $\psi \in \Psi$, and for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{array}{l} \alpha(\xi)\beta(\eta) \geq 1 \quad \Rightarrow \\ \varepsilon \leq \psi(\kappa_\lambda(\xi,\eta)) < \varepsilon + \delta \quad \Rightarrow \quad \kappa_\lambda(V\xi, V\eta) < \varepsilon. \end{array}$$

Then, $V$ admits a fixed point. Moreover, if $\alpha(\xi)\beta(\eta) \geq 1$ for all $\xi, \eta \in Fix(V)$, then $V$ admits a unique fixed point.
**Example 2.6** Let $M_\kappa = \mathbb{R}$, $\kappa_\lambda(\xi,\eta) = \frac{1}{\lambda}|\xi - \eta|$ for all $\xi, \eta \in M_\kappa$. Define the mappings $V, Z : M_\kappa \to M_\kappa$ as follows:

$$V\xi = \frac{\xi}{2}, \quad Z\eta = \frac{\eta}{3}, \quad \xi, \eta \in M_\kappa$$

If we consider the function $\psi(t) = \frac{t}{6}$, then all the hypotheses of Theorem 2.2 are satisfied and 0 is a unique common fixed point of $V$ and $Z$.

## 3 Conclusion

In this work, we prove the existence and uniqueness of the common fixed points of $(\alpha, \beta) - \psi-$Meir-Keeler contraction on non-Archimedean modular metric spaces. Our results generalize and extend various comparable results in the literature. The fixed point technique is used to solve mathematical problems as it gets involved with differential and integral equations, integro-differential equations, game theory, economics, and more disciplines. Thus, in future work, we can give such applications using Meir-Keeler contraction in abstract spaces.


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# g-$I$-closed Sets

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**Abstract:** We introduce the notion of $g-I-$closed set by using $I-$open sets as a new $g-$closed sets type in ideal topological spaces. We also investigate of these sets.

**Keywords:** $g-$closed sets, $I-$open sets, $g-I-$closed sets, $\Lambda_I-$operation

## 1 Introduction and Preliminaries

Since the notion of set ideals [5, 9], which is known as topological ideal, is interesting subject so it has studied on topology as an important tool for several years. Indeed, the reader find to this subject's basic and value papers in [2], [8], [3], [4] and [1] as cronologically. At the beginning of the 21st century, in [7], authors introduced some sets using $I$-open set [1, 4]. If $A$ is a subset of a topological space $(X, \tau)$, $Cl(A)$ and $Int(A)$ denote the closure of $A$ and the interior of $A$, respectively. In [6] introduced the concept of generalized closed sets. This notion has been studied extensively in recent years by many topologists. A subset $A$ of a topological space $(X, \tau)$ is said to be generalized closed ( briefly g-closed ) if $Cl(A) \subset U$ whenever $A \subset U$ and $U$ is open in $(X, \tau)$. In this paper, we introduce and study the concept of g-closed sets with respect to $I$-open set [1, 4], which is the extension of the concept of g-closed sets.

**Definition 1.** *[5] An ideal $I$ on a set $X$ is non-void subcollection of $P(X)$ satisfiying the following two conditions which known as heredity and finite additivity properties, respectively:*
*(1) If $A \in I$ and $B \subset A$, then $B \in I$;*
*(2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.*

A topological space together with an ideal $I$ is called an ideal topological space and is denoted by $(X, \tau, I)$ [3]. Using the concept of ideal the notion of localization is introduced in [5] as stated in the following.

**Definition 2.** *Let $(X, \tau, I)$ be an ideal topological space. For a subset $A \subset X$,*

$$A^*(I) = \{x \in X : U \cap A \notin I \text{ for each neighbourhood } U \text{ of } x\}$$

*is called the local function of $A$ with respect to $I$ and $\tau$ [5]. $X^*$ is often a proper subset of X. We simply write $A^*$ instead of $A^*(I)$ in case there is no chance for confusion.*

Now, we remember some properties of local function.

**Lemma 1.** *[3] Let $(X, \tau, I)$ be an ideal topological space and $A, B$ subsets of $X$. Then the following properties hold:*
*(1) If $A \subset B$, then $A^* \subset B^*$,*
*(2) $A^* = Cl(A^*) \subset Cl(A)$,*
*(3) $(A^*)^* \subset A^*$,*
*(4) $(A \cup B)^* = A^* \cup B^*$,*
*(5) If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$.*

We note that arbitrary intersection and finite union properties of local function are given by Kuratowski [5] as follows:
(1) $(\cap\{A_\alpha : \alpha \in \Delta\})^* \subset \cap(\{A_\alpha^* \mid \alpha \in \Delta\})$,
(2) $(\cup\{A_\alpha : \alpha \in \Delta\})^* = \cup(\{A_\alpha^* \mid \alpha \in \Delta\})$, for each $A_\alpha \subseteq X$.

**Definition 3.** *A subset $A$ of an its $(X, \tau, I)$ is said to be $I$-open [1] , [4] ( resp. $\tau^*$-closed [3] ) if $A \subset Int(A^*)$ ( resp. $A^* \subset A$ ). A set $A$ will be called $I$-closed iff $A^c$ is $I$-open. We denote the family of all $I$-open sets of $(X, \tau, I)$ by $IO(X, \tau)$.*

One can see two essential properties of $IO(X, \tau)$ in literature as stated the following lemma.

**Lemma 2.** *[4] Let $(X, \tau, I)$ be an ideal topological space with $A \subseteq X$ and $\Delta$ an arbitrary index set.*

*(1) If* $\{A_\alpha : \alpha \in \Delta\} \subseteq IO(X,\tau)$*, then* $\cup\{A_\alpha : \alpha \in \Delta\} \in IO(X,\tau)$.
*(2) If* $U \in \tau$ *and* $A \in IO(X,\tau)$*, then* $(U \cap A) \in IO(X,\tau)$.

In this paper is consists of three sections. In Section 2, we introduce the notion of g-$I$-closed sets and obtain some properties and characterizations. In Section 3, we give definition of g-$I$-open set as complement of g-$I$-closed set and some properties of it. Finally in Section 4, we introduce a new separation axiom is called $I$-$T^*_{1/2}$ spaces in ideal topological spaces and obtain some properties of them.

## 2 g-$I$-closed sets

In this section, we introduce the notion of g-$I$-closed sets and obtain some prperties of these sets.

**Definition 4.** *A subset $A$ of an its $(X,\tau,I)$ is said to be generalized $I$-closed ( briefly, g-$I$-closed ) if $Cl(A) \subset U$ whenever $A \subset U$ and $U$ is an $I$-open set of $(X,\tau,I)$.*

We denote the family of all g-$I$-closed subsets of an its $(X,\tau,I)$ by $GIC(X,\tau)$.
One can obtain the following facts from Definition 4.

**Remark 1.** *Every closed set is g-$I$-closed in any ideal topological space $(X,\tau,I)$. The converse of this implication isn't true in generally as shown the following example.*

**Example 1.** *Let $(X,\tau,I)$ be an ideal topological space such that X={a, b, c, d}, $\tau$={$\varnothing$, X, {a}, {b}, {a,b} } and I={$\varnothing$, {a}}. Set A={a,b,c} is g-$I$-closed but it is not closed. Since neither $A$ nor $X$ is an $I$-open set, there is no $I$-open set containing $A$ and hence $A$ is a g-$I$-closed. But $A \notin \tau^c$.*

We have given the answer to the question of "When is the reverse of the implication in Remark 1 true?" in the following proposition.

**Proposition 1.** *For a subset $A$ of an ideal topological space $(X,\tau,I)$, the following property holds: If $A$ is a g-$I$-closed and $I$-open, then it is closed.*

*Proof:* Let $A$ is a g-$I$-closed and $I$-open. Then, we have $Cl(A) \subset$A and this shows that $A$ is closed. □

**Theorem 1.** *The family of all g-$I$-closed subsets of an ideal topological space $(X,\tau,I)$, i.e., $GIC(X,\tau)$ is closed under finite union.*

*Proof:* Since $\tau^c$ is closed under finite union, this is obtained directly. □

**Remark 2.** *The intersection of two g-$I$-closed sets is not a g-$I$-closed set in generally.*

**Example 2.** *Let $(X,\tau,I)$ be an ideal topological space as Example 1., i.e., X={a, b, c, d}, $\tau$={$\varnothing$, X, {a}, {b}, {a,b} } and I={$\varnothing$, {a}}. Although A={a,b,c} and B={b,d} are each g-$I$-closed, but $(A \cap B)$={b} is not g-$I$-closed.*

**Definition 5.** *An ideal topological space $(X,\tau,I)$ is said to be*
*(1) $Hayashi$ space ([2]) if $X = X^*$,*
*(2) $Samuels$ space ([8]) if $\tau \cap I$ = {$\varnothing$ }.*

It is known that in [3], authors obtained that each of spaces in above definition is coincide to other and they renamed as $Hayashi$-$Samuels$ $space$ (briefly, H.S.S.). We note that in any ideal topological space $(X,\tau,I)$ since $\varnothing^* = \varnothing$, it is clear that $\varnothing$ is $I$-open and hence g-$I$-closed set, but $X$ is not. In really since $X$ is an $I$-open in an $H.S.S.$, $X$ is g-$I$-closed set in only $H.S.S.$.

**Theorem 2.** *Let $(X,\tau,I)$ be any ideal topological space. Every subset of $X$ is a g-$I$-closed set iff $IO(X,\tau) = \tau^c$.*

*Proof:* Sufficiency. Suppose that $IO(X,\tau) = \tau^c$. Let $A \subset U$ and $U \in IO(X,\tau)$. Then we have $Cl(A) \subset Cl(U) = U$. This shows that $A$ is a g-$I$-closed set.
Necessity. Assume that every subset of $X$ is a g-$I$-closed set. Let $U \in IO(X,\tau)$. Then since $U \subset U$ and $U$ is a g-$I$-closed set, we have $Cl(U) \subset U$ and hence $U \in \tau^c$. Hence we obtain that $IO(X,\tau) \subset \tau^c$. According to hypothesis, $F \in GIC(X,\tau)$. If $F \in \tau^c$, then we have $F \in IO(X,\tau)$. Therefore, $\tau^c \subset IO(X,\tau)$ and consequently $IO(X,\tau) = \tau^c$. □

**Definition 6.** *[7] Let A be a subset of an ideal topological space $(X,\tau,I)$. A subset $\Lambda_I(A)$ is defined as follows:*

$$\Lambda_I(A) = \cap\{U \mid A \subset U,\ U \in IO(X,\tau)\}.$$

**Theorem 3.** *A subset $A$ of an ideal topological space $(X,\tau,I)$ is a g-$I$-closed iff $Cl(A) \subset \Lambda_I(A)$.*

*Proof:* Necessity. Let $A$ be a g-$I$-closed. Then, according Definition 4, for any $U \in IO(X,\tau)$ such that $A \subset U$, we have $Cl(A) \subset U$. Therefore, $Cl(A) \subset \cap\{U : A \subset U \in IO(X,\tau)\}) = \Lambda_I(A)$ and hence $Cl(A) \subset \Lambda_I(A)$.
Sufficiency. Let $A \subset U \in IO(X,\tau)$ and $Cl(A) \subset \Lambda_I(A)$. Therefore, we obtain $Cl(A) \subset \Lambda_I(A) \subset$U. Hence, A is a g-$I$-closed. □

**Definition 7.** *[7] A subset $A$ of an ideal topological space $(X,\tau,I)$ is a $\Lambda_I$-set if $A = \Lambda_I(A)$.*

 

**Lemma 3.** *[7] Every $I$-open set is $\Lambda_I$-set.*

We have the following consequence from Theorem 3 and Lemma 3. So we give it without proof.

**Corollary 1.** *For a subset $A$ of an its $(X, \tau, I)$, the following property holds: If $A$ is a g-$I$-closed set and $\Lambda_I$-set, then it is closed.*

It is obvious that Corollary 1 is a generalization of Proposition 1.

**Theorem 4.** *Let $(X, \tau, I)$ be an ideal topological space and $A$, $B \subset X$. If $A$ is g-$I$-closed and $A \subset B \subset Cl(A)$, then $B$ is g-$I$-closed.*

*Proof:* Let $B \subset U$ and $U$ be $I$-open. Since $A \subset B$, we have $A \subset U$. Besides since $A$ is g-$I$-closed, we have $Cl(A) \subset U$ using Definition 4. Finally, we have $Cl(A) \subset Cl(B) \subset (Cl(Cl(A)) = Cl(A)$ and $Cl(A) = Cl(B)$. This shows that $B$ is g-$I$-closed. □

**Proposition 2.** *Let $(X, \tau, I)$ be an ideal topological space and $A$ a subset of $X$. Then $(Cl(A) - A)$ does not contain any non-empty open set.*

*Proof:* Let $A$ be a subset of $X$. Assume that $U \in \tau$ and $U \subset (Cl(A) - A)$. Since $U \subset (Cl(A) - A) \subset (X - A)$, we have $A \subset (X - U) \in \tau^c$. Then, $Cl(A) \subset (X - U)$ and hence $U \subset (X - Cl(A))$. But, since $U \subset (Cl(A) - A) \subset (X - A)$, we have $U \subset Cl(A)$. Therefore, we have $U \subset (X - Cl(A))$ is true for only $U = \varnothing$. □

**Proposition 3.** *Let $A$ be a subset in any ideal topological space $(X, \tau, I)$. $A$ is a g-$I$-closed iff $(Cl(A) - A)$ does not contain any non-empty $I$-closed set.*

*Proof:* Necessity. Let $A$ be g-$I$-closed. Assume that $F$ is an $I$-closed set and $F \subset (Cl(A) - A)$. Since $F \subset (Cl(A) - A) \subset (X - A)$, we have $A \subset (X - F) \in IO(X, \tau)$. According to hypothesis since $A$ is a g-$I$-closed set, we have $Cl(A) \subset (X - F)$ by Definition 4. Hence, $F \subset (X - Cl(A))$. Consequently, we have $F \subset Cl(A)$ and $F \subset (X - Cl(A))$. This shows that $F \subset (Cl(A) \cap (X - Cl(A))) = \varnothing$ and hence $F = \varnothing$.

Sufficiency. Assume that $(Cl(A) - A)$ does not contain any non-empty $I$-closed set and $A \subset U$, $U$ is an $I$-open. If $Cl(A)$ isnt a subset of $U$, then we have $(Cl(A) \cap (X - U))$ is non-empty $I$-closed subset of $(Cl(A) - A)$. This is contradiction to hypothesis. Therefore $Cl(A) \subset U$ and hence $A$ is g-$I$-closed. □

Since the notions of open set and $I$-open set are independent of each other, similarly the next state is hold: "Closed sets and $I$-closed sets are independent of each other." So, next theorem is important and interesting as it gives a characterization of closed set via $I$-closed.

**Theorem 5.** *Let $A$ be an g-$I$-closed set in any its $(X, \tau, I)$. Then, $A \in \tau^c$ iff $(Cl(A) - A)$ is an $I$-closed set.*

*Proof:* Necessity. Let $A$ is a closed and g-$I$-closed in X. Then, we have $Cl(A) = A$ and hence $(Cl(A)\text{-}A) = \varnothing$. Since $A$ is g-$I$-closed, via Proposition 3, we have $(Cl(A) - A)$ is an $I$-closed.

Sufficiency. Let $(Cl(A) - A)$ is an $I$-closed set. Since $A$ is g-$I$-closed, we have $(Cl(A) - A) = \varnothing$ using Proposition 2. This shows that $A$ is a closed. □

**Theorem 6.** *Let $A$ be an g-$I$-closed set and $B$ be a closed set in any its $(X, \tau, I)$. Then, $(A \cap B)$ is a g-$I$-closed in X.*

*Proof:* Let $(A \cap B) \subset U$ and $U \in \tau$. Then, $A \subset (U \cup (X - B))$. Since $A$ is g-$I$-closed, we have $Cl(A) \subset (U \cup (X - B))$ and hence $(Cl(A) \cap B) \subset U$. Since $B$ is closed set, $Cl(A) \cap Cl(B) = (Cl(A) \cap B) \subset U$ and $Cl(A) \cap Cl(B) \subset U$ and hence $Cl(A \cap B) \subset U$. This shows that $(A \cap B)$ is a g-$I$-closed. □

**Theorem 7.** *Let $(X, \tau, I)$ be an ideal topological space and $A \subset Y \subset X$. If $A$ is an g-$I$-closed set in $X$, then $A$ is an g-$I$-closed set relative to $Y$.*

*Proof:* Assume that $A \subset (Y \cap U)$ and $U$ is an $I$-open set of $(X, \tau, I)$. Since $A$ is an g-$I$-closed set, $A \subset U$ and hence $Cl(A) \subset U$. It follows that $(Y \cap Cl(A)) \subset (Y \cap U)$. Since $(Y \cap U)$ is an $I$-open set of $(X, \tau_Y, I_Y)$ according to definition of subspace topology and Lemma 2.2, the proof is omitted. □

## 3 g-$I$-open sets

In this section we give a new definition of open set as complement of g-$I$-closed set. Besides, we obtain some properties of these sets.

**Definition 8.** *A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be generalized $I$-open set ( briefly, g-$I$-open ) if and only if $(X - A)$ is g-$I$-closed.*

**Theorem 8.** *A subset $A$ is a g-$I$-open in $(X, \tau, I)$ if and only if the following condition holds:*

*"$F \subset Int(A)$ whenever $F \subset A$ and $F$ is an $I$-closed set."*

*Proof:* Necessity. Assume that $A$ is an g-$I$-open and $F \subset A$, $F$ is an $I$-closed. Then, $(X - F)$ is an $I$-open and $(X - A) \subset (X - F)$. Since $A$ is an g-$I$-open, $(X - A)$ is an g-$I$-closed and hence $Cl(X - A) \subset (X - F)$. Therefore, we obtain that $F \subset (X - Cl(X - A)) = Int(A)$.

Sufficiency. Let $F \subset Int(A)$ whenever $F \subset A$ and $F$ is an $I$-closed set. We take $(X - A) = B$. Assume that $B \subset U$ where $U = (X - F)$ is an $I$-open set of $(X, \tau, I)$. Since $(X - A) \subset U$, $F \subset A$ and hence $F$ is an $I$-closed which $F \subset Int(A)$. Besides, we have $(X - Int(A)) \subset (X - F) = U$ and $(X - Int(X - B)) \subset U$. We have equivalently $Cl(B) \subset U$. This shows that $B$ is g-$I$-closed and hence $B$ is g-$I$-open. □

**Theorem 9.** *Let $(X, \tau, I)$ be an ideal topological space and $A, B$ subsets of $X$. If $A$ and $B$ are separated g-$I$-open sets, then $(A \cap B)$ is g-$I$-open.*

*Proof:* Let $F$ be a closed subset of $(A \cap B)$. Then, we have $(F \cap Cl(A)) \subset A$ and hence using Theorem 8, $(F \cap Cl(A)) \subset Int(A)$. Similarly, we obtain $(F \cap Cl(B)) \subset Int(B)$. Therefore, $F = (F \cap (A \cap B)) \subset ((F \cap Cl(A)) \cap (F \cap Cl(B))) \subset (Int(A) \cap Int(B)) = Int(A \cap B)$. Hence we have $F \subset (Int(A \cap B))$ and by Theorem 8 , $(A \cap B)$ is g-$I$-open. □

**Remark 3.** *The union of two g-$I$-open sets is not a g-$I$-open set in generally.*

**Example 3.** *Let $(X, \tau, I)$ be an ideal topological space as Example 1., i.e., X={a, b, c, d}, $\tau$={$\varnothing$, X, {a}, {b}, {a,b} } and I={$\varnothing$, {a}}. Although A={d} and B={a,c} are each g-$I$-open, but $(A \cup B)$={a,c,d} is not g-$I$-open.*

**Theorem 10.** *Let $A$ and $B$ are subsets of an ideal topological space $(X, \tau, I)$. If $(Int(A)) \subset B \subset A$ and $A$ is g-$I$-open, then $B$ is g-$I$-open.*

*Proof:* Since $(Int(A)) \subset B \subset A$ and $A$ is g-$I$-open, we have $(X - A) \subset (X - B) \subset Cl(X - A)$ and $X - A$ is g-$I$-closed. Using Theorem 4, we obtain that $X - B$ is g-$I$-closed. Therefore, $B$ is g-$I$-open. □

We conclude this section with the next theorem.

**Theorem 11.** *Let $A$ be a subset of an ideal topological space $(X, \tau, I)$. If $A$ is g-$I$-closed, then $(Cl(A) - A)$ is g-$I$-open.*

*Proof:* Assume that $A$ is g-$I$-closed and $F \subset (Cl(A) - A)$, $F$ is closed. According to Proposition 3, $F = \varnothing$ and hence $F \subset Int(Cl(A) - A)$. Finally, we obtain that $(Cl(A) - A)$ is g-$I$-open using Theorem 8.

□

We must state that the question of when is the converse of Theorem 11 true is left as an open question.

## 4 $I$-$T^*_{1/2}$ spaces

In this section, we introduce a new separation axiom which is called $I$-$T^*_{1/2}$ space and give some properties of it.

**Definition 9.** *[7] An ideal topological space $(X, \tau, I)$ is said to be $I - T_1$ if for each pair of distinct points $x$ and $y$ of $X$, there exists an $I$-open set $U$ of $X$ containing $x$ but not $y$ and $I$-open set $V$ of $X$ containinig $y$ but not $x$.*

**Lemma 4.** *[7] $(X, \tau, I)$ is $I$-$T_1$ iff every subset of $X$ is a $\Lambda_I$-set.*

We have the following conclusion using Corollary 1 and Lemma 4. In it, we partially have answered the question of when the inverse of Remark 1 is true. Therefore, we think that it is important.

**Corollary 2.** *In every $I$-$T_1$-space, the notions of g-$I$-closed set and closed set are coincident.*

Now, we introduce a new separation axiom which is related to Remark 1.

**Definition 10.** *An ideal topological space $(X, \tau, I)$ is said to be $I$-$T^*_{1/2}$ if every g-$I$-closed set is closed.*

**Theorem 12.** *For an ideal topological space $(X, \tau, I)$, the following property holds:"$(X, \tau, I)$ is $I$-$T^*_{1/2}$-space iff each singleton of X is open or $I$-closed.*

*Proof:* Necessity.Let $(X, \tau, I)$ be $I$-$T^*_{1/2}$-space and $x \in X$. Suppose that $\{x\}$ is not $I$-closed. Then ($X$-$\{x\}$) is not $I$-open and hence ($X$-$\{x\}$) is g-$I$-closed. Since $(X, \tau, I)$ be $I$-$T^*_{1/2}$, ($X$-$\{x\}$) is closed and hence $\{x\}$ is open in $(X, \tau, I)$.

Sufficiency. Suppose that $A$ is a g-$I$-closed set. We will show that $Cl(A) \subset A$. Let $x$ be any point of $Cl(A)$. Then $\{x\}$ is open or $I$-closed.

(a) In case $\{x\}$ is open. According to Proposition 2, $(Cl(A) - A)$ does not contain any nonempty open set. Therefore, $x \notin (Cl(A) - A)$, but $x \in Cl(A)$. Hence $x \in A$.

(b) In case $\{x\}$ is $I$-closed. According to Proposition 3, $Cl(A) \subset A$ does not contain any nonempty $I$-closed set. Therefore, $x \notin Cl(A) \subset A$, but $x \in Cl(A)$. Hence $x \in A$.

According to (a) and (b), we have $Cl(A) \subset A$ and hence $A$ is closed. □

**Theorem 13.** *Every $I$-$T_1$-space is a $I$-$T^*_{1/2}$-space.*

*Proof:* Let $(X, \tau, I)$ be $I$-$T_1$-space and $A$ be a g-$I$-closed set in $(X, \tau, I)$. Using Corollary 1 and Lemma 5, we have $A$ is a closed set. This shows that $(X, \tau, I)$ is a $I$-$T^*_{1/2}$-space. □

 

**Remark 4.** *The converse of Theorem 13 isn't true in generally as shown the following example.*

**Example 4.** *Let $(X, \tau, I)$ be an ideal topological space such that $(X, \tau)$ be a Sierpinski space, i.e., $X$={0,1}, $\tau = \{\varnothing$, X, {O}} and I={$\varnothing$, {0}}. $(X, \tau, I)$ is a $I$-$T^*_{1/2}$-space, but not $I$-$T_1$-space.*

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# Integrated Fuzzy-Analytical Hierarchy Process (F-AHP) and Technique for Preference by Similarity to the Ideal Solution (TOPSIS) in Recommending Extracurricular Program Selection

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**Abstract:** School Extracurricular Program provides students with self-development activities. The extensive options of the extracurricular program are due to the bewilderment for students and parents in selecting the proper program, thus impacting an unoptimal student's potential construction. This paper considers the parents, schools, and Psychiatric Test evaluation through the determination of the student's level of intelligence, concentration, memory, commitment to the task, willingness, creativity, experience, health history, and parents' recommendation as criteria for the Decision Support System (DSS) establishment. Two stakeholders from the counseling experts are asked their opinions on weighting the significant values of the criteria using an F-AHP of a 1 to 9-scale questionnaire. As a result, the powerful values of criteria contribution to selection are then performed. Meanwhile, the TOPSIS approach is used for the alternative proposed ranking that includes basketball, football, volleyball, choir, dance, music, theater, mathematics and natural sciences Olympiad, social science Olympiad, National Flag Hoisting Troop, and debate programs. By touching on the momentous level of F-AHP criteria, the TOPSIS formula ranks 30 students' performance to obtain matchless program preference. A prototype integrated multi attributes DSS system has been successfully developed as an automatic recommendation system in aiding the student's selection for the alternative program. Herein, Black Box, User Acceptance Test (UAT), and sensitive analysis testing showed the potential contribution of this DSS system in presenting the optimal solution.

**Keywords:** Decision Support System, Fuzzy-Analytical Hierarchy Process, Multi-Criteria Decision Making, School Extracurricular Program, Technique for Preference by Similarity to the Ideal Solution.

## 1 Introduction

Student self-development is defined as their self-involvement in making the potential expansion of thought and intelligence through various learning activities thus, it directly guides their energy for improvement [1]. School Extracurricular Program is one sample of school activities supporting the student's self-development. Indonesia Minister of Education and Culture No 62, the Year 2014, regulates elementary and secondary education to carry out students' activities outside of school learning hours and under supervision in terms of the extracurricular program in order to give the optimal cultivation of students' potential, talents, interests, abilities, personality, cooperation, and independence in encouraging the achievement of maximal educational goals [2]. The school extracurricular program affects students' academic and non-academic outcomes. This program habituates the students' values of Sportif, honesty, discipline, and empathy which are effective for the student's future construction [3].

Unfortunately, this program has been underestimated and lacks school and student attention [4]. Moreover, the sheer number of extracurricular programs offered causes the students confusion in finding the proper program [5]. In a nutshell, this program fails to trigger optimal student potential self-development. Therefore, developing the DSS system becomes a new challenge and solution in administering the appropriate multi-criteria analysis for school extracurricular program selection thus, the chosen program fits into the students' potential inquiry [6]. Previous studies found that DSS administers computer-based tools that have been adapted to support and aid complex problem-solving in decision-making. DSS allows the decision-maker to develop and compare different configurations and multi-scenario techniques in a user-friendly computer environment [7]. DSS implementation has been recognized for successfully addressing several real-world case studies in data analytical suggestions. Okfalisa et al. (2018) developed a DSS-based AHP and Objective Matrix (OMAX) within the Balanced Scorecard Dashboard Model approach to measuring the university strategy execution achievement [8]. Okfalisa et al. (2021) have been successfully increasing the effectiveness of decision-making in the Field Experience Program (FEP) placement under the needs and expectations of the

 

entire stakeholders, including management universities, students, and schools [9]. Okfalisa et al. (2022) applied DSS based dashboard model with F-AHP as a new feature in assessing Small and Medium Enterprises' digitalization readiness [10]. Meanwhile, Bakir and Atalik (2021) applied F-AHP and Fuzzy Marcos within the DSS approach for evaluating the e-service quality in the airline industry [11]. In general, the DSS allows researchers to approach different real case studies, testing the effectiveness of models and heuristics in providing performing solutions and creating knowledge over the most critical and recurrent storage issues for optimal analysis.

Based on the above-described functionalities, the proposed DSS in this paper is developed to cover decision-makers perceptions and needs, including school management, parents, and students, to determine the convenient extracurricular program. The criteria proposed are based on the school's psychological test result, including the student's level of intelligence, concentration, memory, commitment to the task, willingness, creativity, experience, health history, and parents' recommendation. Here in, 30 students' case studies are identified to be analyzed and ranked based on the prospective criteria and alternatives using the F-AHP and TOPSIS approach. 11 intended alternatives programs at Senior High School No.1 in Pekanbaru, Riau, as well as basketball (A01), football (A02), volleyball (A03), choir (A04), dance (A05), music (A06), theater (A07), mathematics and natural sciences Olympiad (A08), social science Olympiad (A09), National Flag Hoisting Troop (A10), and debate (A11). This Senior High School No. 1 is chosen due to the high quality of this school's accreditation and performance in Riau Province.

Furthermore, the DSS with integrated F-AHP and TOPSIS approach offers stakeholders the opportunity to simulate the extracurricular program analysis using the criteria weighting of F-AHP and alternatives TOPSIS rank, which results in an optimal recommendation program for the students. The list conducting testing as well as Blackbox, UAT for software testing, and sensitivity analysis for DSS approach testing defined the DSS's success in providing the best solution.

## 2 Literature reviews

### *2.1 F-AHP for criteria weighting*

F-AHP is a multi-attribute decision-making (MADM) approach that accommodates the benefit of fuzzy calculation in enhancing the functional hierarchy of AHP [12, 13]. The F-AHP equips the linguistics reasoning leverages designed in simple equations, easy to understand, high tolerance in data accuracy, and adaptable in the complex nonlinear functional model. In addition, the fuzzy figuring better and more administers the vague decision description than AHP [14]. Besides the potential advances. Criteria Construction of F-AHP accounts for this approach to be integrated with another MADM approach. Awasthi et al. (2018) applied F-AHP and multi-criteria optimization and compromise solution (VIKOR) based approach in selecting the multitier sustainable global supplier [15]. Sirisawat and Kiatcharoenpol (2018) solved the problem by prioritizing solutions for reverse logistics barriers using F-AHP and TOPSIS [16]. Blagojevic et al. (2020) incorporated F-AHP and Data Envelopment Analysis (DEA) in measuring the efficiency of freight transport railway undertaking [17]. In a nutshell, F-AHP was effectively used in weighting the criteria, thus successfully indicating the level of the significant contribution of each hierarchy construct and sub-constructs. Moreover, the fuzzy calculation provides a more precise primary picture of criteria selection thus, it decreases the respondents' or decision makers' ambiguity in evaluating the criteria.

The formula for F-AHP calculation is explained as follows [13, 18]:

1. Creating a hierarchical structure of Extracurricular Program selection and performing pairwise matrix comparisons between nine criteria using the nine Saaty scale.
2. Calculating the maximum values of consistency ($\lambda_{max}$), Consistency Index (CI), and CR using the following formula.

$$\lambda_{max} = \frac{Total\ Matrix\ Sum}{Sum\ of\ Criteria} \tag{1}$$

$$CI = \left(\frac{\lambda_{max} - n}{n - 1}\right) \tag{2}$$

$$CR = \left(\frac{CI}{RI}\right) \tag{3}$$

If $CR \leq 0.1$ indicates a consistent matrix.

Where $n$ defines as the number of elements/criteria; $\lambda_{max}$ designates as the result of multiplying the number of columns with the eigenvectors, and the value of $CR \leq 0.1$ to show the matrix consistency.

3. Converting the AHP pairwise comparison matrix into TFN (Triangular Fuzzy Number) scale.
4. Calculating the synthetic value of the Synthetic fuzzy extent ($Si$) by considering the values of $M$ as $TFN$ numbers, m as total criteria, $i$ and $j$ as line and column in matrix performed respectively, and $g$ as the level of the parameter in low ($l$), middle ($m$), and upper ($u$).

$$Si = \sum_{j=1}^{m} M_{gi}^{j} \frac{1}{\sum_{i=1}^{m} \sum_{j=1}^{m} M_{gi}^{j}} \tag{4}$$

5. Determine the value of the vector ($V$) and the value of the defuzzification ordinate ($d'$). Where by $M_2 = (l_2, m_2, u_2) \geq M_1 = (l_1, m_1, u_1)$ can be defined as a vector value.

$$V(M_2 \geq M_1) = sup[min(\pi M_1(x)), min(\pi M_2(x))] \tag{5}$$

$$V(M_2 \geq M_1) = \begin{cases} 1\ ;\ m_2 \geq m_1 \\ 0\ ;\ l_1 \geq u_2 \\ \dfrac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)}\ ;\ \text{other than above} \end{cases} \tag{6}$$

For $k = 1, 2, \ldots, n$; $k \neq i$, then the vector weight value ($W$) is obtained through the formula below.

$$W' = (d'(A_1), d'(A_2), \ldots, d'(A_n))^T \tag{7}$$

6. Normalization of fuzzy vector weight values is the calculated with the following formula with the conditional value of $W$ is a non-fuzzy number.

$$d(A_n) = \frac{d'}{\sum_{i=1}^{n} d(A_n)} \tag{8}$$

$$W = (d(A_1), d(A_2), \ldots, d(A_n))^T \tag{9}$$

### 2.2 TOPSIS for alternatives ranking

Similar to AHP, TOPSIS is a multi-criteria decision-making method that considers the geometric Euclidean distance using the smallest distance values of positive ideal solutions and the most extensive distance values of negative ideal solutions from the proposed alternatives [19]. Besides, the weighted score for each criterion is also provided by the TOPSIS approach, thus, it administers the distinctive evaluation of criteria [20]. The TOPSIS formula is easy to understand, put forward the rational concept, and efficient, fast, and uncomplicated computational calculations. Moreover, the TOPSIS hands over the measurement of alternative performance and decisions in a simple outcome [21]. The TOPSIS approach has been widely used and applied to solve several cases. Wang et al. (2020) practiced the integration of AHP and TOPSIS in evaluating the internet security system [22]. Lei et al. (2020) developed the mechanism of suppliers' selection with probabilistic linguistic information using TOPSIS [23]. Akram et al. (2020) employed the intertwin of ELECTRE-I and TOPSIS in handling the group decision-making under a complex Pythagorean fuzzy environment [24]. In a nutshell, the above review successfully showed the potential contribution of TOPSIS for multi-criteria decision-making mechanisms and the high challenges of this approach to be integrated with others to enhance the sensitivity values of decision-making. Therefore, the effectiveness of F-AHP in weighting criteria is applied for evaluating the significance of the student's level of intelligence, concentration, memory, commitment to the task, willingness, creativity, experience, health history, and parents' recommendation. Furthermore, reviewing alternatives is impressively calculated using the leverage of TOPSIS.

The step activities of TOPSIS calculation is disclosed bellows [19]:

1. Determine the normalized decision matrix as follows

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \tag{10}$$

2. Calculate the weighted normalized decision matrix. The positive ideal solution $(+)$ and the negative ideal solution can be determined $(-)$ based on the normalized weight rating $(y_{ij})$ as below

$$y_{ij} = w_i r_{ij} \tag{11}$$

3. Calculate the positive ideal solution matrix $(A^+)$ and the negative ideal solution matrix $(A^-)$ as follows

$$A^+ = (y_1^+, y_1^+, \ldots, y_n^+) \tag{12}$$

$$A^- = (y_1^-, y_1^-, \ldots, y_n^-) \tag{13}$$

With assumptions as bellows

$$y_j^+ = \begin{cases} \text{Max } y_{ij} \text{ if } j \text{ is a benefit attribute} \\ \text{Min } y_{ij} \text{ if } j \text{ is the attribute cost} \end{cases} \qquad y_j^- = \begin{cases} \text{Max } y_{ij} \text{ if } j \text{ is a benefit attribute} \\ \text{Min } y_{ij} \text{ if } j \text{ is the attribute cost} \end{cases} \tag{14}$$

4. Calculate the distance $(d)$ between the values of each alternative $(v)$ with the positive ideal solution matrix and the negative ideal solution matrix.

$$d_1^* = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^*)^2}, i = 1, \ldots, m \tag{15}$$

$$d_1^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2}, i = 1, \ldots, m \tag{16}$$

5. Calculating preference values for each alternative as formula below

$$CC_i = \frac{d_1^-}{d_1^* + d_1^-}, i = 1, \ldots, m \tag{17}$$

### 2.3 Criteria construction

The criteria were constructed based on the school's psychiatric test results and several interviewees with the school management, teachers, and parents. The criteria components are described in Table 1.

 

| Code | Criteria | Sub-Criteria | Definition and References | Attributes |
|---|---|---|---|---|
| C01 | The student's level of intelligence | Low ($\leq$ 90)<br>Average (91 – 119)<br>High ($\geq$ 120) | Psychiatric test result [25] | Benefit |
| C02 | Concentration | Low<br>Average<br>High | Psychiatric test result [26] | Benefit |
| C03 | Memory | Low<br>Average<br>High | Psychiatric test result [27] | Benefit |
| C04 | Commitment to the task | Low<br>Average<br>High | Psychiatric test result [28] | Benefit |
| C05 | Willingness | Low<br>Average<br>High | Psychiatric test result [29] | Benefit |
| C06 | Creativity | Low<br>Average<br>High | Psychiatric test result [30] | Benefit |
| C07 | Experience | Low (No competition experience)<br>Average (1 – 3 competition experiences)<br>High (> 3 competition experiences) | [31] | Benefit |
| C08 | Health history | Low (No illness)<br>Average (Minor illness)<br>High (Major illness) | [32] | Cost |
| C09 | Parents' recommendation | Low (No permission)<br>Average (Under supervision)<br>High (High permission) | [33] | Benefit |

**Table 1** The criteria construction.

## 3 Research method

This study accommodates the integration of DSS F-AHP and TOPSIS in handling the domain decision-making to recommend the optimal and appropriate extracurricular program selection for high school students. Herein, nine attributes and eleven alternatives are constructed through thorough theoretical reviews and interviews thus, it was defined as criteria as depicted in Table 2. The criteria are defined by considering the parents, schools, and Psychiatric Test evaluation and results to enhance the optimal suggestion of this approach. Two stockholders from the counseling experts are asked their opinions on weighting the significant values of the criteria using an F-AHP of a 1 to the 9-scale questionnaire. Herein, the F-AHP algorithm in Equation (1-9) is traced to calculate and provide the analysis values of each criterion. As a result, the significant level values of criteria are then performed. Next, by touching on the momentous level of F-AHP criteria, the TOPSIS formula (Eqs. 10 – 17) works to rank 30 students' performance to obtain matchless program preference. In ensuring the quality analysis of this hybrid approach, a sensitivity value is determined based on the minimum range values of variables ($X1$ as the first alternative, $X2$ as the second alternative, and $X$ as the value of the variable) in regression. The formulas are explained below.

$$Total\ sensitivity = (X1 - X2) \tag{18}$$

$$Total\ sensitivity = \frac{X_i}{\Sigma X} \tag{19}$$

$$Total\ sensitivity = \frac{1}{2} \times (X1 + X2) \tag{20}$$

The automatic mechanism of the integrated F-AHP and TOPSIS approach is evolved through the development of DSS software, namely Extracurricular Program Recommendation (EPR). This application applied prototyping software model development and has tested Blackbox and UAT for the software efficacy. The entire respondents, including the experts and students, were designed as actors for this software.

## 4 Discussion

### *4.1 Criteria performance analysis - F-AHP*

Following the step process of F-AHP from Equation 1 to 9:

1. The hierarchical structure is performed in Figure 1.

Figure 1 explains the hierarchy analysis level for this recommendation system, including level 1 as the system objective and level 2 for criteria. Meanwhile, level 3 for possible alternatives is proposed. Then, the comparison matrix with the AHP scale is defined in Table 2 with the maximum Ratio Index, Consistency Ratio ($CR$), and Consistency Index ($CI$) values are 1.46, 0.084, and 0.122, respectively. This result showed the consistency of this matrix whereby $CR \leq 0.1$.

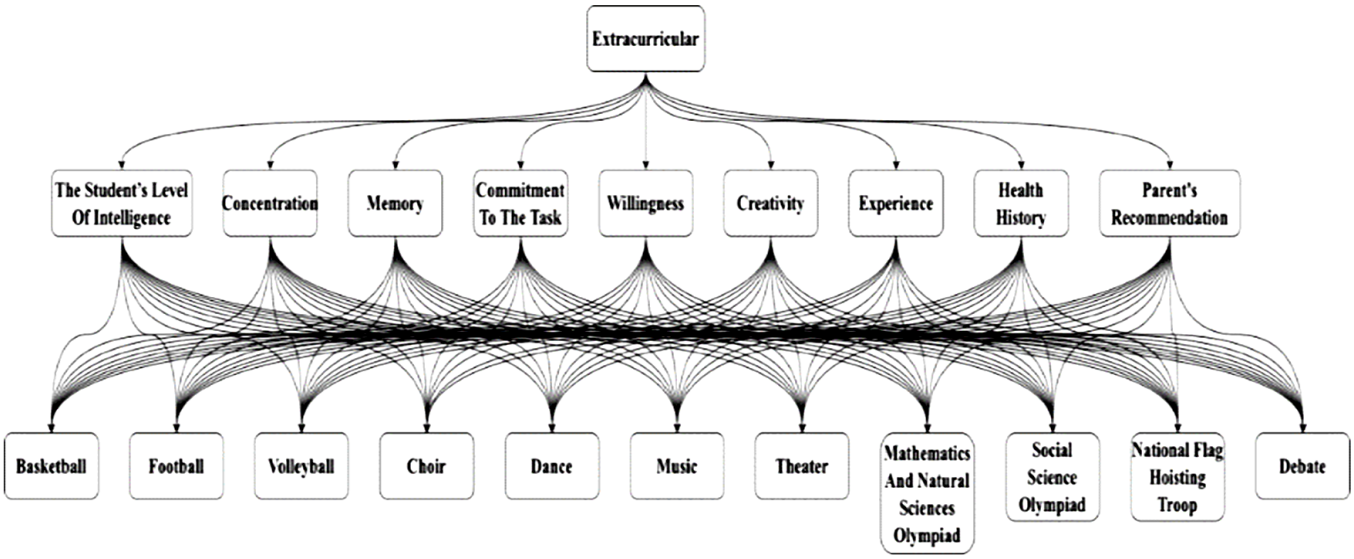


**Fig. 1**: Hierarchical structure of extracurricular program recommendation.

| | | | Compari | son Matrix | between | Criteria | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Criteria** | **C01** | **C02** | **C03** | **C04** | **C05** | **C06** | **C07** | **C08** | **C09** |
| C01 | 1 | 3 | 2 | 3 | 2 | 2 | 3 | 3 | 3 |
| C02 | 0.333 | 1 | 3 | 0.5 | 0.333 | 0.333 | 3 | 3 | 3 |
| C03 | 0.5 | 0.333 | 1 | 0.333 | 0.333 | 0.333 | 3 | 3 | 3 |
| C04 | 0.333 | 2 | 3 | 1 | 0.333 | 0.5 | 3 | 3 | 3 |
| C05 | 0.5 | 3 | 3 | 3 | 1 | 2 | 3 | 3 | 3 |
| C06 | 0.5 | 3 | 3 | 2 | 0.5 | 1 | 3 | 3 | 3 |
| C07 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 1 | 2 | 3 |
| C08 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.5 | 1 | 3 |
| C09 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 0.333 | 1 |

**Table 2** AHP comparison matrix between criteria.

2. Matrix TFN and Synthetic Calculation The calculation of Equation 4 to 9 are described in Table 3. Table 3 pointed out that the normalized criteria weight values is on scale of 0 – 1 with priority performance reaches C05 (willingness) as the most significant criteria than following by C06 (creativity), C01 (The student's level of intelligence), C04 (commitment to the task), C02 (concentration), C03 (memory), C08 (health history), C07 (experience), and C09 (parents' recommendation), respectively.

| **Criteria** | | **TFN** | | | **Si** | | **W'** | **W** |
|---|---|---|---|---|---|---|---|---|
| | **L** | **M** | **U** | **L** | **M** | **U** | | |
| C01 | 7.5 | 11.5 | 15.5 | 0.06 | 0.13 | 0.25 | 0.97 | 0.13 |
| C02 | 7.16 | 10 | 14 | 0.06 | 0.12 | 0.23 | 0.87 | 0.12 |
| C03 | 6.66 | 9.16 | 13 | 0.05 | 0.11 | 0.21 | 0.82 | 0.11 |
| C04 | 7.16 | 10.3 | 14.5 | 0.06 | 0.12 | 0.24 | 0.89 | 0.12 |
| C05 | 8.16 | 12 | 16.5 | 0.07 | 0.14 | 0.27 | 1 | 0.13 |
| C06 | 7.83 | 11.5 | 16.5 | 0.06 | 0.13 | 0.27 | 0.97 | 0.13 |
| C07 | 5.5 | 7.5 | 10.5 | 0.05 | 0.087 | 0.17 | 0.66 | 0.09 |
| C08 | 5.66 | 7.5 | 11 | 0.05 | 0.087 | 0.18 | 0.68 | 0.09 |
| C09 | 5 | 6.33 | 9 | 0.04 | 0.074 | 0.15 | 0.58 | 0.07 |

**Table 3** F-AHP calculation.

### *4.2 Criteria performance analysis - F-AHP*

TOPSIS ranks the alternatives by considering the 30 students matching and weighting criteria set from 1 to 3 for low to high-performance assessment. Ensuing the TOPSIS formula at Equations 10 to 17, the analysis is performed as explained in Table 4.

The three highest score recommendations for 30 students are disclosed in Table 5. For example, student S01 suggested Football (0.565) as the first system recommendation, followed by MIPA Olympiad (0.532) and debate (0.529), respectively.

 

| Alternative | Ideal Solution | Distance | Preference | Value Ranking |
|---|---|---|---|---|
| | Positive | Negative | Preference Value | Rank |
| A01 | 0.080 | 0.0723 | 0.474 | 8 |
| A02 | 0.065 | 0.0856 | 0.565 | 1 |
| A03 | 0.083 | 0.0687 | 0.452 | 9 |
| A04 | 0.075 | 0.077 | 0.504 | 5 |
| A05 | 0.0765 | 0.0762 | 0.499 | 6 |
| A06 | 0.0845 | 0.0672 | 0.443 | 10 |
| A07 | 0.0883 | 0.0622 | 0.413 | 11 |
| A08 | 0.0714 | 0.081 | 0.532 | 2 |
| A09 | 0.0796 | 0.073 | 0.478 | 7 |
| A010 | 0.0746 | 0.0781 | 0.511 | 4 |
| A011 | 0.0719 | 0.0806 | 0.529 | 3 |

**Table 4** Topsis calculation.

| Student ID | | Alternative Recommendation | |
|---|---|---|---|
| | Alternative Code | Alternative Name | Preference Value |
| S01 | A02 | Football | 0.565 |
| | A08 | MIPA Olympiad | 0.532 |
| | A11 | Debate | 0.529 |
| S02 | A08 | MIPA Olympiad | 0.582 |
| | A09 | Social Science Olympiad | 0.511 |
| | A10 | National Flag Hoisting Troop | 0.479 |
| S03 | A02 | Football | 0.528 |
| | A11 | Debate | 0.506 |
| | A04 | Choir | 0.498 |
| ... | ... | ... | ... |
| S30 | A08 | MIPA Olympiad | 0.554 |
| | A09 | Social Science Olympiad | 0.497 |
| | A10 | National Flag Hoisting Troop | 0.491 |

**Table 5** students program recommendation.

### *4.3 Sensitivity test analysis*

Pursuing the sensitivity analysis in Equations 18 to 20, the average values for 30 students are 0.03, 0.349, and 0.540 for sensitivity 1, 2, and 3, respectively. This calculation shows the positive value of the sensitivity test for this approach.

### *4.4 EPR software development*

ERP Software is developed through the development of system architecture and prototyping software development life cycle model. Two kinds of the mechanism of this integration for F-AHP and TOPSIS employ two interfaces as well as questionnaires to accommodate the communication between actors. One questionnaire form for F-AHP interaction, and the other for TOPSIS. The generated interface form points out the student-suggested program based on the integration approach analysis and rank. The Blackbox test found that the entire function and modules run well, including login, questionnaires, criteria, alternatives, alternative weight, calculation, historical, graph, and password management. Twenty-two questions are delivered for User Acceptance Test (UAT) to identify the user compliance regarding the usefulness and friendly use of ERP applications. As a result, 91.45% of thirty-five users agree that this application can aid in recommending the optimal extracurricular program for the students.

## 5 Conclusion

The application of the hybrid DSS F-AHP and TOPSIS has been successfully developed. This ERP system analyzes the extracurricular program for the students by forasmuch as the student's level of intelligence, concentration, memory, commitment to the task, willingness, creativity, experience, health history, and parents' recommendation. ERP system administers the first three ranks alternatives based on the preference values to suggest the optimal matching extracurricular program for students. A list of software testing has been conducted to ensure the efficacy of the ERP system's calculation, usability, and functionality. As this application's actors, students and parents show their satisfaction with the recommendation provided.

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# On Gould-Hopper Based Fully Degenerate Type2 Poly-Euler Polynomials with a $q$ Parameter

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**Abstract:** In this paper, we consider the Gould-Hopper based fully degenerate type2 poly-Euler polynomials with a $q$ parameter and provide some of their properties. Moreover, we derive multifarious correlations and identities for these polynomials, including recurrence relations, symmetric property, and implicit summation formulas.

**Keywords:** Euler polynomials, Gould-Hopper polynomials, Poly-Euler polynomials, Stirling numbers of the first kind.

## 1 Introduction

For $\lambda \in \mathbb{C}$, the $\lambda$-falling factorial $(x)_{n,\lambda}$ is defined by (see [2, 3, 7, 8, 11, 13])

$$(x)_{n,\lambda} = \begin{cases} x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), & n=1,2,\ldots \\ 1, & n=0. \end{cases} \tag{1}$$

In this case $\lambda = 1$, the $\lambda$-falling factorial reduces to the familiar falling factorial as follows

$$(x)_{n,1} := (x)_n = x(x-1)\cdots(x-n+1) \text{ and } (x)_0 = 1.$$

The $\Delta_{\lambda,x}$ difference operator with respect to $x$ is defined by (cf. [2, 3, 7, 8, 11, 13])

$$\Delta_{\lambda,x} f(x) = \frac{1}{\lambda}(f(x+\lambda) - f(x)), \quad \lambda \neq 0. \tag{2}$$

The degenerate exponential function $e_\lambda^x(t)$ is defined as follows (see [2, 3, 7, 8, 11, 13])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \text{ and } e_\lambda^1(t) = e_\lambda(t). \tag{3}$$

It is readily seen that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$. From (1) and (3), we obtain the following relation

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{4}$$

which satisfies the following difference rule

$$\Delta_{\lambda,x} e_\lambda^x(t) = t e_\lambda^x(t). \tag{5}$$

Note that $e_\lambda^1(t) := e_\lambda(t)$. The degenerate logarithm function is defined as follows

$$log_\lambda(1+t) := \frac{(1+t)^\lambda - 1}{\lambda} = \frac{1}{\lambda}\sum_{\ell=1}^{\infty} (\lambda)_\ell \frac{t^\ell}{\ell!},$$

which holds the following relations with the degenerate exponential function:

$$log_\lambda(e_\lambda(1+t)) = 1+t \text{ and } e_\lambda(log_\lambda(1+t)) = 1+t.$$

The classical Euler polynomials $E_n(x)$ and the degenerate Euler $E_{n,\lambda}(x)$ polynomials are given as follows:

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt} \text{ and } \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1}e_\lambda^x(t).$$

One can look at the references [4, 5, 7, 8], [10]-[12], [14] to see the various applications of Euler polynomials.

The Gould-Hopper polynomials are defined by the following generating function (see [2, 3, 6, 10], [13]-[15] )

$$\sum_{n=0}^{\infty} GH_n^{(j)}(x,y)\frac{t^n}{n!} = e^{xt+yt^j}, \tag{6}$$

where $j \in \mathbb{N}$ with $j \geq 2$. In this case $j = 1$, the corresponding bivariate polynomials are simply expressed by the Newton binomial formula. Upon setting $j = 2$ in (6) gives the classical Hermite polynomials $GH_n^{(2)}(x,y)$ (see [2, 3, 6, 10], [13]-[15]) and the mentioned polynomials have been used to define bivariate extensions of some special polynomails, such as Bernoulli and Euler polynomials.

The Stirling numbers of the first kind $S_1(n,k)$ and the Stirling numbers of the second kind $S_2(n,k)$ are defined (*cf.* [2, 3, 6, 9, 10],[13]-[16]) by means of the following generating functions:

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!} \text{ and } \frac{\left(e^t-1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{7}$$

From (7), we get the following relations for $n \geq 0$:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k. \tag{8}$$

Recently, Kim-Kim [9] performed to generalize the degenerate Bernoulli polynomials by using the polyexponential function

$$\mathrm{Ei}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!n^k} \tag{9}$$

as inverse to the polylogarithm function

$$Li_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k} \qquad (|t| < 1; k \in \mathbb{Z}) \tag{10}$$

provided by

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{11}$$

Upon setting $x = 0$ in (11), $\beta_{n,\lambda}^{(k)}(0) := \beta_{n,\lambda}^{(k)}$ are called the degenerate poly-Bernoulli numbers. Kim et al. [9] studied the degenerate poly-Bernoulli polynomials and also gave some explicit expressions and several formulas for those polynomials. Since $\mathrm{Ei}_1(t) = e^t - 1$, it is worthy to note that

$$\beta_{n,\lambda}^{(1)}(x) := B_{n,\lambda}(x).$$

The degenerate form of the Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ is defined by means of the following generating function:

$$\frac{(log_\lambda(1+t))^m}{m!} = \sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!}$$

Let $n, j \in \mathbb{Z}$ with $n \geqq 0$, $j > 0$ and let $q \in \mathbb{R}/\{0\}$ with $q \neq 0$. The fully degenerate Gould-Hopper polynomials with a $q$ parameter are defined by the following generating function to be

$$\sum_{n=0}^{\infty} GH_{n,\lambda;q}^{(j)}(x,y)\frac{t^n}{n!} = e_\lambda^x(qt)\,e_\lambda^y\left(qt^j\right). \tag{12}$$

When $\lambda \to 0$ and $q \to 1$, we have the Gould-Hopper polynomials denoted by $GH_n^{(j)}(x,y)$ (*cf.* [2, 3, 13]).

 

The fully degenerate Gould-Hopper polynomials with a $q$ parameter have the following representation

$$GH_{n,\lambda;q}^{(j)}(x,y) = n! \sum_{k=0}^{\lfloor n/j \rfloor} \frac{(x)_{n-jk,\lambda}(y)_{k,\lambda}}{(n-jk)!k!} q^{n-(j-1)k},$$

where $\lfloor \cdot \rfloor$ is the Gauss notation, and represents the maximum integer which does not exceed the number in the square brackets and where we used the following elementary series manipulation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/j \rfloor} A(k,n-jk).$$

Also note that the following difference rules hold (*cf.* [2, 3, 13])

$$\Delta_{\lambda,x} GH_{n,\lambda;q}^{(j)}(x,y) = qn\, GH_{n-1,\lambda;q}^{(j)}(x,y), \tag{13}$$

$$\Delta_{\lambda,y} GH_{n,\lambda;q}^{(j)}(x,y) = q(n)_j\, GH_{n-j,\lambda;q}^{(j)}(x,y). \tag{14}$$

## 2 The Gould-Hopper based fully degenerate type2 poly-Euler polynomials with a $q$ parameter

In this section, we deal with the Gould-Hopper based fully degenerate type2 poly-Euler polynomials with a $q$ parameter. Then, we investigate their diverse relations and properties.

**Definition 1.** *Let $n, m \in \mathbb{N}_0$ and $j \in \mathbb{N}$ and let $q \in \mathbb{R}/\{0\}$. The Gould-Hopper based fully degenerate Stirling polynomials of the first kind with a q parameter are defined as follows (cf. [13]):*

$$\sum_{n=m}^{\infty} S_{1,\lambda;q}^{(j)}(n,m:x,y)\frac{t^n}{n!} = \frac{(\log_\lambda(1+qt))^m}{m!} e_\lambda^x(qt)\, e_\lambda^y\left(qt^j\right). \tag{15}$$

Some properties of Gould-Hopper based fully degenerate Stirling polynomials of the first kind with a $q$ parameter are as follows:

$$S_{1,\lambda;q}^{(j)}(n,m:x,y) = \sum_{s=0}^{n}\binom{n}{s} S_{1,\lambda;q}(s,m)\, GH_{n-s,\lambda;q}^{(j)}(x,y),$$

$$S_{1,\lambda;q}^{(j)}(n,m:x,y) = \sum_{s=0}^{n}\binom{n}{s}\lambda^s q^s \left(\frac{x}{\lambda}\right)_s S_{1,\lambda;q}^{(j)}(n-s,m:0,y),$$

$$S_{1,\lambda;q}^{(j)}(n,m:x,y) = n! \sum_{s=0}^{\lfloor n/j \rfloor} \frac{\lambda^s q^s}{s!(n-js)!}\left(\frac{y}{\lambda}\right)_s S_{1,\lambda;q}^{(j)}(n-js,m:x),$$

$$\Delta_{\lambda,x} S_{1,\lambda;q}^{(j)}(n,m:x,y) = qnS_{1,\lambda;q}^{(j)}(n-1,m:x,y)$$

and

$$\Delta_{\lambda,y} S_{1,\lambda;q}^{(j)}(n,m:x,y) = q(n)_j\, S_{1,\lambda;q}^{(j)}(n-j,m:x,y)$$

hold for $j \in \mathbb{N}$ and $q \in \mathbb{R}/\{0\}$.

Now we give the following definition.

**Definition 2.** *Let $n, k, j \in \mathbb{Z}$ with $n \geqq 0$, $k, j > 0$ and let $q \in \mathbb{R}/\{0\}$ with $q \neq 0$. We introduce the Gould-Hopper based fully degenerate type2 poly-Euler polynomials with a q parameter by means of the following generating function:*

$$\frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)}{t\left(1+e_\lambda^{-1}(qt)\right)} e_\lambda^x(qt)\, e_\lambda^y\left(qt^j\right) = \sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^n}{n!}. \tag{16}$$

Upon setting $x = 0 = y$, we then get ${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(0,0) := {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}$ which are called the Gould-Hopper based fully degenerate type2 poly-Euler numbers with a $q$ parameter, see [8].

Some special cases of ${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)$ are listed in the following remark.

**Remark 1.** 1. *When $\lambda \to 0$, we obtain the Gould-Hopper based type2 poly-Euler polynomials with a q parameter denoted by ${}_{GH}\mathfrak{E}_{n;q}^{(k,j)}(x,y)$.*
2. *When $q \to 1$ , we get the Gould-Hopper based fully degenerate type2 poly-Euler polynomials denoted by ${}_{GH}\mathfrak{E}_{n,\lambda}^{(k,j)}(x,y)$.*

3. *When $y=0$, we have the fully degenerate type2 poly-Euler polynomials with a q parameter denoted by $\mathfrak{E}_{n,\lambda;q}^{(k)}(x)$:*

$$\frac{2q\,\mathrm{Ei}\left(\frac{log_{\lambda}(1+qt)}{q}\right)}{t\left(1+e_{\lambda}^{-1}(qt)\right)}e_{\lambda}^{x}(qt)=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda;q}^{(k)}(x)\frac{t^{n}}{n!}.$$

4. *When $x=y=0$, we have the fully degenerate type2 poly-Euler numbers with a q parameter denoted by $\mathfrak{E}_{n,\lambda;q}^{(k)}$:*

$$\frac{2q\,\mathrm{Ei}_{k}\left(\frac{log_{\lambda}(1+qt)}{q}\right)}{t\left(1+e_{\lambda}^{-1}(qt)\right)}=\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda;q}^{(k)}\frac{t^{n}}{n!}.$$

5. *When $\lambda\to 0$ and $q\to 1$, we reach the Gould-Hopper based type2 poly-Euler polynomials denoted by ${}_{GH}\mathfrak{E}_{n}^{(k,j)}(x,y)$.*
6. *When $k=1$, we get the Gould-Hopper based fully degenerate Euler polynomials with a q parameter denoted by ${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(j)}(x,y)$.*
7. *When $\lambda\to 0$ and $k=1$, we reach the Gould-Hopper based Euler polynomials with a q parameter denoted by ${}_{GH}\mathfrak{E}_{n}^{(k,j)}(x,y)$.*
8. *Upon setting $k=1$ and $q\to 1$, we get the Gould-Hopper based fully degenerate Euler polynomials denoted by ${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)$.*
9. *When $k=q\to 1$ and $y=0$, we obtain the fully degenerate Euler polynomials denoted by $\mathfrak{E}_{n,\lambda}(x)$.*
10. *When $k=q\to 1$, and $\lambda\to 0$, we have the Gould-Hopper based Euler polynomials denoted by ${}_{GH}\mathfrak{E}_{n}(x,y)$ (cf. [14]).*
11. *For $k=q\to 1$, $\lambda\to 0$ and $y=0$, we reach the classical Euler polynomials denoted by $E_{n}(x)$ (see [4,5,7,8,10-12,14]).*

We give the following theorem.

**Theorem 1.** *(Summation formulas) We have*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)=\sum_{s=0}^{n}\binom{n}{s}\mathfrak{E}_{s,\lambda;q}^{(k)}GH_{n-s,\lambda;q}^{(j)}(x,y)$$

*and*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)=n!\sum_{s=0}^{\lfloor n/j\rfloor}\left(\frac{y}{\lambda}\right)_{s}\frac{\lambda^{s}q^{s}}{s!(n-js)!}\mathfrak{E}_{n-js,\lambda;q}^{(k)}(x).$$

*Proof:* Indeed, by (16), we get

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^{n}}{n!} &= \frac{2q\,\mathrm{Ei}_{k}\left(\frac{log_{\lambda}(1+qt)}{q}\right)}{t\left(1+e_{\lambda}^{-1}(qt)\right)}e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}\mathfrak{E}_{s,\lambda;q}^{(k)}GH_{n-s,\lambda;q}^{(j)}(x,y)\right)\frac{t^{n}}{n!}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^{n}}{n!} &= \frac{2q\,\mathrm{Ei}_{k}\left(\frac{log_{\lambda}(1+qt)}{q}\right)}{t\left(1+e_{\lambda}^{-1}(qt)\right)}e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right)\\ &= \left(\sum_{n=0}^{\infty}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x)\frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{y}{\lambda}\right)_{s}q^{n}\lambda^{n}\frac{t^{jn}}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\lfloor n/j\rfloor}\mathfrak{E}_{n-js,\lambda;q}^{(k,j)}(x)\frac{t^{n-js}}{(n-js)!}\left(\frac{y}{\lambda}\right)_{s}q^{s}\lambda^{s}\frac{t^{js}}{s!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\lfloor n/j\rfloor}\mathfrak{E}_{n-js,\lambda;q}^{(k,j)}(x)\left(\frac{y}{\lambda}\right)_{s}\frac{q^{s}\lambda^{s}n!}{s!(n-js)!}\right)\frac{t^{n}}{n!},\end{aligned}$$

which give the desired results. □

**Theorem 2.** *($\lambda$-Difference Rules for ${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)$)*

$$\Delta_{\lambda,x}\,{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)=nq\,{}_{GH}\mathfrak{E}_{n-1,\lambda;q}^{(k,j)}(x,y)$$

*and*

$$\Delta_{\lambda,y}\,{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)=q\,(n)_{j}\,{}_{GH}\mathfrak{E}_{n-j,\lambda;q}^{(k,j)}(x,y).$$

*Proof:* The proof can be done by using some series manipulation methods. So, we omit them. □

Here, we give multifarious connection formulas including the Gould-Hopper based fully degenerate type2 poly-Euler polynomials with a $q$ parameter, the Gould-Hopper based fully degenerate Stirling polynomials of the first kind with a $q$ parameter by the following theorems.

**Theorem 3.** *We have*

$$\begin{aligned} & n\,{}_{GH}\mathfrak{E}_{n-1,\lambda;q}^{(k,j)}(x,y)+n\sum_{\ell=0}^{n-1}\binom{n-1}{l}(-1)_{\ell,\lambda}\,q^{\ell}\,{}_{GH}\mathfrak{E}_{n-1-l,\lambda;q}^{(k,j)}(x,y) \\ = \; & 2\sum_{u=0}^{n}\binom{n}{u}\sum_{m=0}^{u}\left(\sum_{\ell=0}^{u}\binom{u}{l}S_{1,\lambda}(\ell,m)(\lambda)_{u-\ell}-S_{1,\lambda}(u,m)\right) \\ & \times\frac{q^{n-m}}{\lambda(m+1)^{k}}GH_{n-u,\lambda;q}^{(j)}(x,y). \end{aligned}$$

*Proof:* We observe that

$$t\left(1+e_{\lambda}^{-1}(qt)\right)\left(\sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^{n}}{n!}\right)=2q\,\mathrm{Ei}_{k}\left(\frac{log_{\lambda}(1+qt)}{q}\right)e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right). \tag{17}$$

Let $LHS$ and $RHS$ be the left hand-side and the right hand-side of (17), respectively. Then, we get

$$\begin{aligned} LHS &= \sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^{n+1}}{n!}+\sum_{n=0}^{\infty}(-1)_{n,\lambda}\frac{q^{n}t^{n}}{n!}\sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^{n+1}}{n!}+\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{l}(-1)_{\ell,\lambda}\,q^{\ell}\,{}_{GH}\mathfrak{E}_{n-l,\lambda;q}^{(k,j)}(x,y)\right)\frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty}\left({}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)+\sum_{\ell=0}^{n}\binom{n}{l}(-1)_{\ell,\lambda}\,q^{\ell}\,{}_{GH}\mathfrak{E}_{n-l,\lambda;q}^{(k,j)}(x,y)\right)\frac{t^{n}}{n!} \end{aligned}$$

and

$$\begin{aligned} RHS &= 2q\,\mathrm{Ei}_{k}\left(\frac{log_{\lambda}(1+qt)}{q}\right).e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right) \\ &= 2q\sum_{m=1}^{\infty}\frac{(log_{\lambda}(1+qt))^{m}.q^{-m}}{(m-1)!.m^{k}}.e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right) \\ &= 2\sum_{m=0}^{\infty}\frac{(log_{\lambda}(1+qt))^{m+1}.q^{-m}}{m!\,(m+1)^{k}}.e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right) \\ &= 2\sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{q^{n}t^{n}}{n!}\right)\frac{1}{\lambda}\left(\sum_{n=0}^{\infty}(\lambda)_{n}\frac{q^{n}t^{n}}{n!}-1\right)\frac{q^{-m}}{(m+1)^{k}}e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right) \\ &= 2\sum_{m=0}^{\infty}\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{l}S_{1,\lambda}(\ell,m)(\lambda)_{n-\ell}-S_{1,\lambda}(n,m)\right)\frac{q^{n}t^{n}}{n!}\frac{q^{-m}}{(m+1)^{k}}e_{\lambda}^{x}(qt)\,e_{\lambda}^{y}\left(qt^{j}\right) \\ &= 2\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}\sum_{m=0}^{n}\sum_{\ell=0}^{n}\binom{n}{l}S_{1,\lambda}(\ell,m)(\lambda)_{n-\ell}-S_{1,\lambda}(n,m)\right)\frac{q^{n-m}}{(m+1)^{k}}\frac{t^{n}}{n!}\sum_{n=0}^{\infty}GH_{n,\lambda;q}^{(j)}(x,y)\frac{t^{n}}{n!} \\ &= 2\sum_{n=0}^{\infty}\sum_{u=0}^{n}\binom{n}{u}\sum_{m=0}^{u}\left(\sum_{\ell=0}^{u}\binom{u}{l}S_{1,\lambda}(\ell,m)(\lambda)_{u-\ell}-S_{1,\lambda}(u,m)\right) \\ &\quad\times\frac{q^{n-m}}{\lambda(m+1)^{k}}GH_{n-u,\lambda;q}^{(j)}(x,y)\frac{t^{n}}{n!}. \end{aligned}$$

□

**Theorem 4.** *For* $\left|e_{\lambda}^{-1}(qt)\right|<1$, *we have*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)=2\sum_{m=1}^{n}\sum_{u=0}^{\infty}(-1)^{u}\frac{q^{-m+1}}{m^{k-1}}S_{1,\lambda;q}^{(j)}(n,m;x-u,y).$$

*Proof:* By (16), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^n}{n!} &= 2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)\sum_{u=0}^{\infty}(-1)^u e_\lambda^{-u}(qt)\,e_\lambda^{x}(qt)\,e_\lambda^{y}\left(qt^j\right)\\
&= 2\sum_{m=0}^{\infty}\frac{(log_\lambda(1+qt))^{m-1}q^{-m+1}}{(m-1)!m^k}\sum_{u=0}^{\infty}(-1)^u e_\lambda^{x-u}(qt)\,e_\lambda^{y}\left(qt^j\right)\\
&= 2\sum_{m=1}^{\infty}\frac{(log_\lambda(1+qt))^{m}}{m!}\frac{q^{-m+1}}{m^{k-1}}\sum_{u=0}^{\infty}(-1)^u e_\lambda^{x-u}(qt)\,e_\lambda^{y}(qt^j)\\
&= 2\sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{q^nt^n}{n!}\right)\frac{q^{-m+1}}{m^{k-1}}\sum_{u=0}^{\infty}(-1)^u e_\lambda^{x-u}(qt)\,e_\lambda^{y}\left(qt^j\right)\\
&= 2\sum_{m=1}^{\infty}\sum_{u=0}^{\infty}(-1)^u\frac{q^{-m+1}}{m^{k-1}}\left(\sum_{n=m}^{\infty}S_{1,\lambda;q}^{(j)}(n,m;x-u,y)\,q^n\frac{t^n}{n!}\right)\\
&= 2\sum_{n=m}^{\infty}\left(\sum_{m=1}^{n}\sum_{u=0}^{\infty}(-1)^u\frac{q^{-m+1}}{m^{k-1}}S_{1,\lambda;q}^{(j)}(n,m;x-u,y)\right)\frac{t^n}{n!},
\end{aligned}$$

which is the desired result. □

**Theorem 5.** *We have*

$$n\,{}_{GH}\mathfrak{E}_{n-1,\lambda;q}^{(k,j)}(x,y)+n\sum_{\ell=0}^{n-1}\binom{n-1}{l}(-1)_{\ell,\lambda}\,q^{\ell}\,{}_{GH}\mathfrak{E}_{n-l-1,\lambda;q}^{(k,j)}(x,y)=2\sum_{m=1}^{\infty}\frac{q^{1-m}}{m^{k-1}}S_{1,\lambda;q}^{(j)}(n,m;x,y)$$

*Proof:* Let $LHS$ and $RHS$ be the left hand-side and the right hand-side of (17), respectively. Then, we get

$$LHS=\sum_{n=0}^{\infty}\left({}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)+\sum_{\ell=0}^{n}\binom{n}{l}(-1)_{\ell,\lambda}\,q^{\ell}\,{}_{GH}\mathfrak{E}_{n-l,\lambda;q}^{(k,j)}(x,y)\right)\frac{t^{n+1}}{n!}$$

and

$$\begin{aligned}
RHS &= 2\sum_{m=1}^{\infty}\frac{q^{1-m}}{m^{k-1}}\frac{log_\lambda(1+qt)^m}{m!}e_\lambda^{x}(qt)\,e_\lambda^{y}\left(qt^j\right)\\
&= 2\sum_{m=1}^{\infty}\frac{q^{1-m}}{m^{k-1}}\sum_{n=m}^{\infty}S_{1,\lambda;q}^{(j)}(n,m;x,y)\frac{t^n}{n!}\\
&= 2\sum_{n=m}^{\infty}\left(\sum_{m=1}^{\infty}\frac{q^{1-m}}{m^{k-1}}S_{1,\lambda;q}^{(j)}(n,m;x,y)\right)\frac{t^n}{n!},
\end{aligned}$$

which means the asserted result. □

We give the following result.

**Theorem 6.** *We have*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)=\frac{1}{n+1}\sum_{u=0}^{n+1}\binom{n+1}{u}\left(\sum_{m=1}^{u}S_{1,\lambda}(u,m)\frac{q^{u-1-m}}{m^{k-1}}\right){}_{GH}\mathfrak{E}_{n+1-u,\lambda;q}^{(j)}(x+1,y),$$

*where* ${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(j)}(x,y)$ *denotes the Gould-Hopper based degenerate Euler polynomials with a q parameter defined by*

$$\sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,\lambda;q}^{(j)}(x,y)\frac{t^n}{n!}=\frac{2}{e_\lambda(qt)+1}e_\lambda^{x}(qt)\,e_\lambda^{y}\left(qt^j\right).$$

*Proof:* By (16), we get

$$\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^n}{n!} = q\frac{Ei\left(\frac{log_\lambda(1+qt)}{q}\right)}{t}\frac{2}{e_\lambda(qt)+1}e_\lambda^{x+1}(qt)\,e_\lambda^{y}\left(qt^j\right)$$
$$= \frac{q}{t}\sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{q^n t^n}{n!}\right)\frac{q^{-m}}{m^{k-1}}\left(\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(j)}(x+1,y)\frac{t^n}{n!}\right)$$
$$= \frac{1}{t}\left(\sum_{n=m}^{\infty}\sum_{m=1}^{n} S_{1,\lambda}(n,m)\frac{q^{n+1-m}t^n}{m^{k-1}n!}\right)\left(\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(j)}(x+1,y)\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left[\sum_{u=0}^{n}\binom{n}{u}\left(\sum_{m=1}^{u} S_{1,\lambda}(u,m)\frac{q^{u-1-m}}{m^{k-1}}\right) {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(j)}(x+1,y)\right]\frac{t^{n-1}}{n!},$$

which gives the desired result. □

**Theorem 7.** *We have*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x+1,y) + {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y) = 2n\sum_{m=1}^{n-1}\frac{q^{-m+1}}{m^{k-1}}S_{1,\lambda;q}^{(j)}(n-1,m;x+1,y).$$

*Proof:* Multiplying $(e_\lambda(qt)+1)$ to both sides of (16), we observe that

$$\sum_{n=0}^{\infty}\left({}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x+1,y) + {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\right)\frac{t^{n+1}}{n!} = 2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)e_\lambda^{x+1}(qt)\,e_\lambda^{y}\left(qt^j\right)$$
$$= 2\sum_{m=1}^{\infty}\left(\frac{(log_\lambda(1+qt))^m}{m!}\right)\frac{q^{-m+1}}{m^{k-1}}e_\lambda^{x+1}(qt)\,e_\lambda^{y}\left(qt^j\right)$$
$$= 2\sum_{m=1}^{\infty}\frac{q^{-m+1}}{m^{k-1}}\left(\sum_{n=m}^{\infty} S_{1,\lambda;q}^{(j)}(n,m;x+1,y)\frac{t^n}{n!}\right)$$
$$= 2\sum_{n=m}^{\infty}\left(\sum_{m=1}^{n}\frac{q^{-m+1}}{m^{k-1}}S_{1,\lambda;q}^{(j)}(n,m;x+1,y)\right)\frac{t^n}{n!}.$$

□

**Theorem 8.** *We have*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y) = \frac{1}{n+1}\sum_{u=0}^{m-1}\sum_{s=0}^{n+1}\binom{n+1}{s}\sum_{m=1}^{s}(-1)^u\frac{q^{s-m+1}}{m^{k-1}}$$
$$\times S_{1,\lambda}(s,m)\ {}_{GH}\mathfrak{E}_{n+1-s,\lambda/m;qm}^{(j)}\left(\frac{x+u-1}{m}-1,\frac{y}{m}\right).$$

*Proof:* By (16), we acquire

$$\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^n}{n!} = \frac{e_\lambda^m(qt)+1}{e_\lambda(qt)+1}\frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)}{t\left(e_\lambda^m(qt)+1\right)}e_\lambda^{x+1}(qt)\,e_\lambda^y\left(qt^j\right)$$

$$= \frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)}{t\left(1+e_\lambda^{-m}(qt)\right)}\sum_{u=0}^{m-1}(-1)^u e_\lambda^{x+u+1-m}(qt)\,e_\lambda^y\left(qt^j\right)$$

$$= \sum_{u=0}^{m-1}(-1)^u\frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)}{1+e_\lambda^{-m}(qt)}e_\lambda^{x+u+1+m}(qt)\,e_\lambda^y\left(qt^j\right)$$

$$= \sum_{u=0}^{m-1}(-1)^u\frac{q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)}{t}\frac{2}{1+e_{\lambda/m}^{-1}(qmt)}e_{\lambda/m}^{\frac{x+u-1}{m}-1}(qmt)\,e_{\lambda/m}^{\frac{y}{m}}\left(qmt^j\right)$$

$$= \left(\sum_{u=0}^{m-1}\frac{(-1)^u}{t}\left(\sum_{n=m}^{\infty}\sum_{m=1}^{n}\frac{q^{-m+1+n}}{m^{k-1}}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}\right)$$

$$\times\left(\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda/m;qm}^{(j)}\left(\frac{x+u-1}{m}-1,\frac{y}{m}\right)\frac{t^n}{n!}\right)$$

$$= \sum_{u=0}^{m-1}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{m=1}^{s}\binom{n}{s}(-1)^u\frac{q^{s-m+1}}{m^{k-1}}S_{1,\lambda}(s,m)$$

$$\times {}_{GH}\mathfrak{E}_{n-s,\lambda/m;qm}^{(j)}\left(\frac{x+u-1}{m}-1,\frac{y}{m}\right)\frac{t^{n-1}}{n!}.$$

□

**Theorem 9.** *We have*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x_1+x_2,y_1+y_2) = \sum_{u=0}^{n}\binom{n}{u} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x_1,y_1)\,GH_{n-u,\lambda;q}^{(j)}(x_2,y_2).$$

*Proof:* From (16), we attain

$$\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x_1+x_2,y_1+y_2)\frac{t^n}{n!} = \frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)}{t\left(1+e_\lambda^{-1}(qt)\right)}e_\lambda^{x_1+x_2}(qt)\,e_\lambda^{y_1+y_2}\left(qt^j\right)$$

$$= \left(\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x_1,y_1)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}GH_{n,\lambda;q}^{(j)}(x_2,y_2)\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\binom{n}{u} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x_1,y_1)\,GH_{n-u,\lambda;q}^{(j)}(x_2,y_2)\right)\frac{t^n}{n!}$$

which means the claimed result. □

**Theorem 10.** *We have*

$${}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y) = \sum_{s=0}^{\lfloor n/j\rfloor}\sum_{m=0}^{n-js}\binom{n-js}{m}\mathfrak{E}_{n-js-m,\lambda;q}^{(k)}\left(\frac{x}{\lambda}\right)_m\left(\frac{y}{\lambda}\right)_s\lambda^{m+s}q^{m+s}\frac{n!}{s!(n-js)!}.$$

*Proof:* By (16), we get

$$\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k,j)}(x,y)\frac{t^n}{n!} = \frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+qt)}{q}\right)}{t\left(1+e_\lambda^{-1}(qt)\right)} e_\lambda^x(qt)\, e_\lambda^y\left(qt^j\right)$$

$$= \left(\sum_{n=0}^{\infty} {}_{GH}\mathfrak{E}_{n,\lambda;q}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{x}{\lambda}\right)_n \lambda^n q^n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{y}{\lambda}\right)_n \lambda^n q^n \frac{t^{jn}}{n!}\right)$$

$$= \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_{GH}\mathfrak{E}_{n-m,q}^{(k)}(x,y;\lambda)\left(\frac{x}{\lambda}\right)_m \lambda^m q^m \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{y}{\lambda}\right)_n \lambda^n q^n \frac{t^{jn}}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\left\lfloor\frac{n}{j}\right\rfloor}\sum_{m=0}^{n-js}\binom{n-js}{m} {}_{GH}\mathfrak{E}_{n-js-m,q}^{(k,j)}(x,y;\lambda)\left(\frac{x}{\lambda}\right)_m\left(\frac{y}{\lambda}\right)_s \lambda^{m+s} q^{m+s}\frac{n!}{s!\,(n-js)!}\right)\frac{t^n}{n!}.$$

□

We note that the following series manipulation formulas hold (*cf.* [15]):

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{18}$$

and

$$\sum_{k,l=0}^{\infty} A(l,k) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} A(l,k-l). \tag{19}$$

We give the following theorem.

**Theorem 11.** *(Implicit Summation Formula) We have*

$$ {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(\tau,y) = \sum_{n,m=0}^{s,l}\binom{s}{n}\binom{l}{m} {}_{GH}\mathfrak{E}_{s+l-n-m,\lambda;q}^{(k,j)}(x,y)(\tau-x)_{n+m,\lambda}. \tag{20}$$

*Proof:* Upon setting $t$ by $t + u$ in (16), we derive

$$\frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+q(t+u))}{q}\right)}{t\left(1+e_\lambda^{-1}(q(t+u))\right)} e_\lambda^y\left(q(t+u)^j\right) = e_\lambda^{-x}(q(t+u))\sum_{s,l=0}^{\infty} {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(x,y)\frac{t^s}{s!}\frac{u^l}{l!}.$$

Again replacing $\tau$ by $x$ in the last equation, and using (18), we get

$$e_\lambda^{-\tau}(q(t+u))\sum_{s,l=0}^{\infty} {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(\tau,y)\frac{t^s}{s!}\frac{u^l}{l!} = \frac{2q\,\mathrm{Ei}_k\left(\frac{log_\lambda(1+q(t+u))}{q}\right)}{t\left(1+e_\lambda^{-1}(q(t+u))\right)} e_\lambda^y\left(q(t+u)^j\right).$$

By the last two equations, we obtain

$$\sum_{s,l=0}^{\infty} {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(\tau,y)\frac{t^s}{s!}\frac{u^l}{l!} = e_\lambda^{\tau-x}(q(t+u))\sum_{s,l=0}^{\infty} {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(x,y)\frac{t^s}{s!}\frac{u^l}{l!}$$

which yield

$$\sum_{s,l=0}^{\infty} {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(\tau,y)\frac{t^s}{s!}\frac{u^l}{l!} = \sum_{n,m=0}^{\infty}(\tau-x)_{n+m,\lambda}\frac{t^n}{n!}\frac{u^m}{m!}e_\lambda^{\tau-x}(q(t+u))\sum_{s,l=0}^{\infty} {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(x,y)\frac{t^s}{s!}\frac{u^l}{l!}$$

Utilizing (19), we acquire

$$\sum_{s,l=0}^{\infty} {}_{GH}\mathfrak{E}_{s+l,\lambda;q}^{(k,j)}(\tau,y)\frac{t^s}{s!}\frac{u^l}{l!} = \sum_{s,l=0}^{\infty}\sum_{n,m=0}^{s,l}\frac{{}_{GH}\mathfrak{E}_{s+l-n-m,\lambda;q}^{(k,j)}(x,y)(\tau-x)_{n+m,\lambda}}{n!m!\,(s-l)!\,(l-m)!}t^s u^l$$

which implies the asserted result (20). □

**Corollary 1.** *Letting $s=0$ in (20), the following implicit summation formula holds:*

$$ {}_{GH}\mathfrak{E}_{l,\lambda;q}^{(k,j)}(\tau,y)=\sum_{m=0}^{l}\binom{l}{m}{}_{GH}\mathfrak{E}_{l-m,\lambda;q}^{(k,j)}(x,y)(\tau-x)_{m,\lambda}. $$

**Corollary 2.** *Upon setting $s=0$ and replacing $\tau$ by $\tau+x$ in (20), we attain*

$$ {}_{GH}\mathfrak{E}_{l,\lambda;q}^{(k,j)}(\tau+x,y)=\sum_{m=0}^{l}\binom{l}{m}{}_{GH}\mathfrak{E}_{l-m,\lambda;q}^{(k,j)}(x,y)(\tau)_{m,\lambda}. $$

Now, we give the following theorem.

**Theorem 12.** *(Symmetric Property) The following symmetric identity*

$$ \begin{aligned} & \sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{u=0}^{n-m}\binom{n}{m}\binom{m}{s}\binom{n-m}{u}\left(a^{j}y\right)_{u,a^{j}\lambda}\left(b^{j}y\right)_{s,b^{j}\lambda} \\ & \times {}_{GH}\mathfrak{E}_{n-m-u,a\lambda;q}^{(k,j)}(ax)\ {}_{GH}\mathfrak{E}_{m-s,b\lambda;q}^{(k,j)}(bx)\,a^{m}b^{n-m} \\ = & \sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{u=0}^{n-m}\binom{n}{m}\binom{m}{s}\binom{n-m}{u}\left(b^{j}y\right)_{u,b^{j}\lambda}\left(a^{j}y\right)_{s,a^{j}\lambda} \\ & \times {}_{GH}\mathfrak{E}_{n-m-u,b\lambda;q}^{(k,j)}(bx)\ {}_{GH}\mathfrak{E}_{m-s,a\lambda;q}^{(k,j)}(ax)\,b^{m}a^{n-m} \end{aligned} \tag{21} $$

*holds for $\alpha\in\mathbb{N}$, $a,b\in\mathbb{R}$ and $n\geq 0$.*

*Proof:* Let

$$ \Upsilon=\frac{2^{2}q^{2}Ei_{k}\left(\frac{log_{\lambda}(1+qat)}{q}\right)Ei_{k}\left(\frac{log_{\lambda}(1+qbt)}{q}\right)}{t^{2}\left(1+e_{\lambda}^{-1}(qat)\right)\left(1+e_{\lambda}^{-1}(qbt)\right)}e_{\lambda}^{2x}(qabt)\,e_{\lambda}^{2y}\left(q(abt)^{j}\right) $$

Then, the expression for $\Upsilon$ is symmetric in $a$ and $b$, and we derive the following two expansions of $\Upsilon$:

$$ \begin{aligned} \Upsilon = & \sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,b\lambda;q}^{(k,j)}(bx)\frac{(at)^{n}}{n!}\sum_{n=0}^{\infty}\left(b^{j}y\right)_{n,b^{j}\lambda}\frac{(at)^{n}}{n!} \\ & \times\sum_{n=0}^{\infty}{}_{GH}\mathfrak{E}_{n,a\lambda;q}^{(k,j)}(ax)\frac{(bt)^{n}}{n!}\sum_{n=0}^{\infty}\left(a^{j}y\right)_{n,a^{j}\lambda}\frac{(bt)^{n}}{n!} \\ = & \sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}\left(b^{j}y\right)_{s,b^{j}\lambda}{}_{GH}\mathfrak{E}_{n-s,b\lambda;q}^{(k,j)}(bx)\frac{(at)^{n}}{n!} \\ & \times\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}\left(a^{j}y\right)_{s,a^{j}\lambda}{}_{GH}\mathfrak{E}_{n-s,a\lambda;q}^{(k,j)}(ax)\frac{(bt)^{n}}{n!} \\ = & \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\left(\sum_{s=0}^{m}\binom{m}{s}\left(b^{j}y\right)_{s,b^{j}\lambda}{}_{GH}\mathfrak{E}_{m-s,b\lambda;q}^{(k,j)}(bx)\right) \\ & \times\left(\sum_{s=0}^{n-m}\binom{n-m}{s}\left(a^{j}y\right)_{s,a^{j}\lambda}{}_{GH}\mathfrak{E}_{n-m-s,a\lambda;q}^{(k,j)}(ax)\right)a^{m}b^{n-m}\frac{t^{n}}{n!} \end{aligned} $$

and similarly

$$ \begin{aligned} \Upsilon = & \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\left(\sum_{s=0}^{m}\binom{m}{s}\left(a^{j}y\right)_{s,a^{j}\lambda}{}_{GH}\mathfrak{E}_{m-s,a\lambda;q}^{(k,j)}(ax)\right) \\ & \times\left(\sum_{s=0}^{n-m}\binom{n-m}{s}\left(b^{j}y\right)_{s,b^{j}\lambda}{}_{GH}\mathfrak{E}_{n-m-s,b\lambda;q}^{(k,j)}(bx)\right)a^{n-m}b^{m}\frac{t^{n}}{n!}, \end{aligned} $$

which gives the desired result (21).

□

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# A Surplus Calculation of Quadratic Demand and Supply Functions with Trapezoidal Fuzzification Method and Graded Mean Defuzzification Method

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**Abstract:** Let the quadratic demand function be $p(x) = a_0 - b_0x - c_0x^2$ and the quadratic supply function be $p(x) = d_0 + e_0x + g_0x^2$ where $x$ is a quantity and $a_0, b_0, c_0, d_0, e_0$ and $g_0$ are coefficients. In this study, we fuzzify these coefficients using trapezoidal fuzzy numbers. Then we estimate consumer surplus and producer surplus. We use the graded mean defuzzification method to obtain the crisp values. Finally we compare our results with not only ordinary case but also triangular fuzzy case.

**Keywords:** Consumer surplus, Producer surplus, Quandratic demand function, Quadratic supply function, Trapezoidal fuzzy number.

## 1 Introduction

In recent years, fuzzy set theory has become a main tool to study economic problems such as estimating optimal revenue [1, 2] and optimal profit [4, 8], calculating the best prices of two and three mutually complementary merchandises [10, 11] and calculating the consumer and producer surplus [7, 9].

In [9], Yao and Wu considered linear demand function and linear supply function in which the demand quantity and the supply quantity are triangular fuzzy numbers. Then they calculated the consumer surplus and producer surplus. Following paper, Wu [7] estimated these surpluses taking into consideration demand function and supply function to be linear or quadratic. He fuzzify the constants instead of quantity. In the both of the papers [7, 9], triangular fuzzy numbers has been used for fuzzification.

In this paper, we use the quadratic demand $p = a_0 - b_0x - c_0x^2$ of function and quadratic supply $p = d_0 + e_0x + g_0x^2$ of function to calculate the consumer surplus and producer surplus. Then we fuzzify the quantity $a_0, b_0, c_0, d_0, e_0$, and $g_0$ by using trapezoidal fuzzy number. Yao and Wu [7, 9] have used the centroid method for defuzzification. Here we use the graded mean defuzzification method. Finally, we showed that our trapezoidal fuzzy model gives more better results than Wu's [7] triangular fuzzy model.

## 2 Preliminaries

A *fuzzy number* is a function $X$ from $\mathbb{R}$ to $[0, 1]$, satisfying:

(i) $X$ is *normal*, i.e., there exists an $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;
(ii) $X$ is *fuzzy convex*, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, $X(\lambda x + (1 - \lambda)y) \geq \min\{X(x), X(y)\}$;
(iii) $X$ is *upper semi-continuous*;
(iv) the *closure* of $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by $X^0$, is *compact*.

We denote the set of all fuzzy numbers by $\mathcal{F}(\mathbb{R})$. Note that the fuzzy point $a_1$ defined by

$$a_1(x) := \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise.} \end{cases}$$

A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ represented with the membership function

$$\widetilde{A}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ \frac{d-x}{d-c}, & c \leq x \leq d, \\ 0. & \text{otherwise.} \end{cases}$$

 

If we take $a = b = c = d$, then the trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is identical to the fuzzy point $a_1$. We denote the set of all trapezoidal fuzzy numbers by $\mathcal{F}(T)$ [3]. It is clear that $\mathcal{F}(T) \subset \mathcal{F}(\mathbb{R})$.

Now let us briefly review the operations of *summation* and *scalar multiplication* on the set $\mathcal{F}(T)$ of trapezoidal fuzzy numbers. For $\widetilde{A}, \widetilde{B} \in \mathcal{F}(T)$, the fuzzy number $\widetilde{C} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$ is called the *sum* of $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$, and we write $\widetilde{C} = \widetilde{A} + \widetilde{B}$. Let $k$ be a real number. Then *scalar multiplication* is defined as

$$k\widetilde{A} = \begin{cases} (ka_4, ka_3, ka_2, ka_1), & k < 0, \\ (ka_1, ka_2, ka_3, ka_4), & k > 0, \end{cases} \text{[3].}$$

The $\alpha-$level set of a trapezoidal fuzzy number $\widetilde{A}$ is

$$\begin{aligned} \widetilde{A}_\alpha &= \left[\widetilde{A}_L(\alpha), \widetilde{A}_R(\alpha)\right] \\ &= [a + \alpha(b - a), d - \alpha(d - c)]. \end{aligned}$$

This set is a closed interval for each $\alpha \in [0, 1]$ [3]. If $\widetilde{A} \in \mathcal{F}(T)$ then the graded mean of $\widetilde{A} = (a, b, c, d)$ is defined as

$$G\left(\widetilde{A}\right) = \frac{\frac{1}{2}\int_0^1 [A_L(\alpha) + A_R(\alpha)]\, d\alpha}{\int_0^1 \alpha d\alpha} = \frac{1}{6}(a + 2b + 2c + d) \text{ [10].}$$

## 3 Consumer Surplus and Producer Surplus

We consider the demand function $p = a_0 - b_0 x - c_0 x^2$, $0 \le x \le x_*$ and the and the supply function $p = d_0 + e_0 x + g_0 x^2$, $0 \le x$ where $a_0 > d_0 > 0$, $b_0 > 0$, $c_0 > 0$, $e_0 > 0$, $g_0 > 0$ and we know that $x_* = \left(-b_0 + \sqrt{b_0^2 + 4a_0c_0}\right)/2c_0 > 0$. Now we fuzzify the positive coefficients of demand and supply functions as

$$\begin{array}{lll} \widetilde{a} = (a_0 - \varepsilon_{11}, a_0 - \varepsilon_{12}, a_0 + \varepsilon_{13}, a_0 + \varepsilon_{14}), & 0 \le \varepsilon_{11} \le \varepsilon_{12} \le a_0, & 0 \le \varepsilon_{13} \le \varepsilon_{14} \\ \widetilde{b} = (b_0 - \varepsilon_{21}, b_0 - \varepsilon_{22}, b_0 + \varepsilon_{23}, b_0 + \varepsilon_{24}), & 0 \le \varepsilon_{21} \le \varepsilon_{22} \le b_0, & 0 \le \varepsilon_{23} \le \varepsilon_{24} \\ \widetilde{c} = (c_0 - \varepsilon_{31}, c_0 - \varepsilon_{32}, c_0 + \varepsilon_{33}, c_0 + \varepsilon_{34}), & 0 \le \varepsilon_{31} \le \varepsilon_{32} \le c_0, & 0 \le \varepsilon_{33} \le \varepsilon_{34} \\ \widetilde{d} = (d_0 - \varepsilon_{41}, d_0 - \varepsilon_{42}, d_0 + \varepsilon_{43}, d_0 + \varepsilon_{44}), & 0 \le \varepsilon_{41} \le \varepsilon_{42} \le d_0, & 0 \le \varepsilon_{43} \le \varepsilon_{44} \\ \widetilde{e} = (e_0 - \varepsilon_{51}, e_0 - \varepsilon_{52}, e_0 + \varepsilon_{53}, e_0 + \varepsilon_{54}), & 0 \le \varepsilon_{51} \le \varepsilon_{52} \le e_0, & 0 \le \varepsilon_{53} \le \varepsilon_{54} \\ \widetilde{g} = (g_0 - \varepsilon_{61}, g_0 - \varepsilon_{62}, g_0 + \varepsilon_{63}, g_0 + \varepsilon_{64}), & 0 \le \varepsilon_{61} \le \varepsilon_{62} \le g_0, & 0 \le \varepsilon_{63} \le \varepsilon_{64} \end{array}.$$

Hence $\widetilde{P}_D = \widetilde{a} - \widetilde{b}x - \widetilde{c}x^2 = (D_1, D_2, D_3, D_4)$ is *trapezoidal fuzzy demand function* where

$$\begin{aligned} D_1 &= a_0 - \varepsilon_{11} - x(b_0 + \varepsilon_{24}) - x^2(c_0 + \varepsilon_{34}) \\ D_2 &= a_0 - \varepsilon_{12} - x(b_0 + \varepsilon_{23}) - x^2(c_0 + \varepsilon_{33}) \\ D_3 &= a_0 + \varepsilon_{13} - x(b_0 - \varepsilon_{22}) - x^2(c_0 - \varepsilon_{32}) \\ D_4 &= a_0 + \varepsilon_{14} - x(b_0 - \varepsilon_{21}) - x^2(c_0 - \varepsilon_{31}) \end{aligned}$$

and its graded mean can be easily calculated as

$$\begin{aligned} G\left(\widetilde{P}_D\right) &= E_1(x) = \frac{1}{6}\begin{bmatrix} \left(a_0 - \varepsilon_{11} - x(b_0 + \varepsilon_{24}) - x^2(c_0 + \varepsilon_{34})\right) \\ +2\left(a_0 - \varepsilon_{12} - x(b_0 + \varepsilon_{23}) - x^2(c_0 + \varepsilon_{33})\right) \\ +2\left(a_0 + \varepsilon_{13} - x(b_0 - \varepsilon_{22}) - x^2(c_0 - \varepsilon_{32})\right) \\ +\left(a_0 + \varepsilon_{14} - x(b_0 - \varepsilon_{21}) - x^2(c_0 - \varepsilon_{31})\right) \end{bmatrix} \\ &= a_0 - b_0 x - c_0 x^2 + \frac{1}{6}\left(\Delta_1 - \Delta_2 x - \Delta_3 x^2\right) \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \varepsilon_{14} + 2\varepsilon_{13} - 2\varepsilon_{12} - \varepsilon_{11} \\ \Delta_2 &= \varepsilon_{24} + 2\varepsilon_{23} - 2\varepsilon_{22} - \varepsilon_{21} \\ \Delta_3 &= \varepsilon_{34} + 2\varepsilon_{33} - 2\varepsilon_{32} - \varepsilon_{31} \end{aligned}$$

Since the demand quantity is $x$, $E_1(x)$ is the estimator of price for demand in the fuzzy sense. Similarly $\widetilde{P}_S = \widetilde{d} + \widetilde{e}x + \widetilde{g}x^2 = (S_1, S_2, S_3, S_4)$ is *trapezoidal fuzzy supply function* where

$$\begin{aligned} S_1 &= d_0 - \varepsilon_{41} + x\left(e_0 - \varepsilon_{51}\right) + x^2\left(g_0 - \varepsilon_{61}\right) \\ S_2 &= d_0 - \varepsilon_{42} + x\left(e_0 - \varepsilon_{52}\right) + x^2\left(g_0 - \varepsilon_{62}\right) \\ S_3 &= d_0 + \varepsilon_{43} + x\left(e_0 + \varepsilon_{53}\right) + x^2\left(g_0 + \varepsilon_{63}\right) \\ S_4 &= d_0 + \varepsilon_{44} + x\left(e_0 + \varepsilon_{54}\right) + x^2\left(g_0 + \varepsilon_{64}\right) \end{aligned}$$

and its graded mean can be easily calculated as

$$G\left(\widetilde{P}_S\right) = E_2(x) = d_0 + e_0 x + g_0 x^2 + \frac{1}{6}\left(\Delta_4 + \Delta_5 x + \Delta_6 x^2\right)$$

where

$$\begin{aligned} \Delta_4 &= \varepsilon_{44} + 2\varepsilon_{43} - 2\varepsilon_{42} - \varepsilon_{41} \\ \Delta_5 &= \varepsilon_{54} + 2\varepsilon_{53} - 2\varepsilon_{52} - \varepsilon_{51} \\ \Delta_6 &= \varepsilon_{64} + 2\varepsilon_{63} - 2\varepsilon_{62} - \varepsilon_{61}. \end{aligned}$$

Since the supply quantity is $x$, $E_2(x)$ is the estimator of price for demand in the fuzzy sense.

The classical surpluses is in the following form for the quadratic demand function and quadratic supply function.

**Theorem 1** ([6]). *If the demand equation is $p = a_0 - b_0 x - c_0 x^2$ and the supply function is $p = d_0 + e_0 x + g_0 x^2$ then the consumer surplus for the crisp case is*

$$CSC = \int_0^{x_0} \left(a_0 - b_0 x - c_0 x^2 - p_0\right) dx = \frac{1}{2} b_0 x_0^2 + \frac{2}{3} c_0 x_0^3$$

*and the producer surplus for the crisp case is*

$$PSC = \int_0^{x_0} \left(p_0 - \left(d_0 + e_0 x + g_0 x^2\right)\right) dx = \frac{1}{2} e_0 x_0^2 + \frac{2}{3} g_0 x_0^3$$

*where $x_0$ is the equilibrium quantity.*

The main results of the paper is the following theorem. This is the trapezoidal fuzzy version of the classic surpluses.

**Theorem 2** (CSF and PSF). . *If the demand equation is $\widetilde{P}_D = \widetilde{a} - \widetilde{b}x - \widetilde{c}x^2$ and the supply function is $\widetilde{P}_S = \widetilde{d} + \widetilde{e}x + \widetilde{g}x^2$ then the consumer surplus for the fuzzy case is*

$$CSF = \frac{1}{2} b_0 x_*^2 + \frac{2}{3} c_0 x_*^3 + \frac{1}{12} \Delta_2 x_*^2 + \frac{1}{9} \Delta_3 x_*^3$$

*and the producer surplus for the fuzzy case is*

$$PSF = \frac{1}{2} e_0 x^2 + \frac{2}{3} g_0 x^3 + \frac{1}{12} x^2 \Delta_5 + \frac{1}{9} x^3 \Delta_6.$$

*Proof:* First we should find the equilibrium quantity. We set $E_1(x)$ equal to $E_2(x)$

$$a_0 - b_0 x - c_0 x^2 + \frac{1}{6}\left(\Delta_1 - \Delta_2 x - \Delta_3 x^2\right) = d_0 + e_0 x + g_0 x^2 + \frac{1}{6}\left(\Delta_4 + \Delta_5 x + \Delta_6 x^2\right).$$

Then, we have

$$Ax^2 + Bx + C = 0$$

where

$$\begin{aligned} A &= c_0 + g_0 + \frac{1}{6}\left(\Delta_6 + \Delta_3\right) \\ B &= b_0 + e_0 + \frac{1}{6}\left(\Delta_5 + \Delta_2\right) \\ C &= d_0 - a_0 + \frac{1}{6}\left(\Delta_4 - \Delta_1\right) \end{aligned}$$

and its discriminant is $D = B^2 - 4AC$. It is clear to see that $A > 0$ and $B > 0$.

(1) If $C < 0$ then $D > 0$. Then this system has negative root $\left(-B - \sqrt{D}\right)/2A < 0$ and positive root $x_* = \left(-B + \sqrt{D}\right)/2A$. Hence $x_*$ is the solution if $0 < \left(-B + \sqrt{D}\right)/2A \leq x_*$.

 

(2) If $C = 0$ then $D = B^2$. This system has two roots $(-B - B)/2A < 0$ and $(-B + B)/2A = 0$. These are not solution.

(3) If $C > 0$ then $D = B^2 - 4AC < B^2$. But these roots $\left(-B - \sqrt{D}\right)/2A < 0$ and $\left(-B + \sqrt{D}\right)/2A < 0$ are not solution because they are negative.

Therefore if $C < 0$ and $0 < \left(-B + \sqrt{D}\right)/2A \leq x_*$ then equilibrium quantity is $x_* = \left(-B + \sqrt{D}\right)/2A\, C < 0$. The equilibrium price is

$$p_* = E_1(x_*) = a_0 - b_0 x_* - c_0 x_*^2 + \frac{1}{6}\left(\Delta_1 - \Delta_2 x_* - \Delta_3 x_*^2\right)$$

or

$$p_* = E_2(x_*) = d_0 + e_0 x_* + g_0 x_*^2 + \frac{1}{6}\left(\Delta_4 + \Delta_5 x_* + \Delta_6 x_*^2\right).$$

Hence we can calculate the consumer surplus:

$$\begin{aligned} \text{CSF} &= \int_0^{x_*} \left[E_1(x) - p_*\right] dx \\ &= \frac{1}{2} b_0 x_*^2 + \frac{2}{3} c_0 x_*^3 + \frac{1}{12}\Delta_2 x_*^2 + \frac{1}{9}\Delta_3 x_*^3 \end{aligned}$$

and we get the producer surplus:

$$\begin{aligned} \text{PSF} &= \int_0^{x_*} \left[p_* - E_2(x)\right] dx \\ &= \frac{1}{2} e_0 x^2 + \frac{2}{3} g_0 x^3 + \frac{1}{12} x^2 \Delta_5 + \frac{1}{9} x^3 \Delta_6. \end{aligned}$$

□

## 4 Application

We use [7, Example 2.7] in order to compare our results and Wu's results [7]. Let the demand function be

$$p = 100 - 48x - 4x^2,\ 0 \leq x \leq \left(-24 + \sqrt{976}\right)/4$$

and supply function be

$$p = 10 + 8x + 2x^2,\ x \geq 0,$$

where the coefficients are $a_0 = 100$, $b_0 = 48$, $c_0 = 4$, $d_0 = 10$, $e_0 = 8$ and $g_0 = 2$.

**Case 1** (**[7]**): We set $100 - 48x - 4x^2 = 10 + 8x + 2x^2$. Then the discriminant of $3x^2 + 28x - 45 = 0$ is $D = 1.810$. The equilibrium quantity is $x_0 = 1.398 \in (0, 1.810)$ and equilibrium price is $p_0 = 25.093$. Then we have the consumer surplus and producer surplus:

$$\begin{aligned} \text{CSC} &= \frac{1}{2} b_0 x_*^2 + \frac{2}{3} c_0 x_*^3 = 54.192 \\ \text{PSC} &= \frac{1}{2} e_0 x_*^2 + \frac{2}{3} g_0 x_*^3 = 11.461. \end{aligned}$$

We note that if we require that the constants $\varepsilon_{i4} - \varepsilon_{i1}$ $(i = 1, ..., 6.)$ to be compatible with Wu's [7] triangular fuzzy numbers, i.e., they can not be chosen arbitrarily.

**Case 2.** We use trapezoidal fuzzy number. If we choose

$$\begin{array}{lll} \varepsilon_{14} - \varepsilon_{11} = 1 & \varepsilon_{13} - \varepsilon_{12} = 1.8 & \Delta_1 = 4.6 \\ \varepsilon_{24} - \varepsilon_{21} = 1 & \varepsilon_{23} - \varepsilon_{22} = 1.8 & \Delta_2 = 4.6 \\ \varepsilon_{34} - \varepsilon_{31} = 1 & \varepsilon_{33} - \varepsilon_{32} = -0.8 & \Delta_3 = -0.6 \\ \varepsilon_{44} - \varepsilon_{41} = -1 & \varepsilon_{43} - \varepsilon_{42} = -1.8 & \Delta_4 = -4.6 \\ \varepsilon_{54} - \varepsilon_{51} = -1 & \varepsilon_{53} - \varepsilon_{52} = 0.8 & \Delta_5 = 0.6 \\ \varepsilon_{64} - \varepsilon_{61} = 1 & \varepsilon_{63} - \varepsilon_{62} = -0.8 & \Delta_6 = -0.6 \end{array}$$

then

$$\begin{aligned} A &= 5.8 \\ B &= 56.867 \\ C &= -91.533 \end{aligned}$$

and its discriminant is

$$D = 5357.04.$$

Hence the equilibrium quantity is

$$x_* = 1.4576.$$

The equilibrium price $p_*$ can be calculated as

$$p_* = 25.077.$$

The consumer surplus and producer surplus estimate as

$$\text{CSF} = 59.856,$$

$$\text{PSF} = 12.527.$$

Finally we get

| | $x_*$ | | CSF | | PSF | |
|---|---|---|---|---|---|---|
| Case | [7] | Our results | [7] | Our results | [7] | Our results |
| Crisp | 1.398 | 1.398 | 54.192 | 54.192 | 11.461 | 11.461 |
| Fuzzy | 1.389 | 1.4576 | 54.367 | 59.856 | 11.564 | 12.527 |

**Table 1** Crisp and Fuzzy Surpluses

## 5 Conclusion

This study is a generalization not only of the triangular fuzzy case, but also of the crisp case. That is, if we take $\varepsilon_{i1} = \varepsilon_{i2}$ and $\varepsilon_{i3} = \varepsilon_{i4}$ for each $1 \leq i \leq 6$, then we get the triangular fuzzy case. If we take $\varepsilon_{i1} = \varepsilon_{i2} = \varepsilon_{i3} = \varepsilon_{i4} = 0$ for each $1 \leq i \leq 6$ then we get the crisp case. Moreover in comparison with the crisp model and Wu's model, the trapezoidal fuzzy model is giving the better optimal solution (see Table 1).

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# New Statistical Concepts and Their Applications

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**Abstract:** If a correlation is mentioned between datasets, it is understood from this expression that it measures how well these datasets are related. Meanwhile, this coefficient is a prominent measure to evaluate the relationship between two sets. The Fermatean fuzzy set is an in fluently widening of the available intuitionistic and Pythagorean fuzzy sets, whose benefit is to better exhaustively characterize ambiguous information. That is, Fermatean fuzzy sets are powerful and valuable tools to represent imprecise information. This study, it is aimed to give new correlation coefficients by using Fermatean fuzzy sets. These coefficients identify the degree as well as the nature of correlation( positive or negative) between two Fermatean fuzzy sets. The new coefficient values will also be in the closed interval of $[-1; 1]$. Pairs of membership and non-membership degree as a vector representation with the two elements have been considered during formulation. In addition, the new method was compared with known methods.

**Keywords:** Correlation coefficient; covariance; Fermatean fuzzy set; Pearson correlation coefficient; Variance.

## 1 Introduction

The Fermatean fuzzy set (FFS) is an influential widening of the available intuitionistic fuzzy sets(IFS) and Pythagorean fuzzy sets(PFS), whose benefit is to better exhaustively characterize ambiguous information. That is, Fermatean fuzzy sets are powerful and valuable tools to represent imprecise information. These sets are extensions of the Fuzzy set(FS). There are a lot of research on FS and various extensions of FS, in the literature ([1–16]).

Since FFNs have a great strong ability to model uncertain and vague information in real-life implementations, this study improves the $KK$ based on FFNs to solve MCGDM problems with FFNs. Here, the average and variance of one FFS are defined, along with the covariance of two FFS, and then new $KK$s are defined between the two FFS with new informational energy. The new $KK$s defined between the two FFS have been obtained by considering MD and ND. In this study, new informational energy was first defined in the fermatean fuzzy environment. The informational energy measures the amount of uncertainty of a random variable but augments when randomness decreases. The informational energy is always strictly convex. Since the new $KK$s and different sets have different weights in real life, weighted $KK$s have been defined based on informational energy. A lot of approaches have similar been emerged to find the $KK$s between FSs, such as interval-valued FSs, type 2 FSs, IFSs, PFSs, and hesitant FSs. However, these techniques cannot operate the status in which some values are possible as MDs of the element as well as NDs of the same element. This study has been interested in finding out the correlation between FFE, which can disclose the connection between the FFSs. Generally, its value ranges between $[-1, 1]$, but as we are working under a fuzzy environment, we have excluded the negative part which is also referred to as a reverse correlation; our results lie in $[0, 1]$ interval. The concept of $KK$ belonging to FFSs was developed and presented, as FFSs are effective tools for obtaining relationships between elements with uncertain information. A numerical example such as medical diagnosis was given to show the efficiency of the new approaches. The new $KK$s were compared with the previous $KK$s.

**The originality:** There have been various extensions of the classical $KK$s such as fuzzy $KK$, IF $KK$, PF $KK$. These extensions have improved the performance of the $KK$s. FFSs can handle the problems with ambiguity and incomplete information more efficiency than that of IFSs and PFSs(Figure 1). In this study, the Fermatean fuzzy $KK$s were developed considering the intuitionistic fuzzy $KK$s and Pythagorean fuzzy $KK$s studies. Since the $MD^3 + ND^3 \leq 1$ requirement is satisfied for an object in the use of FFSs, there will be the possibility to cover more elements than IFSs and PFSs. A medical application regarding the new $KK$s is shown. $KK$s based on IFS and PFS given in previous studies were compared with newly proposed $KK$s.

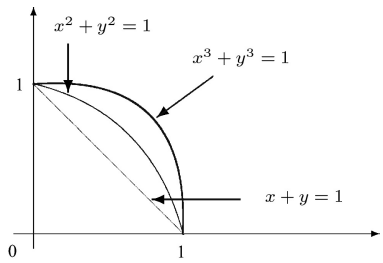


**Fig. 1**: Comparison of space of FMGs, PMGs and IFGs [11]

## 2 Preliminaries

**Definition 1.** *For $\mathcal{X} = \{x_1, x_2, \cdots, x_n\}$, if*

$$S = \{(x, \rho_S(x), \tau_S(x)) : x \in \mathcal{X}\}$$

*satisfies the following conditions, then the set $S$ is called FFS:*

$$\rho_S, \tau_S \in [0,1], \quad 0 \leq \rho_S^3 + \tau_S^3 \leq 1.$$

$\theta_S = (1 - \rho_S^3 + \tau_S^3)^{1/3}$ *shows the hesitation degree.*

The pair $(\rho_S(x), \tau_S(x))$ in the FFS $S$ is given as a Fermatean Fuzzy Number(FFN). The set of all FFs in $\mathcal{X}$ is denoted by $\Omega(\mathcal{X})$. Choose the FFNs $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$.

a. $\overline{\mathcal{F}} = (\tau_\mathcal{F}, \rho_\mathcal{F})$,
b. $\mathcal{F} \boxplus G = ((\rho_F^3 + \rho_\mathcal{G}^3 - \rho_\mathcal{F}^3 \rho_{\mathcal{G}\mathcal{F}}^3)^{1/3}, \tau_\mathcal{F}\tau_\mathcal{G})$,
c. $\mathcal{F} \boxtimes \mathcal{G} = (\rho_\mathcal{F}\rho_\mathcal{G}, (\tau_\mathcal{F}^3 + \tau_\mathcal{G}^3 - \tau_\mathcal{F}^3\tau_\mathcal{G}^3)^{1/3})$,
d. $z.\mathcal{F} = ((1 - (1 - \rho_\mathcal{F}^3)^z)^{1/3}, \tau_\mathcal{F}^z)$,
e. $\mathcal{F}^z = (\rho_\mathcal{F}^z, (1 - (1 - \eta_{\mathcal{F}F}^3)^z)^{1/3})$.

**Definition 2.** *Consider the two $FFNs$ $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$. For $\mathcal{F}$ and $\mathcal{G}$. The operation laws between them as follows:*

*i.* $\mathcal{F} \cup \mathcal{G} = (\max\{\rho_\mathcal{F}, \rho_\mathcal{G}\}, \min\{\tau_\mathcal{F}, \tau_\mathcal{G}\})$
*ii.* $\mathcal{F} \cap \mathcal{G} = (\min\{\rho_\mathcal{F}, \rho_\mathcal{G}\}, \max\{\tau_\mathcal{F}, \tau_\mathcal{G}\})$
*iii.* $\mathcal{F}^C = (\tau_\mathcal{F}, \rho_\mathcal{F})$
*iv.* $\mathcal{F} \preceq \mathcal{G}$ *if and only if* $\rho_\mathcal{F} \leq \rho_\mathcal{G}, \tau_\mathcal{F} \leq \tau_\mathcal{G}$.

**Definition 3** ([11])**.** *Consider the two $FFNs$ $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$. For $\mathcal{F}$ and $\mathcal{G}$, the score functions $SC(\mathcal{F}) = \rho_\mathcal{F}^3 - \tau_\mathcal{F}^3$ and $SC(\mathcal{G}) = \rho_\mathcal{G}^3 - \tau_\mathcal{G}^3$ and the accuracy functions $AC(\mathcal{F}) = \rho_\mathcal{F}^3 + \tau_\mathcal{F}^3$ and $AC(\mathcal{G}) = \rho_\mathcal{G}^3 + \tau_\mathcal{G}^3$.*

In this definition, the following situations are hold:

**Lemma 1.** *For the two $FFNs$ $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$,*

- *If $SC(\mathcal{F}) < SC(\mathcal{G})$, then $\mathcal{F} < \mathcal{G}$,*
- *If $SC(\mathcal{F}) = SC(\mathcal{G})$, $AC(\mathcal{F}) < AC(\mathcal{G})$, then $\mathcal{F} < \mathcal{G}$,*
- *If $SC(\mathcal{F}) = SC(\mathcal{G})$, $AC(\mathcal{F}) = AC(\mathcal{G})$, then $\mathcal{F} = \mathcal{G}$.*

**Lemma 2.** *Choose any two FSs $\mathcal{F}, \mathcal{G}$. If the conditions [i.]-[iv.] are held, then $S : FS \times FS \to [0,1]$ is said to be a SM between $\mathcal{F}, \mathcal{G}$.*

*i.* $0 \leq S(\mathcal{F}, \mathcal{G}) \leq 1$,
*ii.* $S(\mathcal{F}, \mathcal{G}) = 1 \Leftrightarrow \mathcal{F} = \mathcal{G}$,
*iii.* $S(\mathcal{F}, \mathcal{G}) = S(\mathcal{G}, \mathcal{F})$,
*iv.* $S(\mathcal{F}, \mathcal{H}) \leq S(\mathcal{F}, \mathcal{G})$ *and* $S(\mathcal{F}, \mathcal{H}) \leq S(\mathcal{G}, \mathcal{H})$ *if* $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$.

## 3 Correlation Coefficients with Variance and Covariance

**Definition 4.** *Choose any $\mathcal{F} \in \Omega(\mathcal{X})$.*

$$Ave(\mathcal{F}) \quad = \quad (\overline{\rho_\mathcal{F}}, \overline{\tau_\mathcal{F}}) = \left( \frac{1}{n} \sum_{k=1}^{n} \rho_\mathcal{F}(t_k), \frac{1}{n} \sum_{k=1}^{n} \tau_\mathcal{F}(t_k) \right) \tag{1}$$

*is called the average of $\mathcal{F}$.*

**Definition 5.** *Choose any $\mathcal{F} \in \Omega(\mathcal{X})$.*

$$V(\mathcal{F}) = \frac{1}{n-1}\sum_{k=1}^{n}\left(\left[(\rho_{\mathcal{F}}(t_k))^3 - (\overline{\rho_{\mathcal{F}}})^3\right]^2 + \left[(\tau_{\mathcal{F}}(t_k))^3 - (\overline{\tau_{\mathcal{F}}})^3\right]^2\right)$$

*is called the variance of $\mathcal{F}$.*

**Definition 6.** *Choose any $\mathcal{F}, \mathcal{G} \in \Omega(\mathcal{X})$.*

$$\begin{aligned} Cov(\mathcal{F},\mathcal{G}) = \frac{1}{n-1}\sum_{k=1}^{n}\Bigg(&\left\{\left[(\rho_{\mathcal{F}}(t_k))^3 - (\overline{\rho_{\mathcal{F}}})^3\right] \times \left[(\rho_{\mathcal{G}}(t_k))^3 - (\overline{\rho_{\mathcal{G}}})^3\right]\right\} \\ &+ \left\{\left[(\tau_{\mathcal{F}}(t_k))^3 - (\overline{\tau_{\mathcal{F}}})^3\right] \times \left[(\tau_{\mathcal{G}}(t_k))^3 - (\overline{\tau_{\mathcal{G}}})^3\right]\right\}\Bigg) \end{aligned} \tag{2}$$

*is said to be the covariance of $\mathcal{F}, \mathcal{G}$.*

**Proposition 1.** *For any $\mathcal{F}, \mathcal{G} \in \Omega(\mathcal{X})$, the following items are held:*

*i.* $Cov(\mathcal{F},\mathcal{G}) = Cov(\mathcal{G},\mathcal{F})$
*ii.* $Cov(\mathcal{F},\mathcal{F}) = V(\mathcal{F})$
*iii.* $|V(\mathcal{F})| \leq \sqrt{V(\mathcal{F})V(\mathcal{G})}$.

**Definition 7.** *Take any $\mathcal{F}, \mathcal{G} \in \Omega(\mathcal{X})$.*

$$KK(\mathcal{F},\mathcal{G}) = \frac{Cov(\mathcal{F},\mathcal{G})}{\sqrt{V(\mathcal{F})V(\mathcal{G})}}$$

*is said to be the $KK$ of $\mathcal{F}, \mathcal{G}$.*

Now, we give a new definition of KK.

**Definition 8.** *For any $\mathcal{F}, \mathcal{G} \in \Omega(\mathcal{X})$, then the KK between $\mathcal{F}, \mathcal{G}$, where $Cov(\mathcal{F},\mathcal{G})$ is defined as*

$$KK_H(\mathcal{F},\mathcal{G}) = \frac{Cov(\mathcal{F},\mathcal{G})}{\sqrt{V(\mathcal{F})V(\mathcal{G})}}.$$

*where*

$$Var_H(\mathcal{F}) = \frac{1}{n-1}\sum_{k=1}^{n}\left(\left[\rho_{(\mathcal{F}}(t_k))^3 - \rho_{(\mathcal{F}})^3\right]^2 + \left[\tau_{(\mathcal{F}}(t_k))^3 - \tau_{(\mathcal{F}})^3\right]^2 + \left[\theta_{(\mathcal{F}}(t_k))^3 - \theta_{(\mathcal{F}})^3\right]^2\right)$$

*and*

$$\begin{aligned} Cov_H(\mathcal{F},\mathcal{G}) = \frac{1}{n-1}\sum_{k=1}^{n}\Bigg(&\left\{\left[(\rho_{\mathcal{F}}(t_k))^3 - (\overline{\rho_{\mathcal{F}}})^3\right] \times \left[(\rho_{\mathcal{G}}(t_k))^3 - (\overline{\rho_{\mathcal{G}}})^3\right]\right\} \\ &+ \left\{\left[(\tau_{\mathcal{F}}(t_k))^3 - (\overline{\tau_{\mathcal{F}}})^3\right] \times \left[(\tau_{\mathcal{G}}(t_k))^3 - (\overline{\tau_{\mathcal{G}}})^3\right]\right\} \\ &+ \left\{\left[(\theta_{\mathcal{F}}(t_k))^3 - (\overline{\theta_{\mathcal{F}}})^3\right] \times \left[(\theta_{\mathcal{G}}(t_k))^3 - (\overline{\theta_{\mathcal{G}}})^3\right]\right\}\Bigg) \end{aligned}$$

*and also*

$$\overline{\theta_{\mathcal{F}}} = \frac{1}{n}\sum_{k=1}^{n}\theta_{\mathcal{F}}(t_k),$$

$$\overline{\theta_{\mathcal{G}}} = \frac{1}{n}\sum_{k=1}^{n}\theta_{\mathcal{G}}(t_k).$$

## 4 Pearson Correlation Coefficient Formula

First, let's give some equations that will be used in this section:

$$\lambda_1 = \frac{\sum_{k=1}^{n}\left[\left((\rho_{\mathcal{F}}(t_k))^3 - (\overline{\rho_{\mathcal{F}}})^3\right) \times \left((\rho_{\mathcal{G}}(t_k))^3 - (\overline{\rho_{\mathcal{G}}})^3\right)\right]}{\left[\sqrt{((\rho_{\mathcal{F}}(t_k))^3 - (\overline{\rho_{\mathcal{F}}})^3)^2} \times \sqrt{((\rho_{\mathcal{G}}(t_k))^3 - (\overline{\rho_{\mathcal{G}}})^3)^2}\right]}, \tag{3}$$

$$\lambda_2 = \frac{\sum_{k=1}^{n}\left[\left((\tau_{\mathcal{F}}(t_k))^3 - (\overline{\tau_{\mathcal{F}}})^3\right) \times \left((\tau_{\mathcal{G}}(t_k))^3 - (\overline{\tau_{\mathcal{G}}})^3\right)\right]}{\left[\sqrt{((\tau_{\mathcal{F}}(t_k))^3 - (\overline{\tau_{\mathcal{F}}})^3)^2} \times \sqrt{((\tau_{\mathcal{G}}(t_k))^3 - (\overline{\tau_{\mathcal{G}}})^3)^2}\right]}, \tag{4}$$

and

$$\lambda_3 = \frac{\sum_{k=1}^{n}\left[\left((\theta_{\mathcal{F}}(t_k))^3 - (\overline{\theta_{\mathcal{F}}})^3\right) \times \left((\theta_{\mathcal{G}}(t_k))^3 - (\overline{\theta_{\mathcal{G}}})^3\right)\right]}{\left[\sqrt{\left((\theta_{\mathcal{F}}(t_k))^3 - (\overline{\theta_{\mathcal{F}}})^3\right)^2} \times \sqrt{\left((\theta_{\mathcal{G}}(t_k))^3 - (\overline{\theta_{\mathcal{G}}})^3\right)^2}\right]}. \tag{5}$$

**Definition 9.** *For any $\mathcal{F}, \mathcal{G} \in \Omega(\mathcal{X})$,*

$$KK_P(\mathcal{F}, \mathcal{G}) = \frac{1}{2}(\lambda_1 + \lambda_2)$$

*is called the correlation coefficient, where $\lambda_1$ and $\lambda_2$ are defined as in Equations (3) and (4), respectively.*

This definition can be defined by degree of hesitation as follows:

**Definition 10.** *For any $\mathcal{F}, \mathcal{G} \in \Omega(\mathcal{X})$,*

$$KK_{PH}(\mathcal{F}, \mathcal{G}) = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)$$

*is called the correlation coefficient, where $\lambda_1$, $\lambda_2$ and, $\lambda_3$ are defined as in Equations (3), (4), and (5), respectively.*

**Theorem 1.** *For any $\mathcal{F}, \mathcal{G} \in \Omega(\mathcal{X})$,*

*i.* $KK_P(\mathcal{F}, \mathcal{G}) = KK_P(\mathcal{G}, \mathcal{F})$, *($KK_{PH}(\mathcal{F}, \mathcal{G}) = KK_{PH}(\mathcal{G}, \mathcal{F})$),*
*ii.* $-1 \leq KK_P(\mathcal{F}, \mathcal{G}) \leq 1$, *($-1 \leq KK_{PH}(\mathcal{F}, \mathcal{G}) \leq 1$),*
*iii. If $\mathcal{F} = \alpha\mathcal{G}$ for some $\alpha$, then*

$$KK_P(\mathcal{F}, \mathcal{G}) = \begin{cases} 1 & , \quad \alpha > 0, \\ -1 & , \quad \alpha < 0. \end{cases}$$

$$\left(KK_{PH}(\mathcal{F}, \mathcal{G}) = \begin{cases} 1 & , \quad \alpha > 0, \\ -1 & , \quad \alpha < 0. \end{cases}\right)$$

## 5 Medical Application

The infectious diseases example from Kirisci and Simsek [9] was adapted for this study to represent the application of the suggested method in MCDM.

By considering the symptoms, the disease status of the patients will be calculated with the help of distance/similarity/correlation criteria. According to the results obtained, the disease that the patient suffers most will be determined.
For a set of patients $H = \{H_1, H_2, H_3, H_4\}$, let s

$$\begin{aligned} U &= \{\text{ Hepatitis C, Crimean-Congo Hemorrhagic Fever(CCHF), influenza A(H1N1), sandfly fever, norovirus}\} \\ &= \{U_1, U_2, U_3, U_4, U_5\} \end{aligned}$$

be the set of five alternatives. Alternatives in this set were selected as infectious diseases, which are common in Turkey, before COVID-19. The set of symptoms

$$\begin{aligned} S &= \{\text{chest pain, cough, stomachpain, headache, temperature}\} \\ &= \{s_1, s_2, s_3, s_4, s_5\}. \end{aligned}$$

Maximum correlation/minimum distance/maximum similarity will be examined for the relationships between symptoms-diseases, symptoms-patients.

In Table 3- Table 6, bold places indicate diagnostic results.

 

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $U_1$ | (0.1, 0.9) | (0.2, 0.9) | (0.8, 0.5) | (0.4, 0.5) | (0.9, 0.2) |
| $U_2$ | (0.2, 0.7) | (0.7, 0.6) | (0.7, 0.4) | (0.8, 0.4) | (0.9, 0.1) |
| $U_3$ | (0.4, 0.6) | (0.9, 0.2) | (0.1, 0.7) | (0.7, 0.5) | (0.8, 0.4) |
| $U_4$ | (0.5, 0.7) | (0.2, 0.7) | (0.6, 0.6) | (0.8, 0.3) | (0.9, 0.1) |
| $U_5$ | (0.3, 0.7) | (0.2, 0.8) | (0.8, 0.5) | (0.9, 0.1) | (0.4, 0.6) |

**Table 1** Diseases-Symptoms

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $H_1$ | (0.0, 0.6) | (0.6, 0.3) | (0.8, 0.1) | (0.2, 0.6) | (0.8, 0.3) |
| $H_2$ | (0.1, 0.4) | (0.4, 0.5) | (0.6, 0.3) | (0.7, 0.4) | (0.8, 0.1) |
| $H_3$ | (0.1, 0.5) | (0.8, 0.1) | (0.3, 0.7) | (0.5, 0.7) | (0.8, 0.2) |
| $H_4$ | (0.3, 0.5) | (0.0, 0.8) | (0.2, 0.6) | (0.6, 0.5) | (0.9, 0.1) |

**Table 2** Patients-Symptoms

| | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ |
|---|---|---|---|---|---|
| $H_1$ | 0.6347 | -0.1614 | **0.7791** | -0.0138 | 0.5217 |
| $H_2$ | 0.1290 | -0.3876 | **0.7958** | -0.2451 | 0.4592 |
| $H_3$ | 0.2193 | 0.2682 | **0.5135** | -0.1405 | 0.1470 |
| $H_4$ | 0.5315 | 0.3758 | 0.5016 | 0.4816 | **0.6973** |

**Table 3** Values of $KK$

| | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ |
|---|---|---|---|---|---|
| $H_1$ | 0.4233 | -0.5857 | **0.7225** | -0.0079 | 0.2851 |
| $H_2$ | 0.0894 | -0.3165 | **0.7143** | -0.2017 | 0.1928 |
| $H_3$ | 0.0065 | 0.1436 | **0.2619** | -0.0291 | 0.079 |
| $H_4$ | 0.5315 | 0.3758 | 0.5016 | 0.4816 | **0.6184** |

**Table 4** Values of $KK_H$

| | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ |
|---|---|---|---|---|---|
| $H_1$ | **0.6198** | 0.5983 | 0.3704 | -0.2177 | 0.1912 |
| $H_2$ | **0.7354** | -0.1508 | 0.1005 | 0.1606 | -0.081 |
| $H_3$ | 0.2268 | 0.1325 | **0.4913** | -0.019 | -0.0125 |
| $H_4$ | 0.1798 | 0.6784 | 0.0672 | 0.1609 | **0.7164** |

**Table 5** Values of $KK_P$

| | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ |
|---|---|---|---|---|---|
| $H_1$ | **0.6829** | 0.4136 | 0.1184 | -0.0189 | 0.0463 |
| $H_2$ | 0.3381 | **0.6932** | 0.1057 | 0.2460 | -0.134 |
| $H_3$ | 0.0136 | 0.1695 | **0.2656** | -0.1298 | -0.0449 |
| $H_4$ | 0.1073 | 0.3407 | 0.0506 | -0.2741 | **0.4713** |

**Table 6** Values of $KK_{PH}$

| | $H_1$ | $H_2$ | $H_3$ | $H_4$ |
|---|---|---|---|---|
| $KK$ | $U_3$ | $U_3$ | $U_3$ | $U_5$ |
| $KK_H$ | $U_3$ | $U_3$ | $U_3$ | $U_5$ |
| $KK_P$ | $U_1$ | $U_1$ | $U_3$ | $U_5$ |
| $KK_{PH}$ | $U_1$ | $U_2$ | $U_3$ | $U_5$ |

**Table 7** Results

## 6 Discussion and Conclusion

The implementations of fuzzy and non-standard fuzzy KKs are obvious in image segmentation, clustering analysis, pattern classification, etc. In a specific scenario, these measures appear to have similar implications but satisfy different results. Different results with different similarity/distance/correlation measures are obtained. This difference is due to the inability of these measures to determine the complete uncertainty or precision required in such sets.

We can explain the advantages of the proposed methods as follows:

The FFS approach is a advantageous, practical, and considerable generalized model of IFSs and PFSs. In this case, experts become more independent in expressing their beliefs about the degree of membership. The choice of the best alternative from a set of alternatives in an MCGDM problem is handicapped when uncertain data are strained to adopt the limited form of IFNs and PFNs. The aforenamed cases would cause the mutilation of data.A more generalized version is needed to provide efficient solutions in such crucial cases. FFSs give more correct and exact outcomes when used to cope with practical MCGDM problems including FF information as they are an effective extension of IFSs and PFSs. The first distinctive property of the FF-based $KK$s is that these are more close to the human judgment due to containment of more

dimensions of the cognitive vagueness such as MD, ND, and neutrality. The new $KK$s are capable to appreciate two negatively correlated properties as they get their values in $[-1, 1]$. Considering these advantages, the proposed $KK$s seems to be advantageous in the practical problems regarding knowledge estimation in data. More significant knowledge of associations and patterns can be revealed from the big data if the machine learning algorithms are contrived using the new approaches. Despite these advantages, there are some limitations to the new $KK$s. The $KK$s concerning FFSs are difficult to implement to the crisp data present in repositories and likewise other websites. These can be implemented either with the help of some transformation formulae or by creating a multi-dimensional linguistic database.

First of all, let's explain the advantages of the presented technique and the differences with other techniques. As is known, FFSs can investigate the problems with vagueness and incomplete information more efficiency than of IFSs. Since the sets of Pythagorean and intuitionistic MDs are not as extensive as the sets of Fermatean MDs [11], it is clear that FFSs will have many comprehensive possibilities for identifying and resolving uncertainties than IFS and PFS. IFS is a successful generalization of FS theory in dealing with uncertainty and uncertainty, which is characterized by $MD + ND \leq 1$. However, there are cases where the sum of the MDs and NDs will be greater than 1. In this case, the IFS technique will be insufficient to solve this problem. To solve this inadequacy, PFS which is initiated by Yager has emerged. PFS is a natural generalization of FS theory, with successful results. However, the sum of the squares of MD and ND of DMR of a particular attribute may also be greater than 1, in which case it will not be an appropriate solution method in PFS. There are $KK$s obtained with IFSs and PFSs in the literature, and there are algorithms defined using these $KK$s. As mentioned earlier, some cases cannot be symbolized by IFSs and PFSs, hence appropriate results may not be obtained from their corresponding algorithms. The $KK$s obtained with IFSs and PFSs are a specific situation of the $KK$ of FFSs. Then, the suggested $KK$ is more generalized than existing ones and is appropriate for solving real-life problems more accurately. Correlation is fundamentally a statistical approach that demonstrates the linkage between two elements. The principal outcome of a correlation is called a $KK$. This study is dedicated to defining a $KK$ for FFS. This study has extended the constraint conditions of $MD + ND \leq 1$ for IFS and the $MD^2 + ND^2 \leq 1$ for PFS to the FFS $KK$ theory. The numeric example has been served that represents that the offered $KK$ can easily operate the conditions where the present $KK$s in the IFS and PFS frameworks fail. The main caharacteristic of FFSs is that there is a MF and a NF of an FFS according to the elements in the sample space. Therefore, the correlation between FFSs has its own characteristics. Like the results obtained by many researchers in previous studies, correlation coefficients of FFSs based on both MFs and NFs are discussed. From the illustrative example, it has been accomplished that the offered $KK$ in the FFS framework can conveniently operate the real-life DM problem with their objectives. From the calculated results, the advantages of $KK$s defined in the FFS environment are indicated as: The results obtained using the suggested $KK$s are more sensitive. Thus, computational overheads are reduced and results are more amenable to real-life scenarios.

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# Fermatean Fuzzy Type Similarity and Distance Measures with TOPSIS Method

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**Abstract:** Distance and cosine similarity measures are the most convenient ways to verify the degrees of similarity and distinction between two sets. In this study, new measures of distance and cosine similarity between Fermatean fuzzy sets are given. Initially, the definitions of the new measures based on Fermatean fuzzy sets were given and their properties were examined. There may be cases where the similarity measure conditions of the cosine measure do not apply. In this case, a method is proposed to create new similarity measures between two sets according to the initially given measures. The new method provides the similarity measure condition. The new measure between Fermatean fuzzy sets is derived from the idea of the association between the distance and cosine measures.

**Keywords:** Euclidean distance; Fermatean fuzzy set; multi-criteria decision-making; similarity measure; TOPSIS.

## 1 Introduction

In the field of fuzzy set theory, the similarity metric is a crucial concept. In pattern recognition, medical diagnostics, and other fields, it is commonly employed. On fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), and Pythagorean fuzzy sets (PFSs), several similarity metrics have been investigated ([3], [5, 7, 9–12]).

A series of distance and similarity measurements between two hesitant fuzzy linguistic word sets is provided in [3]. Second, different weighted or ordinal weighted distance and similarity measurements are provided between two collections of hesitant fuzzy linguistic word sets. Following that, these metrics were examined in both discrete and continuous scenarios. In [10], a cosine similarity measure and a weighted cosine similarity measure between IFSs are proposed based on the concept of the cosine similarity measure for fuzzy sets, taking into account the information carried by the membership degree and the non-membership degree in IFSs as a vector representation with the two elements. Zhou et al. [12] developed the heuristic fuzzy ordered weighted cosine similarity measure by combining the heuristic fuzzy ordered weighted cosine similarity measure and the extended ordinal weighted average operator. The intuitionistic fuzzy ordered weighted cosine similarity measure distinguishes itself by not only being an extension of several frequently used similarity measures, but also by dealing with the correlation of distinct decision matrices or multi-dimensional arrays for intuitionistic fuzzy values. The entropy of interval-valued fuzzy sets and similarity measures of interval-valued fuzzy sets were presented by Zeng and Li [11]. Based on their axiomatic definitions, Zeng and Li [11] established three theorems that similarity measure and entropy of interval-valued fuzzy sets may be modified by each other and proposed some formulae to compute entropy and similarity measure of interval-valued fuzzy sets. Wei [6] introduced several unique approaches for determining the similarity of picture fuzzy sets. Some similarity metrics across image fuzzy sets are defined in [6], including cosine similarity, weighted cosine similarity, set-theoretic similarity, weighted set-theoretic cosine similarity, grey similarity, and weighted grey similarity. Wei and Wei [7] proposed 10 similarity metrics between PFSs based on the cosine function, taking into account the degree of membership, nonmembership, and reluctance in PFSs. These similarity and weighted similarity scores between PFSs were applied to pattern recognition and medical diagnostics. The axiom definitions of entropy, distance measure, and similarity measure of fuzzy sets are systematically presented in [9], and essential relationships between these measures are examined. Sridevi and Nadarajan [9] presented a new fuzzy similarity measure to determine the degree of similarity of generalized fuzzy numbers (GFNs). The fuzzy similarity measure is created by combining the notion of center of gravity (COG) points with the fuzzy difference of distance between fuzzy number points. Aydin [1] introduced a new MCDM technique using FFSs that employs entropy theory to compute criterion weights and cosine similarity measurements to select the optimal option. Xu and Shen [8] investigated Fermatean fuzzy set similarity measures. The definitions of the Fermatean fuzzy sets similarity measures and its weighted similarity measures on discrete and continuous universes are provided in turn in this work. The fundamental features of the proposed similarity metrics are then addressed. Following that, a decision-making process based on the TOPSIS approach is constructed in the Fermatean fuzzy environment, and a novel method based on the provided Fermatean fuzzy sets similarity measures is designed to tackle medical diagnosis issues.

The major reason we used FFSs in designing the current study's strategy is because of its flexibility in dealing with unclear information. The supreme tendency of FFSs to address inexact human decisions makes it more feasible and accurate to model two-dimensional (i.e., membership and non-membership) information in a wider space as compared to IFSs and PFSs. The inner product of two vectors divided by the product of their lengths gives the measure of cosine similarity. This study aims to define cosine similarity and weighted cosine similarity measures based on FFSs. The characteristics of the new cosine similarity measures will be examined, and a new decision-making algorithm based on these

measures will be presented. The algorithm is obtained by combining the new cosine similarity measures with the TOPSIS method.

**The originality:** There have been various extensions of the classical cosine similarities such as fuzzy, IF, and PF cosine similarities. These extensions have improved the performance of the cosine similarities. FFSs can handle problems with ambiguity and incomplete information more efficiently than that of IFSs and PFSs. In this study, the Fermatean fuzzy cosine and weighted cosine similarity measures were developed considering the intuitionistic fuzzy and Pythagorean fuzzy cosine similarity measures studies. Since the $MD^3 + ND^3 \leq 1$ requirement is satisfied for an object in the use of FFSs, there will be the possibility to cover more elements than IFSs and PFSs. A medical application regarding the new similarities is shown.

The remainder of this article is structured as follows. In Section 2, we will give the fundamental information that will be used in the study. In Section 3, we will present new cosine similarity and weighted cosine similarity measures and show the properties of these measures. Section 4 is devoted to the MCDM algorithm with respect to cosine similarities and the TOPSIS technique. In the fifth chapter, an application to infectious diseases is presented. The medical decision-making model is shown that the cosine similarities given in the study are easy to use and optimum results can be obtained. From the illustrative example study, it has been accomplished that the offered cosine similarities in the FFS framework can conveniently operate the real-life DM problem with their objectives.

## 2 Preliminaries

Now, some fundamental information that will be used in the study will be given.

**Definition 1.** *For $\mathcal{X} = \{x_1, x_2, \cdots, x_n\}$, if*

$$S = \{(x, \rho_S(x), \tau_S(x)) : x \in \mathcal{X}\}$$

*satisfies the following conditions, then the set $S$ is called FFS:*

$$\rho_S, \tau_S \in [0,1], \quad 0 \leq \rho_S^3 + \tau_S^3 \leq 1.$$

$\theta_S = (1 - \rho_S^3 + \tau_S^3)^{1/3}$ *shows the hesitation degree.*

The pair $(\rho_S(x), \tau_S(x))$ in the FFS $S$ is defined as a Fermatean Fuzzy Number(FFN).
Choose the FFNs $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$.

a. $\overline{\mathcal{F}} = (\tau_\mathcal{F}, \rho_\mathcal{F})$,
b. $\mathcal{F} \boxplus G = ((\rho_F^3 + \rho_\mathcal{G}^3 - \rho_\mathcal{F}^3 \rho_{\mathcal{G}\mathcal{F}}^3)^{1/3}, \tau_\mathcal{F}\tau_\mathcal{G})$,
c. $\mathcal{F} \boxtimes \mathcal{G} = (\rho_\mathcal{F}\rho_\mathcal{G}, (\tau_\mathcal{F}^3 + \tau_\mathcal{G}^3 - \tau_\mathcal{F}^3\tau_\mathcal{G}^3)^{1/3})$,
d. $z.\mathcal{F} = ((1 - (1 - \rho_\mathcal{F}^3)^z)^{1/3}, \tau_\mathcal{F}^z)$,
e. $\mathcal{F}^z = (\rho_\mathcal{F}^z, (1 - (1 - \eta_{\mathcal{F}F}^3)^z)^{1/3})$.

**Definition 2.** *Consider the two $FFNs$ $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$. For $\mathcal{F}$ and $\mathcal{G}$. The operation laws between them are as follows:*

*i.* $\mathcal{F} \cup \mathcal{G} = (\max\{\rho_\mathcal{F}, \rho_\mathcal{G}\}, \min\{\tau_\mathcal{F}, \tau_\mathcal{G}\})$
*ii.* $\mathcal{F} \cap \mathcal{G} = (\min\{\rho_\mathcal{F}, \rho_\mathcal{G}\}, \max\{\tau_\mathcal{F}, \tau_\mathcal{G}\})$
*iii.* $\mathcal{F}^C = (\tau_\mathcal{F}, \rho_\mathcal{F})$
*iv.* $\mathcal{F} \preceq \mathcal{G}$ *if and only if* $\rho_\mathcal{F} \leq \rho_\mathcal{G}, \tau_\mathcal{F} \leq \tau_\mathcal{G}$.

**Definition 3.** *[4] Consider the two $FFNs$ $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$. For $\mathcal{F}$ and $\mathcal{G}$, the score functions $SC(\mathcal{F}) = \rho_\mathcal{F}^3 - \tau_\mathcal{F}^3$ and $SC(\mathcal{G}) = \rho_\mathcal{G}^3 - \tau_\mathcal{G}^3$ and the accuracy functions $AC(\mathcal{F}) = \rho_\mathcal{F}^3 + \tau_\mathcal{F}^3$ and $AC(\mathcal{G}) = \rho_\mathcal{G}^3 + \tau_\mathcal{G}^3$.*

In this definition, the following situations are held:

**Lemma 1.** *For the two $FFNs$ $\mathcal{F} = (\rho_\mathcal{F}, \tau_\mathcal{F})$ and $\mathcal{G} = (\rho_\mathcal{G}, \tau_\mathcal{G})$,*

- *If $SC(\mathcal{F}) < SC(\mathcal{G})$, then $\mathcal{F} < \mathcal{G}$,*
- *If $SC(\mathcal{F}) = SC(\mathcal{G})$, $AC(\mathcal{F}) < AC(\mathcal{G})$, then $\mathcal{F} < \mathcal{G}$,*
- *If $SC(\mathcal{F}) = SC(\mathcal{G})$, $AC(\mathcal{F}) = AC(\mathcal{G})$, then $\mathcal{F} = \mathcal{G}$.*

**Lemma 2.** *Choose any two FSs $\mathcal{F}, \mathcal{G}$. If the conditions [i.]-[iv.] are held, then $S : FS \times FS \to [0,1]$ is said to be an SM between $\mathcal{F}, \mathcal{G}$.*

*i.* $0 \leq S(\mathcal{F}, \mathcal{G}) \leq 1$,
*ii.* $S(\mathcal{F}, \mathcal{G}) = 1 \Leftrightarrow \mathcal{F} = \mathcal{G}$,
*iii.* $S(\mathcal{F}, \mathcal{G}) = S(\mathcal{G}, \mathcal{F})$,
*iv.* $S(\mathcal{F}, \mathcal{H}) \leq S(\mathcal{F}, \mathcal{G})$ *and* $S(\mathcal{F}, \mathcal{H}) \leq S(\mathcal{G}, \mathcal{H})$ *if* $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$.

 

## 3 Cosine Similarity and Distance Measures

**Definition 4.** *Take a fixed set $\mathcal{X} = \{x_1, x_2, \cdots, x_n\}$. Choose any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{G} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in \mathcal{X}\}$. Therefore, a CS measure $C_{FFS}(\mathcal{F}, \mathcal{G})$ between $\mathcal{F}$ and $\mathcal{G}$ can be defined as*

$$C_{FFS}(\mathcal{F}, \mathcal{G}) = \frac{1}{n} \sum_{i=1}^{n} \frac{\rho^3_{\mathcal{F}}(x_i)\rho^3_{\mathcal{G}}(x_i) + \tau^3_{\mathcal{F}}(x_i)\tau^3_{\mathcal{G}}(x_i) + \theta^3_{\mathcal{F}}(x_i)\theta^3_{\mathcal{G}}(x_i)}{\sqrt[3]{\rho^6_{\mathcal{F}}(x_i) + \tau^6_{\mathcal{F}}(x_i) + \theta^6_{\mathcal{F}}(x_i)}\sqrt[3]{\rho^6_{\mathcal{G}}(x_i) + \tau^6_{\mathcal{G}}(x_i) + \theta^6_{\mathcal{G}}(x_i)}}.$$

*For $x_i \in \mathcal{X}$, take the weight $\omega_i$. The weighted cosine similarity(WCS) measure $C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ is given as*

$$C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) = \frac{1}{n} \sum_{i=1}^{n} \omega_i \frac{\rho^3_{\mathcal{F}}(x_i)\rho^3_{\mathcal{G}}(x_i) + \tau^3_{\mathcal{F}}(x_i)\tau^3_{\mathcal{G}}(x_i) + \theta^3_{\mathcal{F}}(x_i)\theta^3_{\mathcal{G}}(x_i)}{\sqrt[3]{\rho^6_{\mathcal{F}}(x_i) + \tau^6_{\mathcal{F}}(x_i) + \theta^6_{\mathcal{F}}(x_i)}\sqrt[3]{\rho^6_{\mathcal{G}}(x_i) + \tau^6_{\mathcal{G}}(x_i) + \theta^6_{\mathcal{G}}(x_i)}}.$$

When take $\omega = \{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\}$, the WCS $C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ is reduced to the CS measure $C_{FFS}(\mathcal{F}, \mathcal{G})$.

**Theorem 1.** *Take any two FFSs $\mathcal{F}$ and $\mathcal{G}$. Therefore the CS measure $C_{FFS}(\mathcal{F}, \mathcal{G})$ (the WCS measure $C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ )satisfies the following conditions:*

*i.* $0 \leq C_{FFS}(\mathcal{F}, \mathcal{G}) \leq 1$, *(*$0 \leq C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) \leq 1$*),*
*ii.* $C_{FFS}(\mathcal{F}, \mathcal{G}) = C_{FFS}(\mathcal{G}, \mathcal{F})$, *(*$C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) = C^{\omega}_{FFS}(\mathcal{G}, \mathcal{F})$*),*
*iii.* $C_{FFS}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *(*$\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$*), (*$C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *that is,* $\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$*).*

If a SM $S(\mathcal{F}, \mathcal{G})$ satisfies the conditions [i]-[iii] of Lemma 2, then $S(\mathcal{F}, \mathcal{G})$ is called genuine SM. It is known that an SM satisfies the conditions of Lemma 2, then we can give the following statement:

Let $D(\mathcal{F}, \mathcal{G})$ show the distance measure(DiMe) between $\mathcal{F}, \mathcal{G}$. Then $D = 1 - S$ is the SM between $\mathcal{F}, \mathcal{G}$.

Since Definition 4 and Definition 4 do not the hold the condition [ii] of Lemma 2 in several situation, these CS measures are not the genuine SMs.

**Definition 5.** *For any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{G} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in \mathcal{X}\}$. The Euclidean DiMe $D_{FFS}(\rho, \tau)$ is defined as*

$$D_{FFS}(\mathcal{F}, \mathcal{G}) = \left( \frac{1}{2n} \sum_{x_i \in X} \left( |\rho^3_{\mathcal{F}} - \rho^3_{\mathcal{G}}|^2 + |\tau^3_{\mathcal{F}} - \tau^3_{\mathcal{G}}|^2 + |\theta^3_{\mathcal{F}} - \theta^3_{\mathcal{G}}|^2 \right) \right)^{1/2}.$$

*For $x_i \in X$, take the weight $\omega_i$. The weighted Euclidean DiMe $D^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ is described as*

$$D^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) = \left( \frac{1}{2} \sum_{x_i \in X} \omega_i \left( |\rho^3_{\mathcal{F}} - \rho^3_{\mathcal{G}}|^2 + |\tau^3_{\mathcal{F}} - \tau^3_{\mathcal{G}}|^2 + |\theta^3_{\mathcal{F}} - \theta^3_{\mathcal{G}}|^2 \right) \right)^{1/2}.$$

**Theorem 2.** *For any two FFSs $\mathcal{F}, \mathcal{G}$, the weighted Euclidean DiMe $D^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ satisfies the [i.]-[iii.] conditions:*

*i.* $0 \leq D^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) \leq 1$
*ii.* $D^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) = D^{\omega}_{FFS}(\mathcal{G}, \mathcal{F})$
*iii.* $D^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *that is,* $\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$.

**Definition 6.** *For any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{F} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in \mathcal{X}\}$. The new SM $S^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ can be given as*

$$S_{FFS}(\mathcal{F}, \mathcal{G}) = \frac{C_{FFS}(\mathcal{F}, \mathcal{G}) + 1 - D_{FFS}(\mathcal{F}, \mathcal{G})}{2}$$

**Definition 7.** *For any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{G} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in X\}$, the WCS measure $S^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ between $\mathcal{F}$ and $\mathcal{G}$ can be defined as*

$$S^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) = \frac{C^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) + 1 - D^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})}{2}$$

*where $\omega_i$ denote the weight of $x_i \in \mathcal{X}$ (*$\sum_{i=1}^{n} \omega_i = 1$, $0 \leq \omega_i \leq 1$*).*

**Theorem 3.** *For the two FFSs $F, G$, the new WCS measure $S^{\omega}_{FFS}(\mathcal{F}, \mathcal{G})$ satisfies the [i.]-[iii.] conditions:*

*i.* $0 \leq S^{\omega}_{FFS}(\mathcal{F}, \mathcal{G}) \leq 1$

*ii.* $S^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) = S^{\omega}_{FFS}(\mathcal{G},\mathcal{F})$
*iii.* $S^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, $(\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i), \tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i))$.

When the SM satisfies the condition of DiMe, then a corresponding DiMe can be obtained concerning the relationship between the DM and SM. Since the suggested SM $S^{\omega}_{FFS}(\mathcal{F},\mathcal{G})$ is a genuine SM, the corresponding DiMe $D^{\omega}_{FFS}(\mathcal{F},\mathcal{G})$ between any two FFSs $\mathcal{F},\mathcal{G}$ is obtained as follows:

**Definition 8.** *For the two FFSs $\mathcal{F},\mathcal{G}$. The weighted distance measure(WDM)*

$$DM^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) = 1 - S^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) = \frac{1 - C^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) + D^{\omega}_{FFS}(\mathcal{F},\mathcal{G})}{2}.$$

*where $\omega_i$ denote the weight of $x_i \in \mathcal{X}$ ($\sum_{i=1}^{n} \omega_i = 1$).*

If take $\omega = (1/n, \cdots, 1/n)$, the DiMe $D_{FFS}(\mathcal{F},\mathcal{G})$ is obtained.

**Theorem 4.** *For the two FFSs $\mathcal{F},\mathcal{G}$, $DM^{\omega}_{FFS}(\mathcal{F},\mathcal{G})$ satisfies the following conditions:*

*i.* $0 \le DM^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) \le 1$
*ii.* $DM^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) = DM^{\omega}_{FFS}(\mathcal{G},\mathcal{F})$
*iii.* $DM^{\omega}_{FFS}(\mathcal{F},\mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *that is,* $\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$.

The distance measure $D_{FFS}(\mathcal{F},\mathcal{G})$ also satisfies the properties of Theorem 4.

## 4 TOPSIS Approach

This section is dedicated to developing a TOPSIS technique for MCDM with FFS.

Consider that the experts evaluate the alternatives $U = \{U_1, U_2, \cdots, U_m\}$ according to the criteria $K = \{K_1, K_2, \cdots, K_n\}$, which are represented by FFSs $U_{ij} = (\rho_{ij}, \tau_{ij})$ such that $\rho_{ij}, \tau_{ij} \in [0,1]$ and $\rho^3_{ij} + \tau^3_{ij} \le 1$.

Let $\omega$ be weight vector of criteria satisfying with $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \ge 0$. Then the FF decision matrix(FFDMT) $E = (U_{ij})_{n\times n} = ((\rho_{ij}, \tau_{ij}))_{M\times n}$ is shown as: For $i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$,

$$E = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ U_{21} & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ U_{m1} & U_{m2} & \cdots & U_{mn} \end{pmatrix}$$

where $U_{ij}$ are FFSs.

The algorithm based on the suggested CM is developed as follows:

**1:** Firstly, we will normalize the decision matrix $E = (U_{ij})_{n\times n} = ((\rho_{ij}, \tau_{ij}))_{m\times n}$. For normalization we will use the following negation operator:

$$\widehat{E} = ((\widehat{\rho}_{ij}, \widehat{\tau}_{ij})) \begin{cases} (\rho_{ij}, \tau_{ij}) & \textit{for benefit type} \quad K_j, \\ (\tau_{ij}, \rho_{ij}) & \textit{for cost type} \quad K_j. \end{cases} \tag{1}$$

This operator is comprehended as follows: If the criterion we are considering is benefit-type, no action is taken. If our criterion is cost-type, we will convert this criterion to benefit-type.

**2:** We will obtain positive and negative ideal solutions determined with the help of the score and accuracy functions and denoted by $U^+ = \{U^+_1, U^+_2, \cdots, U^+_n\}$, $U^- = \{U^-_1, U^-_2, \cdots, U^-_n\}$: For $j = 1, 2, \cdots, n$,

$$\begin{aligned} U^+_j &= \max\{SC(U_{1j}), SC(U_{2j}), \cdots, SC(U_{nj})\}, \\ U^-_j &= \min\{SC(U_{1j}), SC(U_{2j}), \cdots, SC(U_{nj})\}. \end{aligned}$$

If all score values are equal, we need to use accuracy values. That is, we use accuracy values for comparison.

 

**3:** We will compute the separations for each alternative between the obtained $U^+$ and $U^-$ with the suggested DiMe $DM^{\omega}_{FFS}$. The separation measures as follows: For $i = 1, 2, \cdots, m$,

$$DM^{\omega}_{FFS}(U_i, U^+) = \sum_{j=1}^{n} \omega_j DM^{\omega}_{FFS}(U_{ij}, U^+),$$
$$DM^{\omega}_{FFS}(U_i, U^-) = \sum_{j=1}^{n} \omega_j DM^{\omega}_{FFS}(U_{ij}, U^-).$$

Based on these measures, the closeness index $\gamma_i$ connected to the $U_i$ will be as follows:

$$\gamma_i = \frac{DM^{\omega}_{FFS}(U_i, U^+)}{DM^{\omega}_{FFS}(U_i, U^+) + DM^{\omega}_{FFS}(U_i, U^-)}.$$

**4:** We will rank the alternatives according to their $\gamma_i$ values. As the $\gamma_i$ value gets smaller, we will take the alternative $U_i$ with the smallest value of $\gamma_i$ to choose the best alternative.

## 5 Infectious Diseases Application

The infectious diseases example from Kirisci and Simsek [2] was adapted for this study to represent the application of the suggested method in MCDM.

Let s

$$\begin{aligned} D &= \{ \text{ Hepatitis C, Crimean-Congo Hemorrhagic Fever(CCHF), influenza A(H1N1)} \} \\ &= \{U_1, U_2, U_3\} \end{aligned}$$

be the set of three alternatives. Alternatives in this cluster were selected as infectious diseases, which are common in Turkey, before COVID-19. The set of criteria $S = \{s_1, s_2, s_3, s_4, s_5\}$. The criteria $s_1$ is cost type and the other criteria $s_2, s_3, s_4, s_5$ are benefit type. The corresponding weight vector of the attribute is $\omega = (0.25, 0.20, 0.15, 0.18, 0.22)^T$. The evaluation values are represented by FFNs(Table 1).

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $U_1$ | (0.7, 0.4) | (0.8, 0.5) | (0.8, 0.7) | (0.7, 0.5) | (0.9, 0.1) |
| $U_2$ | (0.7, 0.3) | (0.6, 0.5) | (0.8, 0.4) | (0.5, 0.5) | (0.7, 0.2) |
| $U_3$ | (0.8, 0.4) | (0.8, 0.6) | (0.9, 0.3) | (0.6, 0.4) | (0.7, 0.4 |

**Table 1** The FFDMT

**1:** We will normalize the decision matrix $E = (U_{ij})_{n\times n} = ((\rho_{ij}, \tau_{ij}))_{n\times n}$. Transform the FFDMT $E$ into the normalized FFDMT by (1) (Table 2).

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $U_1$ | (0.4, 0.7) | (0.8, 0.5) | (0.8, 0.7) | (0.7, 0.5) | (0.9, 0.1) |
| $U_2$ | (0.3, 0.7) | (0.6, 0.5) | (0.8, 0.4) | (0.5, 0.5) | (0.7, 0.2) |
| $U_3$ | (0.4, 0.8) | (0.8, 0.6) | (0.9, 0.3) | (0.6, 0.4) | (0.7, 0.4 |

**Table 2** The normalized FFDMT

**2:** Now we will find the ideal solutions. These solutions:

$$\begin{aligned} U^+ &= \{(0.4, 0.7), (0.8, 0.5), (0.9, 0.3), (0.7, 0.5), (0.9, 0.1)\}, \\ U^- &= \{(0.3, 0.7), (0.6, 0.5), (0.8, 0.7), (0.5, 0.5), (0.7, 0.4)\}. \end{aligned}$$

**3:** We use the suggested FFDMT $DM^{\omega}_{FFS}$ to compute the separation of each alternative between positive ideal and negative ideal solutions. The closeness index $\gamma_i$ (for all $U_i$) is computed: $\gamma_1 = 0.687$, $\gamma_2 = 0.631$, $\gamma_3 = 0.704$.

**4:** For $j = 1, 2, 3$, the $\gamma_j$ values will help rank the alternatives. Hence, the best alternative is $U_2$.

## 6 Conclusion

We can express the advantages of the proposed method as follows:

(1) Since the main characteristic of FFSs is that the sum of cubes of membership and non-membership value of any object can be less than or equal to 1, then using FFSs, we can cover more elements than that of PFSs and IFSs. In other words, the FFS model is a valuable, practical, and impressive extended form of IFSs and PFSs. In this instance, experts become more autonomous in expressing their views on the level of membership.

(2) The choice of the best alternative from a set of alternatives in an MCDM problem is handicapped when uncertain data are strained to adopt the limited form of IFNs and PFNs. The aforenamed cases would cause the mutilation of data. A more generalized model is required to ensure telling solutions in such crucial cases. FFSs give more correct and exact outcomes when used to cope with practical MCDM problems including FF information as they are an effective extension of IFSs and PFSs.

(3) The measures considered in this study are not limited to CS. It has also been studied with Euclidean DiMes. Working with both measures provides a geometric as well as algebraic point of view in the MCDM problem.

This study focuses on solving an MCDM problem in which measures of CS and cosine distance between FFSs are considered. Based on FFS values, CS measure and Euclidean DiMe were defined and their basic properties were examined. Therefore, we established new SMs between FFSs according to the suggested cosine SM and the Euclidean DiMe, which not only satisfy the condition of SM but also deal with the related decision-making problems from both points of view of geometry and algebra. The usefulness, influence, and versatility of the developed method have been demonstrated in a medical case study.

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# A New Lower Bound for the Randić Energy of Graphs

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**Abstract:** Let $G$ be a simple connected graph of order $n$ with Randić eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The Randić energy of $G$ is defined as $RE\,(G) = \sum_{i=1}^{n} |\rho_i|$ . In this study, we are interested in establishing a new lower bound for $RE\,(G)$.

**Keywords:** Graph, Randić eigenvalues, Randić energy.

## 1 Introduction

Let $G = (V, E)$ , $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with $|V| = n$ vertices and $|E| = m$ edges. If $v_i$ and $v_j$ are two adjacent vertices in $G$, that is denoted by the notation $i \sim j$. Let $d_i$ be the degree of vertex $v_i$ of $G$, where $i = 1, 2, \ldots, n$. Denote with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ the vertex degree sequence of $G$.

The adjacency matrix $A = A\,(G)$ of $G$ is an $n$-square matrix whose $ij$-entry is equal to 1 if $i \sim j$ and zero otherwise. The eigenvalues of the graph $G$ are the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $A$ [4]. As well known [4],

$$\sum_{i=1}^{n} \lambda_i = 0\,, \qquad \sum_{i=1}^{n} \lambda_i^2 = 2m\,, \qquad \prod_{i=1}^{n} \lambda_i = \det A\,.$$

The energy of a graph $G$, first proposed by Gutman [8], is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

This graph invariant possesses its origin in theoretical chemistry where it is closely concerned with the total $\pi$-electron energy in a molecule represented by a (molecular) graph [9, 15]. In this context, the graph energy has been widely studied in chemical/mathematical literature. For more details on $E(G)$, we refer to monograph [13], recent papers [16, 17] and the references quoted therein.

Let $D = D\,(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal degree matrix of $G$. The Randić matrix of $G$ is the matrix defined by $R = R(G) = D^{-1/2} A D^{-1/2}$ [1]. The eigenvalues of $R(G)$, $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, are Randić eigenvalues with the following basic properties [1, 10]

$$\sum_{i=1}^{n} \rho_i = 0\,, \qquad \sum_{i=1}^{n} \rho_i^2 = 2\,R_{-1}(G), \qquad \prod_{i=1}^{n} \rho_i = \det R.$$

Here, $R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}$ is the general Randić index of $G$ [3].

Motivated by the evident success of the graph energy defined in (1), the Randić energy of $G$ was introduced in [1] as:

$$RE(G) = \sum_{i=1}^{n} |\rho_i| = \sum_{i=1}^{n} |\rho_i^*|\,. \tag{2}$$

where $|\rho_1^*| \geq |\rho_2^*| \geq \cdots \geq |\rho_n^*|$ is the non-increasing arrangement of the absolute values of Randić eigenvalues of $G$. In [1, 10], it was pointed out that $RE(G)$ coincides with the normalized signless Laplacian energy [10] and the normalized Laplacian energy [3]. Details on the mathematical properties and various bounds of $RE(G)$ can be found in [5, 10–12, 14, 18]

In this study, we are interested in establishing a new lower bound for $RE\,(G)$.

## 2 Preliminary Lemmas

In this section, we recall some preliminary lemmas that will be used in the main result of this study.

**Lemma 1** ([7]). *For $x_1, x_2, \ldots, x_n \geq 0$ and $a_1, a_2, \ldots, a_n \geq 0$ such that $\sum_{i=1}^{n} a_i = 1$,*

$$\sum_{i=1}^{n} a_i x_i - \prod_{i=1}^{n} x_i^{a_i} \geq n\lambda \left( \frac{1}{n} \sum_{i=1}^{n} x_i - \prod_{i=1}^{n} x_i^{1/n} \right), \tag{3}$$

*where $\lambda = \min\{a_1, a_2, \ldots, a_n\}$. Moreover, the equality in* (3) *holds if and only if $x_1 = x_2 = \cdots = x_n$.*

**Lemma 2** ([14]). *The Randić spectral radius of $G$ is $\rho_1 = 1$.*

The following lemma can be found in the proof of Theorem 3.1 of [2].

**Lemma 3** ([2]). *Let $G$ be a connected graph of order $n$. Then*

$$|\rho_2^*| \geq \sqrt{\frac{2R_{-1}(G) - 1}{n-1}} \geq \left( \frac{|\det A|}{\prod_{i=1}^{n} d_i} \right)^{1/(n-1)}.$$

**Lemma 4** ([10]). *If $G$ has no isolated vertices, then*

$$\det R = \frac{\det A}{\prod_{i=1}^{n} d_i}.$$

Recall that the complete product of two graphs $G$ and $H$, denoted by $G \bigvee H$, is produced from $G \cup H$ by joining every vertex of $G$ with every vertex of $H$.

**Lemma 5** ([6] ). *Let $G$ be a connected graph of order $n$ and maximum vertex degree $\Delta = n - 1$. Then $|\rho_2| = |\rho_3| = \cdots = |\rho_n|$ if and only if $G \cong K_n$, or $G \cong K_1 \bigvee p\, K_2$, with $n = 2p + 1$ $(p \geq 2)$.*

## 3 A New Lower Bound for $RE(G)$

We now establish a new lower bound for Randić energy.

**Theorem 1.** *Let $G$ be a connected graph of order $n$ and let $t \geq 0$ be a real number. Then for any real $k$ with $|\rho_2^*| \geq k \geq \left( \frac{|\det A|}{\prod_{i=1}^{n} d_i} \right)^{1/(n-1)}$, the following lower bound holds*

$$RE(G) \geq 1 + k + (n-2) \left( \frac{(t+1) \left( \frac{|\det A|}{\prod_{i=1}^{n} d_i} \right)^{\frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}}}{k^{\frac{1}{(t+1)(n-2)}}} - t \left( \frac{|\det A|}{\prod_{i=1}^{n} d_i} \right)^{1/(n-1)} \right). \tag{4}$$

*If $\Delta = n - 1$, the equality in* (4) *holds if and only if $G \cong K_n$, or $G \cong K_1 \bigvee p\, K_2$, with $n = 2p + 1$ $(p \geq 2)$.*

*Proof:* Choosing $x_i = |\rho_i^*|$ for $i = 2, \ldots, n$, $a_2 = \frac{t}{(t+1)(n-1)}$ and $a_i = \frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}$ for $i = 3, \ldots, n$, in (3), we obtain that

$$\begin{aligned} &\frac{t}{(t+1)(n-1)} |\rho_2^*| + \frac{(t+1)n - (2t+1)}{(t+1)(n-1)(n-2)} \sum_{i=3}^{n} |\rho_i^*| - |\rho_2^*|^{\frac{t}{(t+1)(n-1)}} \prod_{i=3}^{n} |\rho_i^*|^{\frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}} \\ &\geq \frac{t}{(t+1)(n-1)} \sum_{i=2}^{n} |\rho_i^*| - \frac{t}{t+1} \prod_{i=2}^{n} |\rho_i^*|^{1/(n-1)}, \end{aligned}$$

that is,

$$\begin{aligned} &-\frac{1}{(t+1)(n-2)} |\rho_2^*| + \frac{(t+1)n - (2t+1)}{(t+1)(n-1)(n-2)} \sum_{i=2}^{n} |\rho_i^*| - |\rho_2^*|^{-\frac{1}{(t+1)(n-2)}} \prod_{i=2}^{n} |\rho_i^*|^{\frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}} \\ &\geq \frac{t}{(t+1)(n-1)} \sum_{i=2}^{n} |\rho_i^*| - \frac{t}{t+1} \prod_{i=2}^{n} |\rho_i^*|^{1/(n-1)}. \end{aligned}$$

From the above and by Lemmas 2 and 4, we have that

 

$$\begin{aligned} RE(G) &\geq \rho_1 + \left|\rho_2^*\right| + \frac{(t+1)(n-2)\left(|\det R|\right)^{\frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}}}{\left|\rho_2^*\right|^{\frac{1}{(t+1)(n-2)}}} - t(n-2)\left(|\det R|\right)^{1/(n-1)} \\ &= 1 + \left|\rho_2^*\right| + \frac{(t+1)(n-2)\left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{\frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}}}{\left|\rho_2^*\right|^{\frac{1}{(t+1)(n-2)}}} - t(n-2)\left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{1/(n-1)}. \end{aligned} \tag{5}$$

Consider the following function $\phi(x)$

$$\phi(x) = x + \frac{(t+1)(n-2)\left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{\frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}}}{x^{\frac{1}{(t+1)(n-2)}}}.$$

Observe that

$$\phi'(x) = 1 - x^{-\frac{(t+1)n-(2t+1)}{(t+1)(n-2)}}\left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{\frac{(t+1)n-(2t+1)}{(t+1)(n-1)(n-2)}}.$$

It is not difficult to conclude that $\phi$ is increasing for $x \geq \left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{1/(n-1)}$. Then, for any real $k$ with $|\rho_2^*| \geq k \geq \left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{1/(n-1)}$, from (5), we obtain the inequality (4). The equality in (4) holds if and only if

$$\left|\rho_2^*\right| = k \text{ and } \left|\rho_2^*\right| = \left|\rho_3^*\right| = \cdots = \left|\rho_n^*\right|.$$

The above conditions imply that

$$|\rho_2| = |\rho_3| = \cdots = |\rho_n|.$$

Then, according to Lemma 5, if the maximum vertex degree $\Delta$ is equal to $n-1$, the equality in (4) holds if and only if $G \cong K_n$, or $G \cong K_1 \bigvee p\, K_2$, with $n = 2p+1$ $(p \geq 2)$. □

**Corollary 1** ([5, 11])**.** *Let $G$ be a connected graph of order $n$. Then*

$$RE(G) \geq 1 + (n-1)\left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{\frac{1}{n-1}}. \tag{6}$$

*If $\Delta = n-1$, the equality in* (6) *holds if and only if $G \cong K_n$, or $G \cong K_1 \bigvee p\, K_2$, with $n = 2p+1$ $(p \geq 2)$.*

*Proof:* Taking into Lemma 3 and choosing $k = \left(\frac{|\det A|}{\prod_{i=1}^n d_i}\right)^{1/(n-1)}$ and $t = 0$ in Theorem 1, we get the lower bound (6). □

The lower bound (6) was improved in [2] as the following.

**Corollary 2** ([2])**.** *Let $G$ be a connected graph of order $n$. Then*

$$RE(G) \geq 1 + \sqrt{\frac{2R_{-1}(G)-1}{n-1}} + (n-2)\left(\frac{|\det A|}{\sqrt{\frac{2R_{-1}(G)-1}{n-1}}\prod_{i=1}^n d_i}\right)^{\frac{1}{n-2}}. \tag{7}$$

*If $\Delta = n-1$, the equality in* (7) *holds if and only if $G \cong K_n$, or $G \cong K_1 \bigvee p\, K_2$, with $n = 2p+1$ $(p \geq 2)$.*

*Proof:* Considering Lemma 3 and taking $k = \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$ and $t = 0$ in Theorem 1, we obtain the required result. □

**Remark 1.** *It should be noted that if one can find a new bound $|\rho_2^*| \geq k_1 \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$, then the lower bounds in* (6) *and* (7) *can be improved via the lower bound in Theorem* 1.

## 4 Conclusion

In the present study, a new lower bound for $RE(G)$ has been obtained. It has also been shown that two previously known lower bounds of $RE(G)$ can be obtained as the special cases of our result.

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# A Novel Neutrosophic Score and Accuracy Function Proposal in the Context of Decision Making

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**Abstract:** The collected data for some of the real-life problems can include uncertainty, indeterminacy, and inconsistency together. There are several fuzzy set (FS) alternatives for modelling such scenarios in the literature. The most flexible alternative is Neutrosophic Set (NS) because it does not put a limit for inconsistency unlike the others such as Pythagorean FS. NS theory assumes that the inconsistency is caused by collecting data from multiple sources. For this reason, it considers membership, non-membership, and indeterminacy grades as separate terms and gives ability to set values in [0,1] for them independently. On the other hand, this independency makes the ranking of multiple NSs complicated, so an extra approach is needed. In the literature, ranking approaches have been suggested based on several score and ranking functions. However, the available score and accuracy functions may yield different rankings from the others in some scenarios. These may also result different ranking from the score and accuracy functions of intuitionistic FS (IFS) that is a subset of NS. In this study, several score and accuracy functions are compared for some example NSs, and novel score and accuracy functions are proposed to make the ranking results of IFS and NS theories identical. The performance of the proposed functions is illustrated on numerical examples.

**Keywords:** Accuracy function, Fuzzy set, Neutrosophic set, Score function.

## 1 Introduction

Most of the engineering techniques simplify the real life problems by idealizing the effects of the environment as if they are in isolated environments to obtain deterministic results. This approach brings a big disadvantage for these techniques in terms of usage in real-world. The main issue is caused disregarding of the uncertainties of the environment. Engineering techniques were widely reformulated in the literature by considering the uncertainty via Fuzzy Set Theory (FST). FST is used for modelling the uncertainty by using membership degree that is presented with a decimal number in $[0, 1]$. A Fuzzy Set (FS) is represented with Membership Function (MF) concept. Each set element has a Membership Degree (MD) produced by using MF. If MD is equal to 1, it means full membership and if MD is equal to 0, it means full non-membership. A set member can be partially a member and partially a non-member at the same time. The summation of MD and Non-Membership Degree (NMD) is equal to 1 for traditional FSs. [10]. The uncertainty can be caused by different factors such as process variability, lack of expertness, and input data inconsistency. The nature of the uncertainty can vary depending on its reason. Several FST Extensions (FSTEs) were offered for better representation of uncertainties having different natures. Intuitionistic FS (IFS) is one of the most popular FSTEs that gives ability to model the scenarios with incomplete information. For such cases, NMD cannot be calculated as 1-MD. Because, IFS theory relaxes the validity condition of FST by allowing a margin between 1 and the sum of MD and NMD namely Indeterminacy Degree (IDD). Neutrosophic Set (NS) is a generalization of IFS. In IFS theory, IDD is dependent to MD and NMD but IDD is independent from MD and NMD in NS theory for giving ability to use inconsistent data in modelling.
Two-thirds of FST related studies focuses to the uncertainty of Multi Criteria Decision Making (MCDM) problem [3]. The main focus of MCDM problems is to rank the alternatives depending on several criteria. If the expert evaluations are traditional Fuzzy Numbers (FNs), they can easily be ranked according to their MD values (The concept FN refers to a single value populated from an FS.). However, ranking of Intuitionistic Fuzzy Numbers (IFNs) is a more complicated operations. To be able to rank multiple IFNs, Xu [8] proposed two concepts; (i) score and (ii) accuracy functions. IFNs are ranked depending on some ranking rules based on score and accuracy function values. This approach is not directly applicable for ranking Neutrosophic Fuzzy Numbers (NFNs) because of independency of IDD term. In the literature, several score and accuracy functions were suggested to make these ranking rules applicable for NFNs. However, these score and accuracy functions reach different rankings in some specific scenarios. Since NS theory is a generalization of IFS theory, each IFN is also a NFN. For this reason these score and accuracy functions should be applicable IFNs too. However, when they applied to IFNs, they give different ranking from the functions suggested by Xu [8] in some scenario. This means that the results reached by using these functions have reliability issue. When the keywords "neutrosophic" + "decision making" are searched in academic databases it is seen that thousands of studies have been conducted in this field.
In this study, it is aimed to propose reliable score and accuracy functions for ranking NFNs. For this aim, the available score and accuracy functions are compared for some example IFNs to make the reliability issue clearer. The proposed novel score and accuracy functions are

tested with these IFNs to prove that the ranking results are identical with the functions suggested by Xu [8]. The proposed functions are also tested by using some visual NFN examples to prove the reliability of the results by comparing with the intuitive insights about the ranking. The numerical examples also show that the proposed functions have also ability to produce results for every extreme cases in which some other functions cannot yield results.
Rest of the paper is organized as follows: Theoretical background of NS theory is presented in Section 2, available score and accuracy functions are compared and analyzed in Section 3, novel score and accuracy functions are presented and compared with the existing ones on a numerical example in Section 4. The obtained results and future research directions are discussed in Section 5.

## 2 Preliminaries

While an element is either a member or a non-member of a set in the classical set theory, it can be partially member and non-member simultaneously in FST. The uncertainty is modelled with the MF concept based on a continuous variable $x \in [0,1]$. The level of uncertainty is represented with the term MD. If it is high, which means the uncertainty is low and if it is low, which means high uncertainty [9]. A traditional FS is mathematically represented as shown in Definition 1.
**Definition 1:** Let $X$ be the reference universe, $\mu_{\tilde{A}(x)} \in [0,1]$ be the MF of $x \in X$ and $\nu_{\tilde{A}(x)}$ be the non-membership function (NMF) which is the complement of $\mu_{\tilde{A}(x)}$. An FS $\tilde{A}$ on $X$ is defined as $\tilde{A} = x, \mu_{\tilde{A}(x)} | x \in X$ satisfying Eq. (1) [10]:

$$\mu_{\tilde{A}(x)} + \nu_{\tilde{A}(x)} = 1. \tag{1}$$

The sum of MD and NMD of each set element is equal to 1. This is named "complete information case". However, it may not be possible to determine the MD and NMD values so as to be the summation is equal to 1 because of some factors such as lack of expertness in some cases. This is named "incomplete information case". IFS theory has been developed for modelling the uncertainties including incomplete information case. An IFS is defined as shown in Definition 2.
**Definition 2:** Let $X$ be the reference universe, $\mu_{\tilde{A}(x)} \in [0,1]$ be the MF of $x \in X$ and $\nu_{\tilde{A}(x)}$ be the NMF which is complement of $\mu_{\tilde{A}(x)}$. An IFS $\tilde{\tilde{A}}$ on $X$ is defined as $tilde\ddot{A} = x, \mu_{\tilde{\tilde{A}}(x)} | x \in X$ satisfying Eq. (2) [1]:

$$\mu_{\tilde{\tilde{A}}(x)} + \nu_{\tilde{\tilde{A}}(x)} \leq 1. \tag{2}$$

For each set element, MD and NMD are determined dependent to each other. The gap between 1 and the sum of MD and NMD is named as IDD and is calculated as shown in Eq. (3) for a set element $x \in X$ [1]:

$$\pi_{\tilde{\tilde{A}}(x)} = 1 - (\mu_{\tilde{\tilde{A}}(x)} + \nu_{\tilde{\tilde{A}}(x)}) \leq 1. \tag{3}$$

IFS theory embodies traditional FST and can be used for them by assigning 0 to IDD. The IFS theory is mainly focused on the membership and non-membership terms. Indeterminacy is caused by the lack of information about membership and non-membership. NS theory is a generalization of IFS theory considering the inconsistent information case. It is a flexible modelling option because it does not put a limit for inconsistency level and gives ability to set values for membership, non-membership, and indeterminacy independent from each other. The terminology for NS theory is different than IFS theory: the membership is named “truthiness”, and non-membership is named “falsity”. As a characteristic feature, the indeterminacy is represented with a separate term. An NS is defined as shown in Definition 3.
**Definition 3:** Let $t \in [0,1]$ be truthiness, $i \in [0,1]$ be indeterminacy, and $f \in [0,1]$ be falsity. An NS $\tilde{\tilde{A}}$ is defined as shown in Eq. (4) [5]:

$$\tilde{\tilde{A}} = (t, i, f),\ 0 \leq t + i + f \leq 3. \tag{4}$$

Some problem formulations may not be suitable for modelling with inconsistent data even if the collected data contain inconsistency. For such problems, the data can be normalized by dividing the terms with summation of the terms to satisfy Eq. (5) to turn into IFS.

$$t + i + f = 1. \tag{5}$$

## 3 Neutrosophic score and accuracy functions for ranking

Traditional fuzzy numbers (FN) can be ranked based on their membership function values. This approach is not applicable for intuitionistic FNs (IFNs) and neutrosophic fuzzy numbers (NFNs) because the summation of MD and NMD may not be equal to 1. The most popular ranking approach is using score and accuracy functions based ranking rules for IFNs and NFNs.
**Definition 4:** Let $\tilde{\tilde{A}}$ be an IFN having MD $\mu_{\tilde{\tilde{A}}}$ and NMD $\nu_{\tilde{\tilde{A}}}$. Score ($S(\tilde{\tilde{A}})$) and accuracy ($H(\tilde{\tilde{A}})$) functions are defined as in Eqs. (6)- (7) [8]:

$$S(\tilde{\tilde{A}}) = \mu_{\tilde{\tilde{A}}} - \nu_{\tilde{\tilde{A}}}. \tag{6}$$

$$H(\tilde{\tilde{A}}) = \mu_{\tilde{\tilde{A}}} + \nu_{\tilde{\tilde{A}}}. \tag{7}$$

**Definition 5:** Let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ be two IFNs. These IFNs can be ranked based on the score and accuracy function values by using the rule set given in Eq. (8) [8]:

$$\begin{aligned} S(\tilde{\tilde{A}}) > S(\tilde{\tilde{B}}) \ &\Rightarrow\ \tilde{\tilde{A}} > \tilde{\tilde{B}}, \\ S(\tilde{\tilde{A}}) = S(\tilde{\tilde{B}})\ ,\ H(\tilde{\tilde{A}}) > H(\tilde{\tilde{B}}) \ &\Rightarrow\ \tilde{\tilde{A}} > \tilde{\tilde{B}}, \\ S(\tilde{\tilde{A}}) = S(\tilde{\tilde{B}})\ ,\ H(\tilde{\tilde{A}}) = H(\tilde{\tilde{B}}) \ &\Rightarrow\ \tilde{\tilde{A}} = \tilde{\tilde{B}}. \end{aligned} \tag{8}$$

The ranking rules proposed for IFNs do not work for NFNs because the summation of MD, NMD and IDD may not be equal to 1 for NFNs. For this reason, the score and accuracy functions should be redesigned for NSs. There are several studies proposing score and accuracy functions

 

for NSs in the literature. All these studies use the same rule set with IFS as given in Eq. (8) to rank NFNs.
Score and accuracy functions proposed by Şahin & Küçük [7] is defined as in Eq. (9) for a given NFN $\tilde{\dot{A}}$:

$$\begin{aligned} S(\tilde{\dot{A}}) &= \frac{1+t-2i-f}{2}, \\ H(\tilde{\dot{A}}) &= t-i \times (1-t) - f \times (1-i). \end{aligned} \tag{9}$$

Score and accuracy functions proposed by Nancy & Garg [2] is defined as in Eq. (10) for a given NFN $\tilde{\dot{A}}$:

$$\begin{aligned} S(\tilde{\dot{A}}) &= \frac{1+(t-2i-f)\times(2-t-f)}{2}, \\ H(\tilde{\dot{A}}) &= t-2i-f. \end{aligned} \tag{10}$$

Score and accuracy functions proposed by Singh & Bhat [6] is defined as in Eq. (11) for a given NFN $\tilde{\dot{A}}$:

$$\begin{aligned} S(\tilde{\dot{A}}) &= \frac{1+(t-2i-f)}{2\times(2-t-f)}, \\ H(\tilde{\dot{A}}) &= t-i-2f. \end{aligned} \tag{11}$$

Ye [9] proposed a cosine similarity measure based approach to rank the alternatives. The proposed approach uses only score value which is the similarity measure between the NFN and the ideal fuzzy number (t,i,f)=(1,0,0) to rank the alternatives. For a given NFN $\tilde{\dot{A}}$, score function is expressed as in Eq. (12)

$$S(\tilde{\dot{A}}) = \frac{t}{\sqrt{t^2-i^2-f^2}}. \tag{12}$$

Since IFS is a subset of NS, the proposed score and accuracy functions should provide reasonable results for IFNs too. However, the current score and accuracy functions yield different rankings for some examples. Table 1 presents such an example in a comparative way. According to the table, only Ye [9] has reached with the same ranking result with Xu [8]. The method proposed by Yee [9] performed well for IFNs because it is a distance based method and it considers the relative greatness of the MD and NMD while calculating the distance. The others are reached the same ranking with each other but it is different than the ranking obtained by Xu [8] and Ye [9] since these methods have different assumptions for inconsistent information cases. However, the reliability of the results of these methods affected negatively for the scenarios having consistent information.

| IFNs | | IFN1 | IFN2 | IFN3 |
|---|---|---|---|---|
| t | | 0.648 | 0.648 | 0.633 |
| i | | 0.071 | 0.123 | 0.124 |
| f | | 0.281 | 0.229 | 0.243 |
| t+i+f | | 1 | 1 | 1 |
| Xu | Score | 0.367 | 0.419 | 0.390 |
| | Accuracy | 0.929 | 0.877 | 0.876 |
| | Rank | 3 | 1 | 2 |
| Şahin & Küçük | Score | 0.613 | 0.587 | 0.571 |
| | Accuracy | 0.362 | 0.404 | 0.375 |
| | Rank | 1 | 2 | 3 |
| Nancy & Garg | Score | 0.620 | 0.597 | 0.580 |
| | Accuracy | 0.225 | 0.173 | 0.142 |
| | Rank | 1 | 2 | 3 |
| Singh & Bhat | Score | 0.572 | 0.522 | 0.508 |
| | Accuracy | 0.367 | 0.419 | 0.390 |
| | Rank | 1 | 2 | 3 |
| Ye | Score | 0.913 | 0.928 | 0.918 |
| | Accuracy | - | - | - |
| | Rank | 3 | 1 | 2 |

**Table 1** Comparison of existing score and accuracy functions for example IFNs

## 4 Proposed neutrosophic score and accuracy functions

Normalization presented in Eq. (5) is used for transformation of NFNs to IFNs. can enable using the score and accuracy functions proposed by Xu [8] for NFNs. From this point of view, the score and accuracy functions proposed by Xu [8] are extended for NSs in this study. However,

some issues about normalization should be solved to make possible to use it for ranking of NFNs. The modifications that are made to solve these issues are as follows:

• As a first issue, the normalization is not applicable for the lower asymptotic limit ($t = 0, i = 0, f = 0$) of NSs because of division by zero. This problem can be solved by adding a sufficiently small number ($\epsilon$) to the denominator.
• $t, i$ and $f$ are independent of each other in NS theory. For this reason the summation of them can exceed 1. Normalization protects the relative greatness of $t, i$ and $f$ while converting an NFN to IFN. This brings another weakness for normalization based ranking approach. Because, the relative greatness of $t, i$ and $f$ can be same for different NFNs. For example, $\tilde{\ddot{A}}_1 = (0.5, 0.5, 0.5)$ and $\tilde{\ddot{A}}_2 = (1, 1, 1)$ have the same score and accuracy values when the functions proposed by Xu [8] is used with the help of normalization. However it seems intuitively that, the accuracy of $\tilde{\ddot{A}}_2$ should be higher. To be able to rescale the accuracy function better, a $t$ multiplier can be added to the accuracy function equation.

In addition, the proposed approach should satisfy the below rules that are inspired intuitively:

• Relative greatness of $t$ and $f$ can be considered while calculating the score. If $f$ is greater than $t$, this should cause the loss of score. For example, $\tilde{\ddot{A}}_1 = (0.5, 0.5, 0.5)$ should have better ranking than $\tilde{\ddot{A}}_3 = (0.7, 1, 0.5)$.
• $t, i$ and $f$ are independent of each other but $t$ and $f$ should be dominant to $i$ and the NFN that has smaller $f$ should have better ranking while $t$ values are equal. For example, $\tilde{\ddot{A}}_4 = (0.5, 0.5, 1)$ should have better rank than $\tilde{\ddot{A}}_5 = (0.5, 1, 0.5)$.

**Definition 6:** Let $\tilde{\ddot{A}} = (t, i, f)$ be a NFN. Proposed score ($S(\tilde{\ddot{A}})$) and accuracy ($H(\tilde{\ddot{A}})$) functions are defined as in Eqs. (13)- (14):

$$S(\tilde{\ddot{A}}) = \frac{t - f}{t + i + f + \epsilon}. \tag{13}$$

$$H(\tilde{\ddot{A}}) = \frac{t \times (t + f)}{t + i + f + \epsilon}. \tag{14}$$

The proposed functions have been compared with the functions proposed by Xu [8] for the IFNs given in Table 1. As seen in Table 2, the same ranking result with Xu [8] has been reached. This example shows that the proposed functions yield reliable results for IFNs. As illustrated in

| IFNs | | IFN1 | IFN2 | IFN3 |
|---|---|---|---|---|
| t | | 0.648 | 0.648 | 0.633 |
| i | | 0.071 | 0.123 | 0.124 |
| f | | 0.281 | 0.229 | 0.243 |
| t+i+f | | 1 | 1 | 1 |
| Xu | Score | 0.367 | 0.419 | 0.390 |
| | Accuracy | 0.929 | 0.877 | 0.876 |
| | Rank | 3 | 1 | 2 |
| Proposed | Score | 0.37 | 0.42 | 0.39 |
| | Accuracy | 14.08 | 8.13 | 8.06 |
| | Rank | 3 | 1 | 2 |

**Table 2** Comparison of the proposed score and accuracy functions with Xu [8] for IFNs

Table 1, none of the functions proposed by Şahin & Küçük [7], Nancy & Garg [2], and Singh & Bhat [6] produce reliable ranking results. The method proposed by Ye [9] is not reliable for ranking NFNs under inconsistency because of considering relative greatness of MD and NMD. For example it cannot rank these NFNs: NFN1= (t,f,i)=(0.5, 0.5, 0.5) and NFN1= (t,f,i)=(0.6, 0.6, 0.6). Thus, comparison with these approaches is not sufficient to evaluate the reliability of the proposed functions. For this reason, some NFNs have been ranked intuitively as shown in Fig. 1 and the performance of the proposed functions have been compared with this ranking (Most of the NFNs have been preferred at asymptotic limits of NS definition space to make the intuitive ranking easier.). This comparison results shown in Table 3 shows that:

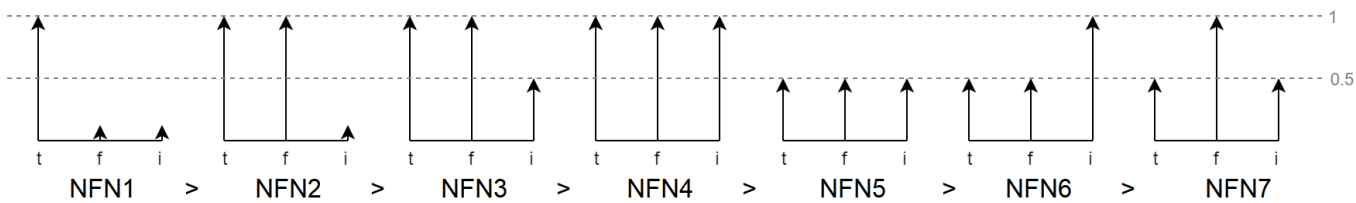


**Fig. 1**: Ranking of some NFNs by using the proposed score and accuracy functions

• Singh & Bhat [6]'s functions cannot produce result because of division by zero error, if $t + f = 2$. For example it does not produce results for NFN1, NFN2, NFN3.
• Normalization and Ye [9]'s methods cannot produce results while t, f, i equal to 0 because of division by zero error.
• The proposed functions gives result for all asymptotic limit values of NS definition space.

 

- Ye [9]'s method and the normalization cannot rank the NFNs where the relative greatness of t,f,i are the same.
- The NFN that should be the worst was found as 2nd by Nancy & Garg [2], 4th by Singh & Bhat [6], and 5th by Şahin & Küçük [7].
- Although all methods finds the 1st NFN truly, only the proposed functions produced the same ranking with the intuitively found ranking presented in Fig. 1.
- Ye [9]'s method handles f and i with the same dominancy so it does not rank NFN6 and NFN7. Functions proposed by Şahin & Küçük [7], and Nancy & Garg [2] handles i as more dominant than f. Thus, they considers the NFNs having a higher f and a lower i are bigger NFNs than the NFNS having a higher i and lower f when t values are the same. For example, they do not rank NFN6 and NFN7 truly. However, a bigger f should mean a more strong assessment opposite to t, so f should be dominant to i. The proposed functions ranks NFN6 and NFN7 truly because of handling f as more dominant than i.

| NFNs | | NFN1 | NFN2 | NFN3 | NFN4 | NFN5 | NFN6 | NFN7 |
|---|---|---|---|---|---|---|---|---|
| t | | 1.0 | 1.0 | 1.0 | 1.0 | 0.5 | 0.5 | 0.5 |
| i | | 0.0 | 0.0 | 0.5 | 1.0 | 0.5 | 1.0 | 0.5 |
| f | | 0.0 | 1.0 | 1.0 | 1.0 | 0.5 | 0.5 | 1.0 |
| t+i+f | | 1.0 | 2.00 | 2.50 | 3.00 | 1.50 | 2.00 | 2.00 |
| Proposed | Score | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.25 |
| | Accuracy | 1.00 | 1.00 | 0.80 | 0.67 | 0.33 | 0.25 | 0.37 |
| | Rank | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Şahin & Küçük | Score | 1.00 | 0.50 | 0.00 | -0.50 | 0.00 | -0.50 | -0.25 |
| | Accuracy | 1.00 | 0.00 | 0.50 | 1.00 | 0.00 | 0.00 | -0.25 |
| | Rank | 1 | 2 | 3 | 6 | 4 | 7 | 5 |
| Nancy & Garg | Score | 1.00 | 0.50 | 0.50 | 0.50 | 0.00 | -0.50 | 0.13 |
| | Accuracy | 1.00 | 0.00 | -1.00 | -2.00 | -1.00 | -2.00 | -1.50 |
| | Rank | 1 | 3 | 4 | 5 | 6 | 7 | 2 |
| Singh & Bhat | Score | 1.00 | N/A | N/A | N/A | 0.00 | -0.50 | -0.50 |
| | Accuracy | 1.00 | -1.00 | -1.50 | -2.00 | -0.50 | -1.00 | -1.50 |
| | Rank | 1 | - | - | - | 2 | 3 | 4 |
| Ye | Score | 1.00 | 0.71 | 0.67 | 0.58 | 0.58 | 0.41 | 0.41 |
| | Accuracy | - | - | - | - | - | - | - |
| | Rank | 1 | 2 | 3 | 4 | 4 | 6 | 6 |
| Normalization | Score | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.25 |
| | Accuracy | 1.00 | 1.00 | 0.80 | 0.67 | 0.67 | 0.50 | 0.75 |
| | Rank | 1 | 2 | 3 | 4 | 4 | 6 | 7 |

**Table 3** Comparison of the proposed score and accuracy functions with the existing ones for example NFNs

In order to make the finding about "the dominance of i over f better", another example has given in Fig. 2. When the NFNs are analyzed it is seen that NFNy has more strong assessments against t compared to NFNx. Thus, NFNx should be considered bigger than NFNy. However, the functions proposed by Nancy & Garg [2] finds NFNy as a better NFN than NFNx. The method proposed by Ye [9] cannot rank them because t=0. The proposed functions ranks them in the right order.

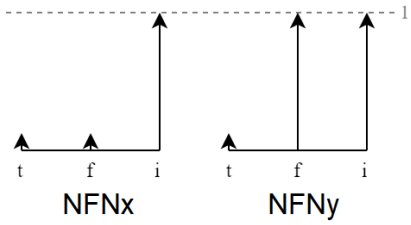


**Fig. 2**: Example NFNs

## 5 Conclusion

FST is an effective way of modelling the uncertainties of real case applications. It enables to make sensitivity analysis about the uncertainties to reach well understanding about the possible outcomes of the event. However, uncertainties can have different natures depending on the causing factors. FS extensions have been offered for modelling of different type uncertainties. IFS theory has been offered for modelling the cases including indeterminacy. NS theory has been developed as a generalization of IFS theory for modelling the cases including both unlimited indeterminacy and inconsistent data. NS theory provides high flexibility in modelling but this flexibility brings some accompanied challenges. Ranking of Neutrosophic expert evaluations in MCDM problems is one of these challenges. In the literature, various score and accuracy functions were offered to rank NFNs. However, these proposals may give different result from each other for some cases. These may also yield different rankings than the score and accuracy functions that are offered for IFNs. Some of them does not produce any results for some limit values of NS definition space.

In this study, novel score and accuracy functions have been offered for ranking NFNs. The proposed functions have yielded the same results with the existing approach for IFSs. In addition, a ranking has been produced for all of the limiting values of NFN space. Quality of the proposed functions have also been examined on a visual numerical example and the same result was obtained with the ranking that is thought to be correct intuitively. As a future study, the proposed score and accuracy functions can be used for real-case MCDM problems.

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# Mathematical Physics Approach to Optical Fibers as Diaphragm Breathing Sensor

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**Abstract:** This study was conducted to detect human diaphragmatic breathing flow using theoretical and experimental approaches. Initially, the lung model was formed using the Navier-Stokes equation and the finite element method by applying the principles of continuity and momentum. Furthermore, single mode fiber (SMF) and fiber Bragg grating (FBG) was experimentally designed in a sinusoidal macro bending pattern as a strain sensor belt applied to the diaphragm. The simulation model shows the airflow velocity increases up to 4 m/s when it flows into smaller branches. While the experimental results show that the largest power loss parameter occurs at a buffer diameter of 0.8 cm. The power loss detected in SMF is a maximum of -0.18 dBm during inhalation and a minimum of -0.28 dBm during expiration. However, the bending of FBG becomes superior with high sensitivity.

**Keywords:** Airflow, Diaphragm, Fiber optic sensor, Navier-Stokes, Sinusoidal bending.

## 1 Introduction

The development of science and technology in this century has had a positive impact on the modern industrial revolution, especially fiber optic sensor (FOS) technology in the medical field. FOS has characteristics such as high bandwidth and transfer speed in transmitting signals. FOS also has physical advantages, namely immune to electromagnetic (EM) wave interference, high sensitivity, and low-cost fabrication [23]-[17]. Therefore, FOS technology has the potential for detection applications in medical fields other than communications.

FOS technology in medical applications can be used to detect several vital human organs such as blood pressure, heart rate, body temperature, and respiratory circulation [3, 10]. The current need for medical detection by the FOS is monitoring the human respiratory circulation. The advantage of this respiratory monitoring is that it can identify early symptoms experienced by patients who are indicated to have lung disease, including kidney failure, stroke, and apnea [2]-[14]. Currently, conventional electronic breathing sensors are not very sensitive in describing respiratory circulation. In addition, there is also a risk of damaging and disturbing comfort when in direct contact with the skin [24]. So FOS technology offers practical and more sensitive detection with the principle of strain at the smallest scale.

Detectors based on the strain principle can use single mode fiber (SMF) and fiber Bragg grating (FBG) by showing high sensitivity qualities and great potential for advances in FOS technology as respiratory circulation detection [15]-[21]. SMF and FBG have characteristics that are superior to other fibers such as micro-sized, resistant to EM interference, easy to modify, and high precision [26]-[9]. The principle of strain in FOS usually uses a bending method with various patterns such as circular, straight, U, and sinusoidal [25]-[8]. However, sinusoidal patterns proved to be more effective than other forms in detecting changes in sensor parameters in the human body [4]. Therefore, the FOS technology developed in this study uses a sinusoidal configuration of SMF and FBG arranged in an elastic belt which is then attached to the diaphragm. The sensitivity value of FOS in the form of an elastic belt can be obtained by calculating the power losses from variations in the diameter of the sinusoidal pattern buffer and changes in the abdominal circumference of the experimental sample. Although the demonstration of FOS on diaphragmatic breathing circulation shows optimal results, a theoretical model with a simulation approach is needed in the form of new information in the form of the distribution of airflow dynamics in lung breathing. Therefore, this paper discusses the phenomenon of airflow vibration for diaphragmatic breathing with a theoretical approach of simulation and experimentation.

## 2 Simulation and experiment model

This study discusses circulating airflow through lung modeling and the demonstration of an experimental approach to diaphragmatic breathing. Initially, the lung model was formed theoretically using the Navier-Stokes equation and the finite element method. In addition, this model also pays attention to the principles of continuity and momentum for incompressible fluids. The equation formed is as follows [5]:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \tag{1}$$

$$\rho\Big(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}\Big)=-\frac{\partial P}{\partial x}+\rho g_x+\mu\Big(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\Big) \tag{2}$$

$$\rho\Big(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}\Big)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\Big(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\Big) \tag{3}$$

$$\rho\Big(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}\Big)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\Big(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}\Big) \tag{4}$$

where $U$, $V$, and $W$ are the velocity of the airflow over the $x$, $y$, and $z$ coordinates, $\rho$ is fluid density, $P$ is pressure, and $\mu$ is viscosity.

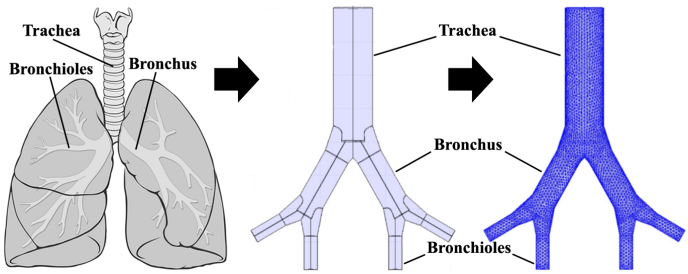


**Fig. 1**: Perspective of lung airflow modeling based on finite element method.

The respiratory circulation model was built using the finite element method with several triangular elements forming the lung tissue as shown in Fig. 1. The lung tissue model consists of a trachea that connects the left and right bronchi, then there are bronchioles that form small branches that connect with the bronchi. The dimensions of the lung tissue model can be seen in Table 1.

| Lung tissue | Diameter (cm) | Length (cm) |
|---|---|---|
| Trachea | 1 | 10 |
| Bronchus | 0.5 | 4 |
| Bronchioles | 0.2 | 1 |

**Table 1** Dimensions of the lung tissue model.

The experimental design and operation of the diaphragmatic breathing circulation detector were carried out using a FOS in the form of an elastic belt attached to the diaphragm as shown in Fig. 2. Inside the belt, the SMF and FBG (1310 nm) were bent sinusoidally with a buffer diameter of 0.8 cm and 1.2 cm. The power source in the form of a laser diode is connected to one end of the optical fiber, then at the other end is connected to an optical power meter (OPM) to measure the input and output power of the optical fiber.

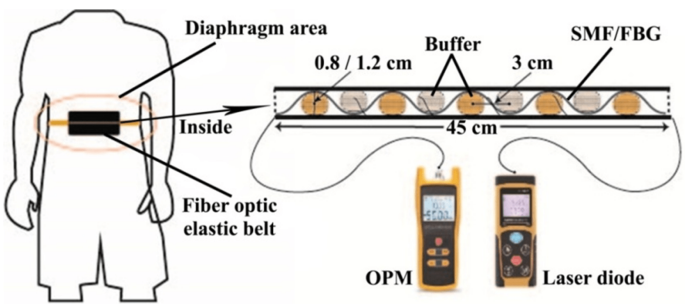


**Fig. 2**: FOS design display in the form of elastic belt.

The sample as the object of the experiment consisted of seven people with variations in abdominal circumference (near the diaphragm) and different ages as shown in Table 2. The sample's position was standing with normal breathing conditions without any previous physical activity. Data collection was carried out for 60 seconds every 5 trials. The resulting data is in the form of power loss and the number of respiratory frequencies in each unit of time. The sensitivity level is then determined based on the change in power loss.

## 3 Results and discussion

The simulation results of the lung airflow modeling can be seen in Fig. 3. The initial parametric settings were carried out by entering the air density value of 1.225 kg/m$^3$, viscosity 1.7894 $\times$ 10$^{-5}$ Pa.s, and velocity of 1 m/s. The airflow velocity in the trachea increases to 1.5 m/s, then in the first branch (bronchus) of the trachea the airflow increases to 2 m/s. The airflow then increases from 3.5 m/s to 4 m/s in the second

 

| Sample | Abdominal circumference (cm) | Age (cm) |
|---|---|---|
| 1 | 64 | 21 |
| 2 | 66 | 21 |
| 3 | 69 | 21 |
| 4 | 72 | 21 |
| 5 | 77 | 22 |
| 6 | 83 | 22 |
| 7 | 97 | 23 |

**Table 2** Variation of experimental object's abdominal circumference.

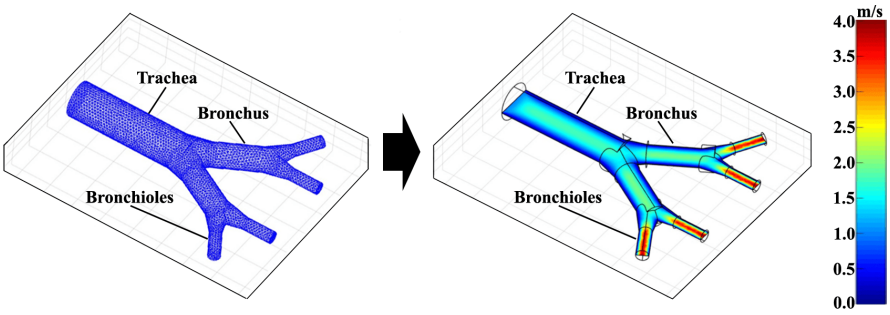


**Fig. 3**: Lung airflow simulation model.

branch (bronchioles) of the bronchus. This explains that the velocity of airflow is influenced by the geometry of the lungs. In addition, airflow velocity is also influenced by density rather than viscosity [20].

The situation that occurs on the inside of the belt during exhalation is shown in Fig. 4. The sinusoidal pattern design of the SMF and FBG optical fibers undergoes a change in position in the diaphragm area. The fiber optic belt changes position during the inhalation state from the initial exhalation state. This change will affect the value that is read on the OPM according to the bending concept in the power loss category [6]. The curved fiber will be pulled back to its original position known as the vibrating belt.

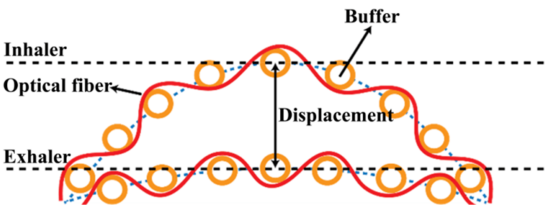


**Fig. 4**: Changes in the position of the fiber optic belt on the diaphragm during the breathing process.

Figure 5 shows a comparison of optical fiber power changes for each experimental sample. The figure illustrates the change in the power of SMF with a buffer diameter of 0.8 cm which has a higher power loss than its type for a diameter of 1.2 cm. An increase in power loss indicates that inhalation is in progress, while power loss decreases during expiration [22]. Inhalation also applies in the seconds before and after although at a peak that is not too high. This shows that there are differences in the breathing carried out by the sample. In addition, the difference in peak power loss in sample 7 which is farther from the other samples is because the fiber optic belt responds more to the large abdominal circumference during inhalation. The further the belt moves during inhalation, the higher the power loss value [11]. This fiber optic belt actually relies on the principle of strain and stress from the buffer that forms the SMF and FBG in a sinusoidal pattern.

The distribution of respiratory frequency for the five experiments is shown in Fig. 6. The median value for the change in power is taken from sample 2 because it has the smallest output power for FBG with a buffer diameter of 1.2 cm. The power changes occurred in the range of -0.40 dBm to -0.55 dBm, while the median area was in the range of -0.45 dBm to -0.50 dBm. These results explain that the sensitivity level of the sample is able to respond to the smallest power changes. Meanwhile, mixed experimental results are reported in the normal radian range.

Figure 7 shows the sensitivity level of SMF and FBG optical fiber for each different buffer diameter. According to the outline of the measurement results, the sinusoidal pattern formed by a buffer diameter of 0.8 cm gives better results than 1.2 cm. This is because the highest and smallest power changes have been successfully detected by the SMF and FBG fiber optic belts. In addition, a critical power threshold factor in optical fiber causes a change in the detected power to depend on the amount of bending applied [13]. In the experimental results on all samples, SMF optical fiber with a smaller buffer diameter has an average sensitivity of 0.26 compared to FBG of 0.23. However, on the other hand, FBG is superior to the measured power loss parameter with the highest value of -1.30 dB compared to SMF of -1.16 dB.

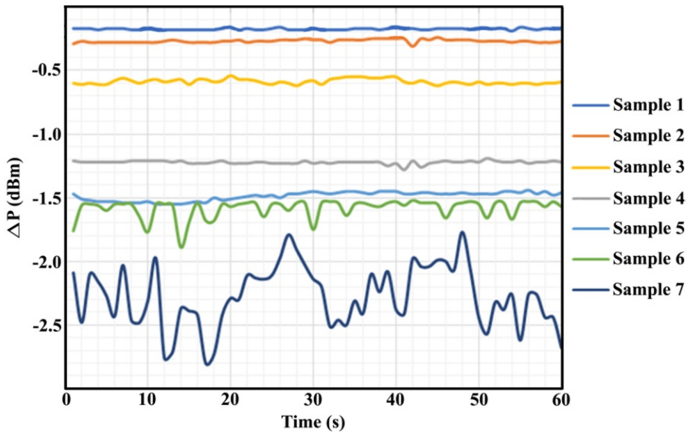


**Fig. 5**: Changes in the power of an SMF optical fiber with a buffer diameter of 0.8 cm.

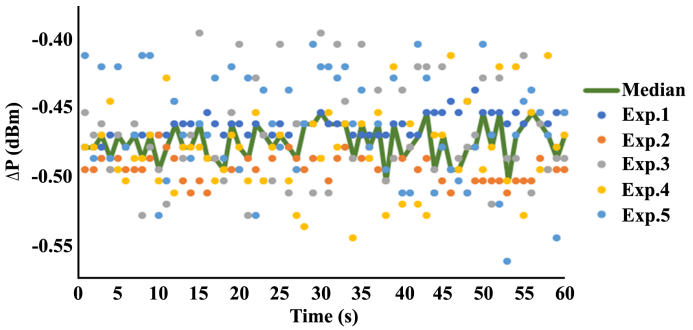


**Fig. 6**: Power change of FBG optical fiber with a buffer diameter of 1.2 cm for sample 2.

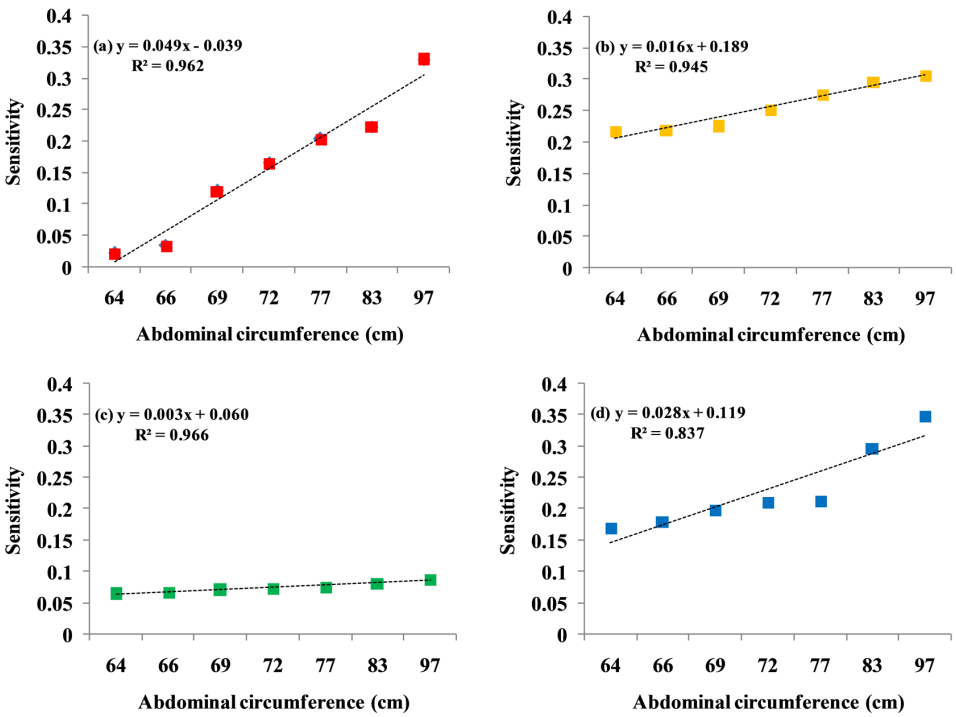


**Fig. 7**: Sensitivity with buffer diameters: SMF (a) 1.2 cm and (b) 0.8 cm; FBG (c) 1.2 cm; and (d) 0.8 cm.

## 4 Conclusion

Detection of human respiratory airflow circulation has been successfully modeled and tested with FOS. Based on the simulation model, the airflow velocity increases from 1.5 m/s to 4 m/s when switching to a smaller branch geometry. In addition, the airflow velocity is also influenced by the density factor rather than the viscosity. Experimentally, the strain variation with the diameter buffer resulted in different diaphragmatic breathing vibration patterns. The power loss tends to be higher with changes in the position of the optical fiber during the inhalation process

and is also influenced by a larger abdominal circumference. The sensitivity produced by SMF and FBG for a buffer diameter of 0.8 cm gives better results than 1.2 cm.


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# Investigation of Linear and Nonlinear Advection-Diffusion Processes by a New Combined Method

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**Abstract:** This study aims to propose a new method to analyze the solutions of linear and nonlinear advection-diffusion equations. The proposed method is constructed to discretize the spatial coordinate by combining the cubic B-spline and fourth-order compact finite difference schemes. Since the present method uses the second-order derivatives in fourth-order accuracy unlike the cubic B-spline method, it appears to be successful in improving solutions. The computed results are seen in good agreement with the exact and literature solutions. Furthermore, the present method is quite easy to implement with minimal computational effort.

**Keywords:** Advection-diffusion equation, Cubic-B spline, Fourth-order compact finite difference.

## 1 Introduction

The advection-diffusion equations play a very important role to describe various real-world phenomena such as heat transfer in a draining film[12], water transfer in soils[30], thermo-hygro transfer in porous media[29], dynamic heterogeneity in cancer invasion[23], the spread of pollutants in rivers and streams[10]. In this study, the nonlinear one-dimensional advection-diffusion equation without the source term will be discussed as

$$u_t + \varepsilon u^{\gamma} u_x = v u_{xx}, \quad a \leq x \leq b, \quad t > 0 \tag{1}$$

subject to the initial condition

$$u(x,0) = g(x), \quad a \leq x \leq b, \tag{2}$$

and the boundary conditions

$$u(a,t) = g_1(t), \tag{3}$$

$$u(b,t) = g_2(t), \quad 0 \leq t \leq T, \tag{4}$$

where, $v$ and $\varepsilon$ are the viscosity coefficient and the velocity component of the fluid, respectively. The terms $g(x)$, $g_1(t)$ and $g_2(t)$ are known functions and the subscripts $x$ and $t$ represent differentiation with respect to space and time, respectively. The advection term $u_x$ depicts the transportation of the quantity $u$ by the velocity field. The diffusion term $u_{xx}$ describes the dissemination of the quantity $u$.

Since these equations have been extensively used in many applications in science, various researchers have paid more attention to investigating the solutions of these problems. However, the analytical solution of these equations could only obtain for some special cases due to the complexities of the velocity field and transport process. Therefore, it has been spent a great deal of effort by researchers to capture the behaviour of these problems by developing various numerical techniques such as finite difference methods[33, 38], Galerkin methods[21], spectral methods[6], finite volume methods[9], B-spline methods[7, 19, 27, 36] and several other techniques [1, 4, 13].

The current study proposes a new combined method based on the cubic B-spline and a fourth-order compact finite difference scheme to investigate the solutions of linear and nonlinear advection-diffusion equations. The main goal of combining these methods is to improve the accuracy of the B-spline method by expressing the second-order derivative with four-order accuracy. The proposed technique uses the finite difference scheme to discretize the temporal derivatives while the new combined method based on the cubic B-spline and a fourth-order compact finite difference scheme is applied in the space coordinate with the help of the Crank-Nicolson method. B-spline methods are widely used to solve partial differential equations due to some significant advantages such as numerical consistency, smoothness, local support of spline curve, good approximation rate and computationally fast. Besides, these methods are also able to approximate analytical solutions up to a certain smoothness [39]. Thus, they provide the flexibility to get the approximation at any point in the domain accurately with more accurate results. However, although B-spline methods can capture the behaviour of partial differential equations efficiently, they should be used together with other powerful techniques to reach the desired accuracy and convergence rate compared to some other methods in the literature. Therefore, in this paper, a fourth-order compact finite difference has been combined with the cubic B-spline method to achieve higher accuracy with minimal computational effort. The computed results revealed that the proposed method produces quantitative and qualitative results when compared with the exact solution and available techniques in the literature. Furthermore, the proposed methodology is easy for programming in any language and, based on the literature review, has not been implemented for the advection-diffusion equations.

 

## 2 Description of the Proposed Method

In this section, the proposed method will be implemented to the advection-diffusion equation given in Eq.(1). Firstly, a uniform partition of the domain $[a,b]\times[0,T]$ is considered by the knots $x_i$, $i=0,1,2...,N$ and $t_n$, $n=0,1,2,...,M$ such that $x_i=a+ih$ and $t_n=ndt$, $h=\frac{b-a}{N}$ and $dt=\frac{T}{M}$.

### 2.1 Time Discretization

Using the Crank-Nicolson method, Eq.(1) is discretized in the temporal direction as follows:

$$\frac{u^{n+1}-u^n}{dt}+\varepsilon\frac{(u^\gamma u_x)^{n+1}+(u^\gamma u_x)^n}{2}=v\frac{u_{xx}^{n+1}+u_{xx}^n}{2} \tag{5}$$

After the nonlinear terms appearing in the above equation are linearized by using Rubin-Graves linearization[34], Eq.(5) becomes as

$$\frac{u^{n+1}-u^n}{dt}+\varepsilon\frac{(u^\gamma)^n u_x^{n+1}+\gamma(u^{\gamma-1})^n u^{n+1}u_x^n-\gamma(u^\gamma)^n u_x^n+(u^\gamma u_x)^n}{2}=v\frac{u_{xx}^{n+1}+u_{xx}^n}{2} \tag{6}$$

Rearranging Eq.(6), it is obtained

$$u^{n+1}\left\{1+\varepsilon\gamma\frac{dt}{2}(u^{\gamma-1})^n u_x^n\right\}+\varepsilon\frac{dt}{2}(u^\gamma)^n u_x^{n+1}-v\frac{dt}{2}u_{xx}^{n+1}=u^n-\varepsilon\frac{dt}{2}(-\gamma(u^\gamma)^n+(u^\gamma)^n)u_x^n+v\frac{dt}{2}u_{xx}^n. \tag{7}$$

### 2.2 Spatial Discretization

In this subsection, it will be discussed the proposed method based on combining the cubic B-spline and a fourth-order compact finite difference scheme to investigate the solution of the advection-diffusion equation (1). The numerical solutions of the problem are approximated as

$$u_N(x,t)=\sum_{i=-1}^{N+1}\delta_i(t)B_i(x), \tag{8}$$

where $\delta_i$ is the time-dependent parameter and $B_i$ represents the well-known cubic B-spline functions given in the following relationship[32]:

$$B_i(x)=\frac{1}{h^3}\begin{cases}(x-x_{i-2})^3, & [x_{i-2},x_{i-1}],\\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2-3(x-x_{i-1})^3, & [x_{i-1},x_i],\\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2-3(x_{i+1}-x)^3, & [x_i,x_{i+1}],\\ (x_{i+2}-x)^3, & [x_{i+1},x_{i+2}],\\ 0, & Otherwise,\end{cases}$$

where $h=x_{i+1}-x_i, i=-1,...,N+1$. The variation of $u_N(x,t)$ over typical element $[x_i,x_{i+1}]$ is expressed by

$$u_N(x,t)=\sum_{j=i-1}^{i+2}\delta_j(t)B_j(x). \tag{9}$$

By using the interpolating conditions, the values at the knots of $u(x,t)$ and its two derivatives $u'(x,t)$ and $u''(x,t)$ at the knots are stated in terms of the time-dependent parameters $\delta_i$ as follows:

$$\begin{aligned} u_i&=u(x_i)=\delta_{i-1}+4\delta_i+\delta_{i+1},\\ u_i^{'}&=u'(x_i)=\frac{3}{h}(\delta_{i+1}-\delta_{i-1}),\\ u_i''&=u''(x_i)=\frac{6}{h^2}(\delta_{i-1}-2\delta_i+\delta_{i+1}). \end{aligned} \tag{10}$$

Substituting Eq.(10) in Eq.(7) for boundary points, $i=0,N$, it is obtained

$$\alpha_1\delta_{i-1}^{n+1}+\alpha_2\delta_i^{n+1}+\alpha_3\delta_{i+1}^{n+1}=\alpha_4\delta_{i-1}^n+\alpha_5\delta_i^n+\alpha_6\delta_{i+1}^n, \tag{11}$$

where

$$\alpha_1=1+\varepsilon\gamma\frac{dt}{2}(u^{\gamma-1})^n u_x^n-\varepsilon\frac{3dt}{2h}(u^\gamma)^n-3v\frac{dt}{h^2},\qquad \alpha_2=4+2\varepsilon\gamma dt(u^{\gamma-1})^n u_x^n+6v\frac{dt}{h^2},$$

$$\alpha_3=1+\varepsilon\gamma\frac{dt}{2}(u^{\gamma-1})^n u_x^n+\varepsilon\frac{3dt}{2h}(u^\gamma)^n-3v\frac{dt}{h^2},\qquad \alpha_4=1+\varepsilon\frac{3dt}{2h}(-\gamma(u^\gamma)^n+(u^\gamma)^n)+3v\frac{dt}{h^2},$$

$$\alpha_5=4-6v\frac{dt}{h^2},\qquad \alpha_6=1-\varepsilon\frac{3dt}{2h}(-\gamma(u^\gamma)^n+(u^\gamma)^n)+3v\frac{dt}{h^2}.$$

Now, the fourth-order compact finite difference formula that was initially proposed by Adam [2, 3] for the second-order derivative terms will be used to improve the accuracy for the solution of Eq.(1) at interior points, $i=1,2,...N-1$. For this, let us introduce the fourth-order

compact finite difference approximations for first and second derivatives, respectively:

$$\frac{1}{4}u'_{i-1} + u'_i + \frac{1}{4}u'_{i+1} = \frac{1}{h}\left[-\frac{3}{4}u_{i-1} + \frac{3}{4}u_{i+1}\right], \tag{12}$$

$$\frac{1}{10}u''_{i-1} + u''_i + \frac{1}{10}u''_{i+1} = \frac{1}{h^2}\left[\frac{6}{5}u_{i-1} - \frac{12}{5}u_i + \frac{6}{5}u_{i+1}\right], \tag{13}$$

where $u'_i$ and $u''_i$ are the first and second derivative approximations of unknown $u$ at point $x_i$, respectively.

By applying the first operator to Eq.(12) again and eliminating $u''_{i-1}$ and $u''_{i+1}$ from the obtained equation and Eq.(13), the new second-order derivative formula is obtained as follows:

$$u''_i = 2\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{u'_{i+1} - u'_{i-1}}{2h}. \tag{14}$$

By this formula, the fourth-order accurate second derivative approximations of unknowns are represented by unknowns themselves and their first derivative approximations [31]. Thus, the second-order derivative is expressed by the convergence of order four.

Now, substituting the approximate value $u$ and its first derivative $u'$ in Eq. (10) and the second derivative $u''$ in Eq. (14) into Eq. (7) at interior points, $i = 1, ..., N-1$, Eq. (7) becomes as

$$\beta_1\delta^{n+1}_{i-2} + \beta_2\delta^{n+1}_{i-1} + \beta_3\delta^{n+1}_{i} + \beta_4\delta^{n+1}_{i+1} + \beta_5\delta^{n+1}_{i+2} = \beta_6\delta^{n}_{i-2} + \beta_7\delta^{n}_{i-1} + \beta_8\delta^{n}_{i} + \beta_9\delta^{n}_{i+1} + \beta_{i+2}\delta^{n}_{i+2}, \tag{15}$$

where

$$\beta_1 = -v\frac{dt}{h^2}, \qquad \beta_2 = 1 + \varepsilon\gamma\frac{dt}{2}(u^{\gamma-1})u^n_x - \varepsilon\frac{3dt}{2h}(u^\gamma)^n - 2v\frac{dt}{h^2}, \qquad \beta_3 = 4 + 2\varepsilon\gamma dt(u^{\gamma-1})u^n_x + v\frac{9dt}{2h^2},$$

$$\beta_4 = 1 + \varepsilon\frac{dt}{2}(u^{\gamma-1})^n u^n_x + \varepsilon\frac{3dt}{2h}(u^\gamma)^n + 2v\frac{dt}{h^2}, \qquad \beta_5 = \beta_1, \qquad \beta_6 = -\beta_1,$$

$$\beta_7 = 1 + \varepsilon\frac{3dt}{h}(-\gamma(u^\gamma)^n + (u^\gamma)^n) + 2v\frac{dt}{h^2}, \qquad \beta_8 = 4 - v\frac{9dt}{2h^2}, \qquad \beta_9 = 1 - \varepsilon\frac{3dt}{2h}(-\gamma(u^\gamma)^n + (u^\gamma)^n) + 2v\frac{dt}{h^2},$$

$$\beta_{10} = -\beta_1.$$

Eq.(11) and Eq.(15) produce a system which consists of $(N+1)$ linear equations in $(N+3)$ unknowns $d^n = \{\delta^n_{-1}, \delta^n_0, \delta^n_1, ..., \delta^n_N, \delta^n_{N+1}\}$. To obtain a unique solution, two additional constraints are acquired from the boundary conditions as:

$$u(x_0) = \delta^{n+1}_{-1} + 4\delta^{n+1}_0 + \delta^{n+1}_1 = g_1(t^{n+1}) \Rightarrow \delta^{n+1}_{-1} = g_1(t^{n+1}) - 4\delta^{n+1}_0 - \delta^{n+1}_1,$$

$$u(x_N) = \delta^{n+1}_{N-1} + 4\delta^{n+1}_N + \delta^{n+1}_{N+1} = g_2(t^{n+1}) \Rightarrow \delta^{n+1}_{N+1} = g_2(t^{n+1}) - 4\delta^{n+1}_N - \delta^{n+1}_{N-1}, \tag{16}$$

Thus, the system combined by Eqs. (11) and (15) is reduce $(N+1)\times(N+1)$ matrix system, which can be solved by using the Thomas algorithm. To approximate $\delta^{n+1}_i$ for a particular time level, the initial vector $d^0$ can be obtained from the initial condition.

## 3 Numerical Experiments

In this section, four test problems have been considered to evaluate the performance of the proposed method. To test the accuracy of the current scheme, the computed results have been compared with the exact solutions and available literature solutions. The computations have been run in a MATLAB environment using version R2021a. The error norms of the present results are calculated by the following definitions:

$$Absolute \quad error = |u^{exact}_i - u^{app}_i|,$$

$$L_\infty = max_i|u^{exact}_i - u^{app}_i|,$$

$$L_2 = \sqrt{h\sum_{i=1}^{N}|u^{exact}_i - u^{app}_i|^2},$$

where $u^{exact}$ and $u^{app}$ represent the exact solution and approximate solution, respectively.

### *3.1 Problem 1*

Consider the linear advection-diffusion equation by taking $\gamma = 0$ in Eq. (1),

$$u_t + \varepsilon u_x = vu_{xx}, \quad 0 \le x \le 1, \quad t > 0 \tag{17}$$

subject the initial condition

$$u(x, 0) = exp(c_1 x). \tag{18}$$

The exact solution is given by

$$u(x, t) = exp(c_1 x + c_2 t), \tag{19}$$

where $c_1$ and $c_2$ are constants.

 

| $x$ | Ismail et al. [18] | Mohammadi [28] | Aminikhah and Alavi [5] | Tassaddiq [37] | Proposed |
|---|---|---|---|---|---|
| | | | $t=1$ | | |
| 0.1 | 2.22E-16 | 6.55E-10 | 2.15E-06 | 2.52E-11 | 1.21E-10 |
| 0.5 | 8.88E-16 | 1.98E-09 | 7.06E-06 | 7.64E-11 | 8.31E-11 |
| 0.9 | 00E+00 | 2.03E-09 | 7.66E-06 | 7.82E-11 | 2.66E-10 |
| | | | $t=2$ | | |
| 0.1 | 2.22E-16 | 8.68E-10 | 2.82E-06 | 3.34E-11 | 1.35E-10 |
| 0.5 | 1.33E-15 | 3.46E-09 | 1.23E-05 | 1.33E-10 | 1.70E-10 |
| 0.9 | 4.44E-16 | 3.06E-09 | 1.16E-05 | 1.18E-10 | 3.20E-10 |
| | | | $t=5$ | | |
| 0.1 | 3.33E-16 | 9.58E-10 | 3.16E-08 | - | 1.26E-10 |
| 0.5 | 2.44E-15 | 4.39E-09 | 1.82E-05 | - | 2.88E-10 |
| 0.9 | 8.88E-16 | 3.71E-09 | 1.63E-05 | - | 3.58E-10 |

**Table 1** Absolute error comparisons of the proposed method and available literature methods for the parameters $\varepsilon = 0.1$, $v = 0.02$,$c_1 = 1.17712434446770$ and $c_2 = -0.09$ at different spatial and temporal points in Problem 1

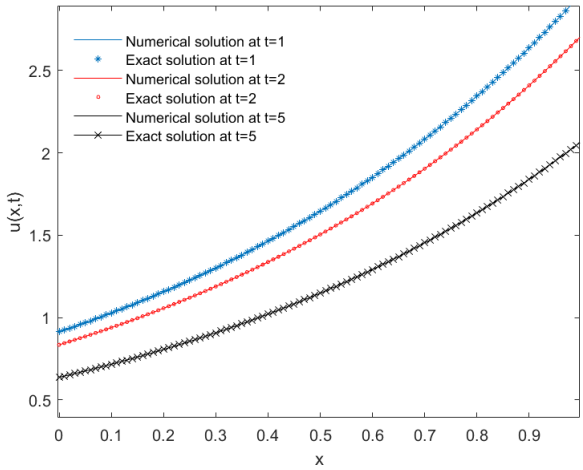


**Fig. 1**: The behaviours of the linear advection-diffusion equation for $\varepsilon = 0.1$, $v = 0.02$ in Problem 1

In this example, the spatial and temporal mesh sizes are taken as $h = 0.01$ and $dt = 0.001$, respectively, as in the references [5, 18, 28, 37]. In Table 1, the absolute errors of results produced by the current scheme have been compared with the literature [5, 18, 28, 37] for the parameters $\varepsilon = 0.1, v = 0.02, c_1 = 1.17712434446770$ and $c_2 = -0.09$ at different spatial and temporal points. As can be seen in the table, the presented method is more accurate in comparison with the works of Mohammadi[28] and Aminikhah and Alavi[5]. In addition, the current results are in reasonable agreement with the results of Tassaddiq [37] but less accurate than those of Ismail et al.[18]. Figure 1 illustrates the behaviour of the problem for the values indicated in Table 1. It can be observed from the figure, the proposed method solutions are very close to the exact solution at different temporal points.

The absolute error comparisons of the currently produced solutions and available literature solutions are listed in Table 2 for the parameters $\varepsilon = 3.5, v = 0.022, c_1 = 0.02854797991928, c_2 = -0.0999$ at different spatial and temporal points. The results in the table revealed that while the proposed method is more accurate than those of Ismail et al.[18] and Aminikhah and Alavi [5], they do not reach the accuracy of the results of Mohammadi [28]. The solutions of Problem 1 for the parameter values in Table 2 are exhibited in Figure 2 and as realized from the figure, the present method can capture the behaviour of the problem accurately.

### *3.2 Problem 2*

Consider the linear advection-diffusion equation by taking $\gamma = 0$ in Eq.(1) in the following form

$$u_t + \varepsilon u_x = v u_{xx}, \quad 0 \le x \le 1, \quad t > 0 \tag{20}$$

subject the initial condition

$$u(x,0) = exp(-\frac{(x-\kappa_0)^2}{2\sigma_0^2}). \tag{21}$$

The exact solution is given by

$$u(x,t) = \frac{\sigma_0}{\sigma} exp\left(-\frac{(x-\kappa_0-\epsilon t)^2}{2\sigma^2}\right), \tag{22}$$

where $\sigma^2 = \sigma_0^2 + 2vt$. The boundary conditions are obtained from the analytical solution.

| $x$ | Ismail et al. [18] | Mohammadi [28] | Aminikhah and Alavi [5] | Proposed |
|---|---|---|---|---|
| $t=1$ | | | | |
| 0.1 | 2.56E-10 | 3.16E-13 | 1.16E-07 | 2.15E-12 |
| 0.5 | 8.39E-10 | 1.61E-12 | 6.41E-07 | 1.09E-11 |
| 0.9 | 1.33E-09 | 2.93E-12 | 1.18E-06 | 1.99E-11 |
| $t=2$ | | | | |
| 0.1 | 2.38E-10 | 2.86E-13 | 1.05E-07 | 1.96E-12 |
| 0.5 | 1.38E-09 | 1.45E-12 | 5.80E-07 | 9.90E-12 |
| 0.9 | 2.83E-09 | 2.65E-12 | 1.07E-06 | 1.80E-11 |
| $t=5$ | | | | |
| 0.1 | 5.65E-10 | 2.59E-13 | 7.81E-08 | 1.45E-12 |
| 0.5 | 1.91E-09 | 1.31E-12 | 4.30E-07 | 7.32E-12 |
| 0.9 | 3.97E-09 | 2.40E-12 | 7.90E-07 | 1.33E-11 |

**Table 2** Absolute error comparisons of the proposed method and available literature methods for the parameters $\varepsilon = 3.5$, $v = 0.022$,$c_1 = 0.02854797991928$ and $c_2 = -0.0999$ at different spatial and temporal points in Problem 1

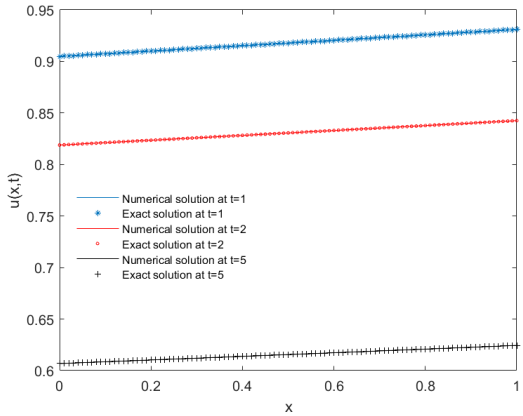


**Fig. 2**: The behaviours of the linear advection-diffusion equation for $\varepsilon = 3.5$, $v = 0.022$ in Problem 1

| | Proposed | | Mohammadi[28] | | Mittal and Jain[24] | |
|---|---|---|---|---|---|---|
| $x$ | $t=1$ | $t=2$ | $t=1$ | $t=2$ | $t=1$ | $t=2$ |
| 0.1 | 1.40E-07 | 1.00E-15 | 1.04E-06 | 1.00E-13 | 1.06E-06 | 1.01E-13 |
| 0.2 | 3.99E-07 | 2.37E-14 | 5.01E-06 | 3.61E-12 | 5.02E-06 | 3.65E-12 |
| 0.3 | 7.43E-07 | 1.18E-13 | 1.82E-05 | 7.49E-11 | 1.82E-05 | 7.56E-11 |
| 0.4 | 3.01E-06 | 4.67E-12 | 1.01E-05 | 1.02E-09 | 1.08E-05 | 1.06E-09 |
| 0.5 | 2.62E-07 | 1.22E-10 | 4.57E-05 | 1.00E-08 | 4.63E-05 | 1.04E-08 |
| 0.6 | 4.36E-06 | 1.53E-09 | 4.04E-06 | 7.24E-08 | 4.17E-06 | 7.27E-08 |
| 0.7 | 2.16E-06 | 1.21E-08 | 3.72E-05 | 3.45E-07 | 3.78E-05 | 3.53E-07 |
| 0.8 | 1.33E-06 | 6.33E-08 | 7.05E-06 | 1.13E-06 | 7.10E-06 | 1.14E-06 |
| 0.9 | 1.29E-06 | 2.19E-07 | 8.88E-06 | 2.09E-06 | 8.98E-06 | 2.12E-06 |

**Table 3** Absolute error comparisons of the proposed method and available literature methods for the parameters $\varepsilon = 1$ and $v = 0.01$ at different spatial and temporal points in Problem 2

The proposed method has been implemented to problem (20)-(22) and has been compared with available literature solutions for different temporal and spatial points in Tables 3-4. In all computations, $\sigma = 0.025$ and $\kappa_0 = -0.5$ are taken as in the references [24, 28]. In Table 3, the absolute errors of the currently produced solutions and available literature solutions have been presented at $t = 1$ and $t = 2$ with $\varepsilon = 1$, $v = 0.01$, $h = 0.01$ and $dt = 0.001$. As seen in the table, the proposed method solutions are more accurate than those of Mohammadi [28] and Mittal and Jain [24]. The numerical and exact solutions have been depicted in Figure 3 for the values $\varepsilon = 1$ and $v = 0.01$. As seen in the figure, the exact solution and proposed method solutions are in good agreement.

The proposed method has been compared with the methods presented in Mohammadi [28] and Mittal and Jain [24] in Table 4 for $\varepsilon = 0.01$, $v = 1$ $h = 0.01$, $dt = 0.001$. The results revealed that the proposed method produces more accurate solutions than the results of Mohammadi [28] and Mittal and Jain [24].

 

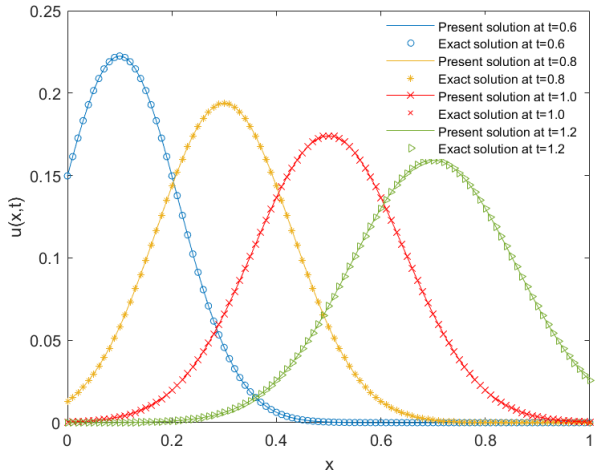


**Fig. 3**: The behaviours of the linear advection-diffusion equation for $\varepsilon = 1, v = 0.01$ in Problem 2

| | Proposed | | | Mohammadi[28] | | | Mittal and Jain[24] | | |
|---|---|---|---|---|---|---|---|---|---|
| | $t = 0.4$ | $t = 0.8$ | $t = 1.2$ | $t = 0.4$ | $t = 0.8$ | $t = 1.2$ | $t = 0.4$ | $t = 0.8$ | $t = 1.2$ |
| 0.1 | 2.09E-09 | 1.71E-10 | 6.16E-13 | 1.29E-07 | 3.31E-08 | 5.46E-10 | 1.35E-07 | 3.55E-08 | 6.49E-10 |
| 0.2 | 4.02E-09 | 3.51E-10 | 8.70E-12 | 4.27E-07 | 3.40E-08 | 4.37E-10 | 4.38E-07 | 3.78E-08 | 7.66E-10 |
| 0.3 | 5.55E-09 | 5.15E-10 | 2.09E-11 | 6.12E-07 | 5.12E-08 | 4.16E-10 | 6.17E-07 | 5.57E-08 | 3.45E-10 |
| 0.4 | 6.54E-09 | 6.43E-10 | 3.41E-11 | 7.19E-07 | 6.59E-08 | 1.11E-09 | 7.30E-07 | 7.03E-08 | 1.78E-09 |
| 0.5 | 6.87E-09 | 7.15E-10 | 4.55E-11 | 7.63E-07 | 7.51E-08 | 1.59E-09 | 7.71E-07 | 7.90E-08 | 3.22E-09 |
| 0.6 | 6.54E-09 | 7.19E-10 | 5.24E-11 | 7.18E-07 | 7.70E-08 | 2.73E-09 | 7.38E-07 | 8.01E-08 | 4.31E-09 |
| 0.7 | 5.59E-09 | 6.47E-10 | 5.26E-11 | 6.20E-07 | 7.04E-08 | 1.19E-09 | 6.34E-07 | 7.27E-08 | 4.74E-09 |
| 0.8 | 4.11E-09 | 4.98E-10 | 4.44E-11 | 4.55E-07 | 5.50E-08 | 1.74E-09 | 4.68E-07 | 5.64E-08 | 4.26E-09 |
| 0.9 | 2.20E-09 | 2.79E-10 | 2.71E-11 | 2.42E-07 | 3.12E-08 | 1.39E-09 | 2.52E-07 | 3.18E-08 | 2.71E-09 |

**Table 4** Absolute error comparisons of the proposed method and available literature methods for the parameters $\varepsilon = 0.01$ and $v = 1$ at different spatial and temporal points in Problem 2.

### *3.3 Problem 3*

Let us now consider the nonlinear advection-diffusion equation, the Burgers equation, by taking $\varepsilon = \gamma = 1$ in Eq.(1)

$$u_t + uu_x = vu_{xx}, \quad 0 \leq x \leq 1, \quad t > 0 \tag{23}$$

with the initial condition

$$u(x, 0) = 2v\frac{\pi sin(\pi x)}{\tau + cos(\pi x)} \tag{24}$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0. \tag{25}$$

The exact solution of this problem is given by

$$u(x, t) = \frac{2v\pi exp(-\pi^2 vt)sin(\pi x)}{\tau + exp(-\pi^2 vt)sin(\pi x)}. \tag{26}$$

Table 5 presents the comparison of the present method with the exact solution and the literature [8, 14, 15, 25] for $\tau = 2$, $v = 0.1, 0.5$, $h = 0.025$ and $dt = 0.0001$ at $t = 0.001$. As can be realized from the table that the computed solutions by the proposed method are very close to exact solutions and while they are more accurate than the results presented by the references [8, 14, 25], the results seem reasonably in agreement with the results of Guo et al. [15].

In Table 6, the results produced by the current scheme have been compared with the results of the studies [15, 20, 25] for $v = 0.005$, $\tau = 100$, $dt = 0.01$ and different spatial discretizations $N = 10, 20, 40, 60$. The results in Table 6 presented that the $L_\infty$ and $L_2$ errors of the proposed method are much lower than Mittal and Jain [25] and Jiwari [20] but the results of the proposed method are less accurate to Guo et al. [15].

Table 7 indicates the comparison of the current scheme with the works of Gou et al. [15] and Mittal and Rohila [25] for $\tau = 2$, $N = 20$, $dt = 0.0001$ and various $v$ at $t = 1$. The $L_\infty$ errors of the currently produced solutions in Table 7 are found to be more accurate than the literature[15, 25].

| $x$ | Exact | Proposed | Guo et al. [15] | Ganaie and Kukreja [14] | Asaithambi [8] | Mittal and Jain[25] |
|---|---|---|---|---|---|---|
| $v = 0.5$ | | | | | | |
| 0.1 | 0.3278695524 | 0.3278695516 | 0.3278695524 | 0.327871 | 0.327874 | 0.327870 |
| 0.2 | 0.6550692222 | 0.6550692203 | 0.6550692220 | 0.655067 | 0.655078 | 0.655071 |
| 0.3 | 0.9784124992 | 0.9784124952 | 0.9784124987 | 0.978418 | 0.978427 | 0.978416 |
| 0.4 | 1.2884634969 | 1.2884634909 | 1.2884634950 | 1.288464 | 1.288485 | 1.288469 |
| 0.5 | 1.5630638524 | 1.5630638514 | 1.5630638350 | 1.563061 | 1.563096 | 1.563074 |
| 0.6 | 1.7566421091 | 1.7566421432 | 1.7566419249 | 1.756648 | 1.756691 | 1.756654 |
| 0.7 | 1.7872063975 | 1.7872065094 | 1.7872065979 | 1.787201 | 1.787281 | 1.787204 |
| 0.8 | 1.5376943907 | 1.5376944436 | 1.5376944007 | 1.537693 | 1.537794 | 1.537649 |
| 0.9 | 0.9168597988 | 0.9168595259 | 0.9168597587 | 0.916880 | 0.916941 | 0.916786 |
| $L_\infty$ | | 2.90E-07 | 8.36E-07 | 2.00E-05 | - | 7.44E-05 |
| $L_2$ | | 9.60E-08 | 2.05E-07 | 3.54E-06 | - | 2.79E-05 |
| $v = 0.1$ | | | | | | |
| 0.1 | 0.0657497591 | 0.0657497590 | 0.0657497591 | 0.065750 | 0.065750 | 0.065750 |
| 0.2 | 0.1313829355 | 0.1313829354 | 0.1313829355 | 0.131383 | 0.131383 | 0.131383 |
| 0.3 | 0.1962808678 | 0.1962808676 | 0.1962808678 | 0.196281 | 0.196281 | 0.196281 |
| 0.4 | 0.2585757378 | 0.2585757375 | 0.2585757375 | 0.258576 | 0.258576 | 0.258576 |
| 0.5 | 0.3138493556 | 0.3138493554 | 0.3138493522 | 0.313848 | 0.313850 | 0.313850 |
| 0.6 | 0.3529718209 | 0.3529718222 | 0.3529717855 | 0.352972 | 0.352972 | 0.352972 |
| 0.7 | 0.3594428596 | 0.3594428642 | 0.3594429072 | 0.359443 | 0.359444 | 0.359443 |
| 0.8 | 0.3095803849 | 0.3095803875 | 0.3095803878 | 0.309580 | 0.309583 | 0.309579 |
| 0.9 | 0.1847537428 | 0.1847537313 | 0.1847537364 | 0.184752 | 0.184756 | 0.184751 |
| $L_\infty$ | | 1.25E-07 | 1.55E-07 | 2.00E-06 | - | 3.08E-06 |
| $L_2$ | | 4.06E-09 | 3.60E-08 | 3.54E-07 | - | 1.15E-06 |

**Table 5** Comparison of the proposed method and available literature methods for $\tau = 2, dt = 0.0001, h = 0.025$ for $v = 0.1, 0.5$ at $t = 0.001$ in Problem 3.

| | Proposed | | Guo et al. [15] | | Mittal and Jain[25] | | Jiwari et al. [20] | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ |
| 10 | 4.6713E-10 | 3.1477E-10 | 2.010E-09 | 8.653E-10 | 1.215E-07 | 8.631E-08 | 4.708E-08 | 6.459E-08 |
| 20 | 8.7149E-11 | 3.1478E-11 | 7.071E-11 | 2.019E-11 | 3.062E-08 | 2.153E-08 | 1.091E-08 | 4.465E-09 |
| 40 | 3.5123E-11 | 8.2949E-12 | 2.629E-12 | 6.460E-13 | 7.644E-09 | 5.378E-09 | 1.980E-09 | 2.786E-10 |
| 80 | 1.6327E-11 | 2.8664E-12 | 8.941E-14 | 2.047E-14 | 1.917E-09 | 1.345E-09 | 1.182E-09 | 2.665E-10 |

**Table 6** Comparison of error norms produced with various $N$ values for $v = 0.005, \tau = 100, dt = 0.01$ at $t = 1$ in Problem 3

| $\tau = 2$ | Proposed | Guo et al.[15] | Mittal and Jain [25] |
|---|---|---|---|
| $v$ | $L_\infty$ | $L_\infty$ | $L_\infty$ |
| 0.1 | 1.0046E-06 | 8.2406E-05 | 1.5E-03 |
| 0.01 | 4.5883E-07 | 3.5459E-06 | 1.6E-04 |
| 0.001 | 1.5162E-08 | 5.6777E-08 | 3.3E-06 |
| 0.0001 | 1.7655E-10 | 8.0326E-10 | 2.7E-08 |
| 0.00001 | 1.7870E-12 | 8.3176E-12 | 5.5E-10 |

**Table 7** Comparison of the error norms produced with $\tau = 2, dt = 0.0001$ and various $v$ values at $t = 1$ in Problem 3.

### *3.4 Problem 4*

Now, consider the Burgers equation with the shock-like wave solution

$$u(x,t) = \frac{x/t}{1 + \sqrt{1/\eta} exp\left(\frac{x^2}{4vt}\right)}, \quad 0 \le x \le 1, \quad t \ge 1, \tag{27}$$

where $\eta = exp(1/8v)$. Initial condition at $t = 1$ is given by

$$u(x,1) = \frac{x}{1 + \sqrt{1/\eta} exp(\frac{x^2}{4v})} \tag{28}$$

and boundary conditions are obtained from the exact solution.

 

| $x$ | $t$ | Exact | Proposed | Hussain[17] | Mittal and Rohila[26] |
|---|---|---|---|---|---|
| 0.25 | 1.5 | 0.0770681272 | 0.0770681271 | 0.0770692371 | 0.07704 |
| | 1.7 | 0.0657641623 | 0.0657641623 | 0.0657666266 | 0.06575 |
| | 2.0 | 0.0534427730 | 0.0534427729 | 0.0534471029 | 0.05343 |
| | 2.5 | 0.0400857043 | 0.0400857043 | 0.0400924946 | 0.04007 |
| 0.50 | 1.5 | 0.1515500560 | 0.1515500559 | 0.1515511339 | 0.15154 |
| | 1.7 | 0.1295265129 | 0.1295265127 | 0.1295287812 | 0.12953 |
| | 2.0 | 0.1054539614 | 0.1054539613 | 0.1054577837 | 0.10545 |
| | 2.5 | 0.0792724660 | 0.0792724660 | 0.0792782030 | 0.07927 |
| 0.75 | 1.5 | 0.2208848921 | 0.2208848920 | 0.2208893879 | 0.22089 |
| | 1.7 | 0.1893076733 | 0.1893076732 | 0.1893142361 | 0.18932 |
| | 2.0 | 0.1546197042 | 0.1546197041 | 0.1546287673 | 0.15031 |
| | 2.5 | 0.1166730661 | 0.1166730661 | 0.1166850089 | 0.11667 |

**Table 8** Comparison of the proposed method and available literature methods for $v = 1$, $N = 40$ and $dt = 0.0001$ at various spatial and temporal points in Problem 4.

| *Methods* | $t = 1.7$ | | $t = 2.1$ | | $t = 3.1$ | |
|---|---|---|---|---|---|---|
| | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ |
| Proposed | 6.0510E-05 | 1.8563E-05 | 2.8825E-05 | 9.6438E-06 | 1.5769E-05 | 5.6142E-06 |
| IMQ+TPS [17] | 7.8182E-05 | 8.5969E-05 | 9.0003E-05 | 1.1891E-04 | 9.1612E-05 | 1.3374E-04 |
| IMQ+S3[17] | 7.7927E-05 | 8.4475E-05 | 9.0240E-05 | 1.1814E-04 | 9.1955E-05 | 1.3346E-04 |
| FVCM [15] | 5.929E-05 | 1.173E-05 | 2.199E-05 | 4.518E-06 | 7.190E-06 | 2.073E-06 |
| PDQM [22] | 2.3E-05 | 1.0E-05 | 3.5E-05 | 1.3E-05 | 4.2E-04 | 4.8E-03 |
| MQ[16] | 0.5654E-03 | 0.0095E-03 | 0.3307E-03 | 0.0029E-03 | 0.0179E-03 | 0.0195E-03 |
| GS[16] | 0.5654E-03 | 0.0019E-03 | 0.3308E-03 | 0.0169E-03 | 0.0216E-03 | 0.0344E-03 |
| S3[16] | 0.7901E-03 | 0.0703E-03 | 4.4576E-03 | 0.0728E-03 | 0.1524E-03 | 0.1255E-03 |
| QBGM [11] | 1.2075E-03 | 0.3513E-03 | 0.8018E-03 | 0.2445E-03 | 4.7906E-03 | 0.6333E-03 |
| CBGM[11] | 1.2072E-03 | 0.3512E-03 | 0.8017E-03 | 0.244E-03 | 0.7906E-03 | 0.6334E-03 |
| QBCM1 [35] | 0.0619E-03 | 0.0170E-03 | 0.0588E-03 | 0.0125E-03 | 4.4346E-03 | 0.6019E-03 |
| QBCM2 [35] | 1.2117E-03 | 0.3589E-03 | 0.8077E-03 | 0.2513E-03 | 4.7906E-03 | 0.6305E-03 |

**Table 9** Comparison of error norms produced with $v = 0.005$, $N = 80$ and $dt = 0.01$ at various time levels.

The results obtained with the proposed method have been compared with the exact solution and available literature [17, 26] in Table 8 for $v = 1$, $N = 40$, $dt = 0.0001$ at various spatial and temporal points. It is seen from the table that the obtained results converge very well to the exact solution and are more accurate than those of the studies [17, 26].

Table 9 tabulates $L_\infty$ and $L_2$ errors of the proposed method and various methods[11, 15–17, 22, 35] in the literature for $v = 0.005$, $N = 80$, $dt = 0.01$ at various time levels. It is clearly seen from the table that the results obtained by the proposed method are superior to the methods presented in the references[11, 16, 17, 35] and the obtained accuracy is compatible with Guo et al.[15] and Korkmaz and Dag[22].

## 4 Conclusion

In this study, a novel numerical method based on cubic B-spline and the fourth-order compact finite difference scheme has been proposed to numerically analyze the linear and nonlinear advection-diffusion equations. In this technique, to improve the accuracy, the second-order derivatives are represented by the convergence of order four by using a modified formula of the fourth-order compact finite difference approximation for the second derivative. The accuracy of the combined method is tested by comparing the computed solutions with the exact solutions and available literature solutions. The results revealed that the proposed method solutions are in good agreement with the exact solution and the literature solutions. Furthermore, the current scheme is quite easy to produce computer codes in any programming language. It is believed that the proposed method is capable of solving a wide range of partial differential equations efficiently.

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