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Preface

It is my great pleasure and honor to welcome you at the 10th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2021). I am pleased to acknowledge the official sponsorship of the conference by Sakarya University, Sakarya University of Applied Science, Kocaeli University, Society of Geometers, and Turkish World Mathematical Society. .

Established since 2012, the series of IECMSA features the latest developments in the field of mathematics and applications. The previous conferences were held as follows: IECMSA-2012, Prishtine, Kosovo, IECMSA-2013, Sarajevo, Bosnia and Herzegovina, IECMSA-2014, Vienna, Austria, IECMSA-2015, Athens, Greece, IECMSA-2016, Belgrade, Serbia, IECMSA-2017, Budapest, Hungary, IECMSA-2018, Kyiv, Ukraine, IECMSA-2019, Baku, Azerbaijan, IECMSA-2020 (online), Skopje, North Macedonia. These conferences gathered a large number of international world-renowned participants.

I would like to thank the members of the scientific committees. They have worked very hard in reviewing process and making valuable suggestions for the authors to improve their work. I also would like to express our gratitude to the external reviewers, for providing extra help in the review process, and the authors for contributing their research result to the conference. At IECMSA-2021, the scientific committee members and the external reviewers accepted 132 virtual presentations. Despite the effects of coronavirus, 147 participants are attending the conference from 23 different countries. The scientific program of the conference features 6 keynote talks, followed by 126 contributed presentations in three parallel sessions.

The conference program represents the efforts of many people. I would like to express my gratitude to all members of the organizing committee, sponsors and, honorary committee for their continued support to the IECMSA. I also thank the invited speakers for presenting their talks on current researches.

Also, the success of IECMSA depends on the effort and talent of researchers that have shared their studies on a variety of topics in mathematics and its applications. So, I would like to sincerely thank ii all participants of IECMSA-2021 for contributing to this great meeting

Wish you all health and safety during this difficult time
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Majorization Problems for Subclasses of Univalent Functions Involving the Jung-Kim-Srivastava Integral Operator

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Abstract: Majorization feature is important for investigating geometric properties of univalent functions. MacGregor investigated a majorization problem for the class of starlike functions in 1967. Later, Altıntaş and his co-authors introduced majorization problems for the classes of starlike and convex functions of complex order. Recently, many researchers have studied several majorization problems for the classes of univalent and multivalent functions involving various linear and non-linear operators, which all are subordinate to the analytic functions having positive real part in the open unit disc.
The main object of this work is to investigate majorization problems for two subclasses $\mathcal{S}^*(\alpha,\beta,\gamma)$ and $\mathcal{C}(\alpha,\beta,\gamma)$ of starlike and convex functions of complex order connected with the Jung-Kim-Srivastava integral operator. For special values of parameters, corresponding consequences of the main results are also presented.

Keywords: Convex functions, Integral operator, Majorization, Starlike functions.

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of all univalent functions in $\mathcal{A}$. A function $f \in \mathcal{S}$ is said to be starlike with respect to the origin in $\mathbb{D}$ if the range $f(\mathbb{D})$ is starlike with respect to the origin. A function $f \in \mathcal{S}$ is said to be convex in $\mathbb{D}$ if the range $f(\mathbb{D})$ is convex. The classes of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively.

Nasr and Aouf [7] defined the class $\mathcal{S}^*(\gamma)$ of starlike functions of complex order γ $(\gamma \in \mathbb{C}\backslash\{0\})$, and Wiatrowski [9] introduced the class $\mathcal{C}(\gamma)$ of convex functions of complex order γ $(\gamma \in \mathbb{C}\backslash\{0\})$ as

$$\mathcal{S}^*(\gamma) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0,\ z \in \mathbb{D} \right\},$$

and

$$\mathcal{C}(\gamma) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D} \right\}.$$

We note that for $\gamma = 1$, these classes reduce to the classes $\mathcal{S}^*$ and $\mathcal{C}$.

Denote by $\mathcal{P}$ the class of functions p which are analytic and having positive real part in $\mathbb{D}$ with $p(0) = 1$. Let Ω be the class of Schwarz functions w which are analytic in $\mathbb{D}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. If f and g are analytic functions in $\mathbb{D}$, then we state f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function $w \in \Omega$ such that $f(z) = g(w(z))$ (see [4]).

Majorization is an important subject for investigating geometric properties of analytic functions. In 1967, MacGregor [6] introduced the concept of majorization as follows:

Definition 1. *Let f and g be analytic functions in $\mathbb{D}$, then we say that f is majorized by g in $\mathbb{D}$, denoted by*

$$f(z) \ll g(z),\ \ (z \in \mathbb{D})$$

if there exists an analytic function φ in $\mathbb{D}$ satisfying

$$|\varphi(z)| \leq 1 \ \text{and}\ \ f(z) = \varphi(z)g(z)\ \ (z \in \mathbb{D}). \tag{2}$$

In 1993, Jung *et al.* [5] introduced the Jung-Kim-Srivastava integral operator $\mathcal{Q}_\beta^\alpha f : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{Q}_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt,$$

where $\alpha > 0$ and $\beta > -1$. They observed that

$$\mathcal{Q}_\beta^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} a_n z^n, \tag{3}$$

where $\Gamma(.)$ is the familiar gamma function. It is easily verified from (3) that

$$z(\mathcal{Q}_\beta^\alpha f(z))' = (\alpha+\beta)\mathcal{Q}_\beta^{\alpha-1} f(z) - (\alpha+\beta-1)\mathcal{Q}_\beta^\alpha f(z). \tag{4}$$

For special values of parameters, the integral operator (3) reduces to the following known integral operators as special cases:

(i) For $\alpha = 1$, the operator $\mathcal{Q}_\beta^\alpha f$ reduces to the Bernardi integral operator defined in [3].
(ii) For $\alpha = 1, \beta = 0$, the operator $\mathcal{Q}_\beta^\alpha f$ reduces to the Alexander integral operator given in [1].

Making use of the integral operator $\mathcal{Q}_\beta^\alpha f$, we define the following new subclasses $\mathcal{S}^*(\alpha, \beta, \gamma)$ and $\mathcal{C}(\alpha, \beta, \gamma)$ of starlike and convex functions of complex order connected with the Jung-Kim-Srivastava integral operator.

Definition 2. *Let $\gamma \in \mathbb{C}\backslash\{0\}$, $\alpha > 0$ and $\beta > -1$. A function $f \in \mathcal{A}$ given by* (1) *is in the class $\mathcal{S}^*(\alpha, \beta, \gamma)$ if and only if*

$$\frac{\mathcal{Q}_\beta^\alpha f(z)}{z} \neq 0 \ \textit{and} \ \ \mathrm{Re}\left(1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{Q}_\beta^\alpha f(z))'}{\mathcal{Q}_\beta^\alpha f(z)} - 1\right)\right) > 0, \ \ (z \in \mathbb{D}) \tag{5}$$

and a function $f \in \mathcal{A}$ is in the class $\mathcal{C}(\alpha, \beta, \gamma)$ if and only if

$$\mathcal{Q}_\beta^\alpha f(z) \neq 0 \ \textit{and} \ \ \mathrm{Re}\left(1 + \frac{1}{\gamma} \frac{z(\mathcal{Q}_\beta^\alpha f(z))''}{(\mathcal{Q}_\beta^\alpha f(z))'}\right) > 0, \ \ (z \in \mathbb{D}) \tag{6}$$

where $\mathcal{Q}_\beta^\alpha f$ is given by (3).

In this paper, we establish majorization features for the classes $\mathcal{S}^*(\alpha, \beta, \gamma)$ and $\mathcal{C}(\alpha, \beta, \gamma)$ of starlike and convex functions of complex order defined by the Jung-Kim-Srivastava integral operator. For special values of parameters, corresponding consequences of the main theorems are also presented.

2 Majorization properties

Theorem 1. *Let the function $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}^*(\alpha, \beta, \gamma)$ with $\mathcal{Q}_\beta^\alpha f(z) \ll \mathcal{Q}_\beta^\alpha g(z)$ for all $z \in \mathbb{D}$, then*

$$|\mathcal{Q}_\beta^{\alpha-1} f(z)| \leq |\mathcal{Q}_\beta^{\alpha-1} g(z)|, \ \ (|z| \leq r_1)$$

where r_1 is given by

$$r_1 = \frac{L - \sqrt{L^2 - 4(|2\gamma-1| + (\alpha+\beta-1))(\alpha+\beta)}}{2(|2\gamma-1| + (\alpha+\beta-1)} \tag{7}$$

with

$$L = 3 + |2\gamma - 1| + (\alpha+\beta-1)$$

is the smallest positive root of the equation

$$(|2\gamma-1| + (\alpha+\beta-1))r^2 - (3 + |2\gamma-1| + (\alpha+\beta-1))r + \alpha + \beta = 0. \tag{8}$$

Proof: Since $g \in \mathcal{S}^*(\alpha, \beta, \gamma)$, we find from (5) that if

$$p(z) = 1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{Q}_\beta^\alpha g(z))'}{\mathcal{Q}_\beta^\alpha g(z)} - 1\right), \tag{9}$$

then p is an analytic function having positive real part $\mathrm{Re}(p(z)) > 0$ in $\mathbb{D}$ with $p(0) = 1$ and

$$p(z) = \frac{1 + w(z)}{1 - w(z)}, \tag{10}$$

where w is the Schwarz function with $w(0) = 0$ and $|w(z)| \leq |z|$. From (9) and (10), we get

$$\frac{z(\mathcal{Q}_\beta^\alpha g(z))'}{\mathcal{Q}_\beta^\alpha g(z)} = \frac{1 + (2\gamma - 1)w(z)}{1 - w(z)}.$$

Hence, by using (4) and applying the virtue $|w(z)| \leq |z|$, we observe

$$|\mathcal{Q}_\beta^\alpha g(z)| \leq \frac{(\alpha + \beta)(1 + |z|)}{1 - |2\gamma - 1||z| + (\alpha + \beta - 1)(1 - |z|)} |\mathcal{Q}_\beta^{\alpha-1} g(z)|. \tag{11}$$

Because $\mathcal{Q}_\beta^\alpha f$ is majorized by $\mathcal{Q}_\beta^\alpha g$, there exists an analytic function φ defined by (2) in $\mathbb{D}$ with $|\varphi(z)| \leq 1$ satisfying

$$\mathcal{Q}_\beta^\alpha f(z) = \varphi(z) \mathcal{Q}_\beta^\alpha g(z). \tag{12}$$

Differentiating on both sides of (12) with respect to z and multiplying by z, we obtain

$$z(\mathcal{Q}_\beta^\alpha f(z))' = z\varphi'(z)\mathcal{Q}_\beta^\alpha g(z) + z\varphi(z)(\mathcal{Q}_\beta^\alpha g(z))' \tag{13}$$

By using (4) in (13) together with (12), we get

$$\mathcal{Q}_\beta^{\alpha-1} f(z) = \frac{1}{\alpha + \beta} z\varphi'(z)\mathcal{Q}_\beta^\alpha g(z) + \varphi(z)\mathcal{Q}_\beta^{\alpha-1} g(z). \tag{14}$$

It is well-known that Schwarz functions satisfy the inequality given by (Nehari [8]);

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}. \tag{15}$$

Substituting (11) and (15) into (14), we get

$$|\mathcal{Q}_\beta^{\alpha-1} f(z)| \leq \left[\frac{|z|(1 - |\varphi(z)|^2)}{(1 - |z|)} \frac{1}{1 - |2\gamma - 1||z| + (\alpha + \beta - 1)(1 - |z|)} + |\varphi(z)|\right] |\mathcal{Q}_\beta^{\alpha-1} g(z)|.$$

Upon setting $|z| = r < 1$ and $|\varphi(z)| = \rho \ (0 \leq \rho \leq 1)$, the last inequality can be written as

$$|\mathcal{Q}_\beta^{\alpha-1} f(z)| \leq \Theta(r, \rho) |\mathcal{Q}_\beta^{\alpha-1} g(z)|,$$

where

$$\Theta(r, \rho) = \frac{r(1 - \rho^2)}{(1 - r)} \frac{1}{1 - |2\gamma - 1|r + (\alpha + \beta - 1)(1 - r)} + \rho.$$

In order to determine r_1, we choose

$$\begin{aligned} r_1 &= \max\{r \in (0, 1) : \Theta(r, \rho) \leq 1, \ \forall \rho \in [0, 1]\} \\ &= \max\{r \in (0, 1) : \chi(r, \rho) \geq 0, \ \forall \rho \in [0, 1]\}, \end{aligned}$$

where

$$\chi(r, \rho) = (1 - r)\big(1 - |2\gamma - 1|r + (\alpha + \beta - 1)(1 - r)\big) - r(1 + \rho).$$

Since $\frac{\partial}{\partial \rho}\chi(r, \rho) = -r < 0$, therefore $\chi(r, \rho)$ takes its minimum for $\rho = 1$, namely

$$\min\{\chi(r, \rho) \geq 0, \ \rho \in [0, 1]\} = \chi(r, 1) =: \chi(r),$$

where

$$\chi(r) = (1 - r)\big(1 - |2\gamma - 1|r + (\alpha + \beta - 1)(1 - r)\big) - 2r.$$

Since $\alpha + \beta - 1 \geq 0$, then $\chi(0) = \alpha + \beta > 0$ and $\chi(1) = -2 < 0$, thus there exists $|z| \leq r_1$ such that $\chi(r) \geq 0$ for all $r \in [0, r_1]$, where r_1 is the smallest positive root of the equation (8). This completes the proof. $\square$

Using $\alpha = 1, \beta = 0$ in Theorem 1, we get the result given in [2].

Remark 1. *Let the function $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}^*(\gamma)$ with $f(z) \ll g(z)$ for all $z \in \mathbb{D}$, then*

$$|f'(z)| \leq |g'(z)|, \ \ (|z| \leq r_2)$$

where r_2 is given by

$$r_2 = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}.$$

The proof of the next theorem is essentially based upon the following lemma.

Lemma 1. *[2] If $f \in \mathcal{C}(\gamma)$, then $f \in \mathcal{S}^*(\frac{1}{2}\gamma)$, that is $\mathcal{C}(\gamma) \subset \mathcal{S}^*(\frac{1}{2}\gamma)$ $(\gamma \in \mathbb{C}\backslash\{0\})$.*

Theorem 2. *Let the function $f \in \mathcal{A}$ and suppose that $g \in \mathcal{C}(\alpha, \beta, \gamma)$ with $\mathcal{Q}_\beta^\alpha f(z) \ll \mathcal{Q}_\beta^\alpha g(z)$ for all $z \in \mathbb{D}$, then*

$$|\mathcal{Q}_\beta^{\alpha-1} f(z)| \leq |\mathcal{Q}_\beta^{\alpha-1} g(z)|, \ \ (|z| \leq r_3)$$

where r_3 is given by

$$r_3 = \frac{M - \sqrt{M^2 - 4(|\gamma - 1| + (\alpha + \beta - 1))(\alpha + \beta)}}{2(|\gamma - 1| + (\alpha + \beta - 1)} \tag{16}$$

and

$$M = 3 + |\gamma - 1| + (\alpha + \beta - 1).$$

Proof: Upon replacing γ in Theorem 1 by $\frac{1}{2}\gamma$, and applying Lemma 1, we get the proof. □

Using $\alpha = 1, \beta = 0$ in Theorem 2, we get the result given in [2].

Remark 2. *Let the function $f \in \mathcal{A}$ and suppose that $g \in \mathcal{C}(\gamma)$ with $f(z) \ll g(z)$ for all $z \in \mathbb{D}$, then*

$$|f'(z)| \leq |g'(z)|, \ \ (|z| \leq r_4)$$

where r_4 is given by

$$r_4 = \frac{3 + |\gamma - 1| - \sqrt{9 + 2|\gamma - 1| + |\gamma - 1|^2}}{2|\gamma - 1|}.$$

3 Conclusion

In this paper, we introduced two new function classes of univalent functions involving the Jung-Kim-Srivastava integral operator and obtained majorization properties of these classes. We concluded that our results are improvement of some results appeared in literature.

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A Study on Bihyperbolic Generalized Fibonacci Numbers

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Abstract: In this paper, Gelin-Cesaro's, Melham's, Vajda's identities have been proven with the help of Binet formula for generalized bihyperbolic Fibonacci numbers. Then, bihyperbolic generating, exponential and Poisson generating functions have been presented for bihyperbolic generalized Fibonacci numbers. Moreover, special cases have been given for all identities and generating functions.

Keywords: Bihyperbolic numbers, generalized bihyperbolic Fibonacci numbers, hyperbolic four-complex numbers.

1 Introduction

A hyperbolic number is defined as $z = x + jy$ where $x, y \in \mathbb{R}$ and j is a unipotent (hyperbolic) imaginary unit such that $j^2 = 1$ and $j \neq \pm 1$, [17]. Algebraic properties of hyperbolic numbers, dual and complex hyperbolic numbers and bihyperbolic numbers are given in [1], [3], [8], [13], [14]. On the other hand, Torunbalcı have defined hyperbolic Fibonacci numbers and hyperbolic Fibonacci vectors. Then, the inner product, cross product and mixed product have been given for these vectors, [16]. Cihan et al. have introduced dual hyperbolic Fibonacci and dual hyperbolic Lucas numbers. Identities for Cassini, Catalan, nega-dual hyperbolic Fibonacci and nega-dual Lucas numbers and Binet formula have been given, [6]. After that, Güngör and Cihan have defined dual-hyperbolic numbers with generalized Fibonacci coefficients. They have proven identities regarding conjugates of these numbers. Also, Catalan, Cassini, Honsberger, Tagiuri, d'Ocagne identities have been found. Lastly, special cases have been presented for these numbers, [7]. Later, Soykan has defined generalized dual hyperbolic Fibonacci numbers. Binet formula, generating function, Catalan, Cassini, d'Ocagne, Gelin-Cesaro, Melham identities, summation formulas and matrices for these numbers have been obtained. Moreover, special cases have been mentioned, [15].
The set of bihyperbolic numbers is defined as follows

$$H_2 = \{t = t_0 + j_1 t_1 + j_2 t_2 + j_3 t_3 : t_0, t_1, t_2, t_3 \in \mathbb{R}\}$$

where $j_1, j_2, j_3 \notin \mathbb{R}$ are hyperbolic imaginary units. Moreover, the multiplication of these units are as follows, [12]

$$j_1^2 = j_2^2 = j_3^2 = 1, \qquad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1.$$

In the literature, bihyperbolic numbers are also called canonical hyperbolic quaternions or hyperbolic four complex numbers, [5], [11].
Brod et al. have defined Fibonacci, Jacobstal and Pell numbers for bihyperbolic numbers. Then, the well-known identities such as Binet, Catalan, Cassini and d'Ocagne and generating functions have been proven, [4]. After, Azak has obtained some identities regarding conjugations, Honsberger's identity and negabihyperbolic numbers for bihyperbolic Fibonacci numbers. Bihyperbolic Lucas numbers, bihyperbolic generalized Fibonacci numbers have been defined. Later, some algebraic properties and well-known identities have been proven for these numbers. Finally, special cases of these identities and formulas have been given, [2].

2 Preliminaries

The Fibonacci number sequence $\{F_n\}$ and and Lucas number sequence $\{L_n\}$ are defined by the recurrence relations

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0,\ F_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2,\ L_1 = 1.$$

The well-known Binet formulas for Lucas and Fibonacci sequences are given by

$$L_n = \alpha^n + \beta^n, \qquad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where α and β are the roots of the equation

$$x^2 - x - 1 = 0$$

so that $\alpha + \beta = 1, \quad \alpha\beta = -1, \quad \alpha - \beta = \sqrt{5}$, [10].
The generating function for Fibonacci sequence is defined by, [10]

$$G(F_n; x) = \frac{x}{1 - x - x^2}.$$

The exponential and Poisson generating functions of a sequence $\{k_n\}$ are defined by

$$k(x) = \sum_{n=0}^{\infty} k_n \frac{x^n}{n!}$$

and

$$\tilde{k}(x) = \sum_{n=0}^{\infty} k_n \frac{e^{-x} x^n}{n!}$$

respectively.
Let W_n be n-th generalized Fibonacci number which has the recurrence relation

$$W_n = W_{n-1} + W_{n-2}; \quad W_0 = a, \quad W_1 = b \quad (n \geq 2)$$

with the nonzero initial values $W_0,\ W_1$. If $W_0 = 0,\ W_1 = 1$, then we obtain Fibonacci numbers and if $W_0 = 2,\ W_1 = 1$, then we obtain Lucas numbers, [9].
Also, Binet formula for generalized Fibonacci numbers is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where $A = W_1 - W_0\beta$ and $B = W_1 - W_0\alpha$, [9].
The set of bihyperbolic Fibonacci numbers is defined by

$$H_2^F = \left\{ BHF_n = F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3} |\, F_n \,\text{is}\, n^{th} \,\text{Fibonacci number} \right\}$$

where

$$j_1^2 = j_2^2 = j_3^2 = 1, \qquad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1.$$

Here $j_s\ (s = 1, 2, 3)$ denote unipotent (hyperbolic) imaginary units such that $j_s \neq \pm 1$ and $j_s \notin \mathbb{R}$, [4].
The recurrence relations for bihyperbolic Lucas and bihyperbolic generalized Fibonacci numbers are presented in [2] as follows:

$$BHL_n = BHL_{n-1} + BHL_{n-2}$$

and

$$BHW_n = BHW_{n-1} + BHW_{n-2}.$$

Let BHL_n be $n - th$ bihyperbolic Lucas number. For $n \geq 1$, Binet Formula is given by

$$BHL_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n \tag{1}$$

such that

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad \hat{\alpha} = 1 + j_1\alpha + j_2\alpha^2 + j_3\alpha^3,\ \hat{\beta} = 1 + j_1\beta + j_2\beta^2 + j_3\beta^3, [2].$$

Binet formula for bihyperbolic generalized Fibonacci number is given by

$$BHW_n = \frac{A\,\hat{\alpha}\alpha^n - B\,\hat{\beta}\beta^n}{\alpha - \beta} \tag{2}$$

where $\hat{\alpha} = 1 + j_1\alpha + j_2\alpha^2 + j_3\alpha^3$ and $\hat{\beta} = 1 + j_1\beta + j_2\beta^2 + j_3\beta^3$, [2].

3 Some identities for the bihyperbolic generalized Fibonacci numbers

In this section, we will give some identities such as Gelin-Cesaro, Melham, Vajda for bihyperbolic generalized Fibonacci numbers. We will mention the special cases which correspond to the identities for bihyperbolic Fibonacci and Lucas.

Theorem 1. *(Gelin-Cesaro's Identity) Let n and m be any integers. Then, the following identity holds for the bihyperbolic generalized Fibonacci number BHW_n*

$$BHW_{n+2}\,BHW_{n+1}\,BHW_{n-1}\,BHW_{n-2}-BHW_n^4=-A^2B^2\left(13+12j_2\right).$$

Proof: Applying the equation (2), we obtain

$$\begin{aligned}&BHW_{n+2}\,BHW_{n+1}\,BHW_{n-1}\,BHW_{n-2}-BHW_n^4\\&=\tfrac{1}{(\alpha-\beta)^4}\Big[A^3B\,\hat{\alpha}^3\hat{\beta}\left(4\alpha^{3n}\beta^n-\alpha^{3n-2}\beta^{n+2}-\alpha^{3n-1}\beta^{n+1}-\alpha^{3n+1}\beta^{n-1}-\alpha^{3n+2}\beta^{n-2}\right)\\&\qquad+A^2B^2\hat{\alpha}^2\hat{\beta}^2\left(\alpha^{2n-3}\beta^{2n+3}+\alpha^{2n-1}\beta^{2n+1}+\alpha^{2n+1}\beta^{2n-1}-4\alpha^{2n}\beta^{2n}+\alpha^{2n+3}\beta^{2n-3}\right)\\&\qquad+AB^3\hat{\alpha}\,\hat{\beta}^3\left(4\alpha^n\beta^{3n}-\alpha^{n-2}\beta^{3n+2}-\alpha^{n-1}\beta^{3n+1}-\alpha^{n+1}\beta^{3n-1}-\alpha^{n+2}\beta^{3n-2}\right)\Big].\end{aligned}$$

Rearranging the above equation, we have

$$\begin{aligned}&BHW_{n+2}\,BHW_{n+1}\,BHW_{n-1}\,BHW_{n-2}-BHW_n^4\\&=\tfrac{1}{(\alpha-\beta)^4}\Big[A^3B\,\tfrac{\hat{\alpha}^3\hat{\beta}}{(\alpha\beta)^2}\alpha^{3n}\beta^n\left(4\alpha^2\beta^2-\beta^4-\alpha\,\beta^3-\alpha^3\beta-\alpha^4\right)\\&\qquad+A^2B^2\tfrac{\hat{\alpha}^2\hat{\beta}^2}{(\alpha\beta)^3}\alpha^{2n}\beta^{2n}\left(\beta^6+\alpha^2\beta^4+\alpha^4\beta^2-4\alpha^3\,\beta^3+\alpha^6\right)\\&\qquad+AB^3\tfrac{\hat{\alpha}\,\hat{\beta}^3}{(\alpha\beta)^2}\alpha^n\beta^{3n}\left(4\alpha^2\beta^2-\beta^4-\alpha\,\beta^3-\alpha^3\beta-\alpha^4\right)\Big].\end{aligned}\tag{3}$$

If we calculate the following equalities, we get

$$\begin{aligned}&4\alpha^2\beta^2-\beta^4-\alpha\,\beta^3-\alpha^3\beta-\alpha^4=0\\&\beta^6+\alpha^2\beta^4+\alpha^4\beta^2-4\alpha^3\,\beta^3+\alpha^6=25\\&\hat{\alpha}^2\hat{\beta}^2=13+12j_2\\&\alpha-\beta=\sqrt{5}\\&\alpha\beta=-1.\end{aligned}$$

Now, substituting the above equalities into the equation (3), it follows that

$$\begin{aligned}BHW_{n+2}\,BHW_{n+1}\,BHW_{n-1}\,BHW_{n-2}-BHW_n^4&=\tfrac{1}{(\alpha-\beta)^4}25\,A^2B^2\tfrac{\hat{\alpha}^2\hat{\beta}^2}{(\alpha\beta)^3}(\alpha\,\beta)^{2n}\\&=-A^2B^2\left(13+12j_2\right).\end{aligned}$$

□

Corollary 1. *Let n be any integer.*
i) *For $W_n=F_n,\quad F_0=0,\quad F_1=1$, Gelin-Cesaro identity for bihyperbolic Fibonacci numbers is given by*

$$BHF_{n+2}\,BHF_{n+1}\,BHF_{n-1}\,BHF_{n-2}-BHF_n^4=-\left(13+12j_2\right).$$

ii) *For $W_n=L_n,\quad L_0=2,\quad L_1=1$, Gelin-Cesaro identity for bihyperbolic Lucas numbers is given by*

$$BHL_{n+2}\,BHL_{n+1}\,BHL_{n-1}\,BHL_{n-2}-BHL_n^4=-25\left(13+12j_2\right).$$

Theorem 2. *(Melham's Identity) Let n and m be any integers. Then, the following identity is given for the bihyperbolic generalized Fibonacci number BHW_n*

$$BHW_{n+1}\,BHW_{n+2}\;BHW_{n+6}\;-BHW_{n+3}^3=(-1)^nAB\left(2j_1+3j_3\right)BHW_n.$$

Proof: Using the equation (2), we get

$$\begin{aligned}&BHW_{n+1}\,BHW_{n+2}\;BHW_{n+6}\;-BHW_{n+3}^3\\&=\tfrac{1}{(\alpha-\beta)^3}\Big[A^2B\,\hat{\alpha}^2\hat{\beta}\left(-\alpha^{2n+8}\beta^{n+1}-\alpha^{2n+7}\beta^{n+2}-\alpha^{2n+3}\beta^{n+6}+3\alpha^{2n+6}\beta^{n+3}\right)\\&\qquad+AB^2\hat{\alpha}\,\hat{\beta}^2\left(\alpha^{n+6}\beta^{2n+3}+\alpha^{n+2}\beta^{2n+7}+\alpha^{n+1}\beta^{2n+8}-3\alpha^{n+3}\beta^{2n+6}\right)\Big].\end{aligned}\tag{4}$$

If some terms of the equation (4) is put in parantheses, it becomes

$$\begin{aligned}BHW_{n+1}\,BHW_{n+2}\;BHW_{n+6}\;-BHW_{n+3}^3=\tfrac{1}{(\alpha-\beta)^3}\Big[&A^2B\,\hat{\alpha}^2\hat{\beta}\,\alpha^{2n+3}\beta^{n+1}\left(-\alpha^5-\alpha^4\beta-\beta^5+3\alpha^3\beta^2\right)\\&+AB^2\hat{\alpha}\,\hat{\beta}^2\alpha^{n+1}\beta^{2n+3}\left(\alpha^5+\alpha\,\beta^4+\beta^5-3\alpha^2\beta^3\right)\Big].\end{aligned}$$

By bringing the term $(\alpha\beta)^{n+1}$ outside the parantheses, we can write

$$BHW_{n+1}\,BHW_{n+2}\;BHW_{n+6}\;-BHW_{n+3}^3=\tfrac{1}{(\alpha-\beta)^3}AB\,\hat{\alpha}\,\hat{\beta}(\alpha\beta)^{n+1}\Big[A\,\hat{\alpha}\,\alpha^{n+2}\left(\alpha^5+\alpha^4\beta+\beta^5-3\alpha^3\beta^2\right) + B\,\hat{\beta}\,\beta^{n+2}\left(\alpha^5+\alpha\,\beta^4+\beta^5-3\alpha^2\beta^3\right)\Big]. \tag{5}$$

Note that

$$\alpha^5+\alpha^4\beta+\beta^5-3\alpha^3\beta^2=5\beta^2,\quad \alpha^5+\alpha\,\beta^4+\beta^5-3\alpha^2\beta^3=5\alpha^2,\quad \alpha\,\beta=-1.$$

Substituting the above equalities into the equation (5) gives us

$$BHW_{n+1}\,BHW_{n+2}\;BHW_{n+6}\;-BHW_{n+3}^3=\frac{1}{(\alpha-\beta)^2}AB\,\hat{\alpha}\hat{\beta}(-1)^{n+2}\left[5\alpha^2\beta^2\left(\frac{A\,\hat{\alpha}\,\alpha^n-B\,\hat{\beta}\,\beta^n}{\alpha-\beta}\right)\right].$$

Finally, we find that

$$BHW_{n+1}\,BHW_{n+2}\;BHW_{n+6}\;-BHW_{n+3}^3=(-1)^n\,(2j_1+3j_3)\,BHW_n$$

where $BHW_n=\frac{A\,\hat{\alpha}\,\alpha^n-B\,\hat{\beta}\,\beta^n}{\alpha-\beta},\quad \alpha-\beta=\sqrt{5},\quad \hat{\alpha}\hat{\beta}=2j_1+3j_3.$ □

Corollary 2. ***i)*** *For any integer n, $W_n=F_n,\ F_0=0,\ F_1=1$, Melham identity for bihyperbolic Fibonacci numbers is given by*

$$BHF_{n+1}\,BHF_{n+2}\;BHF_{n+6}\;-BHF_{n+3}^3=(-1)^n\,(2j_1+3j_3)\,BHF_n.$$

ii) *For any integer n, $W_n=L_n,\ L_0=2,\ L_1=1$, Melham identity for bihyperbolic Lucas numbers is given by*

$$BHL_{n+1}\,BHL_{n+2}\;BHL_{n+6}\;-BHL_{n+3}^3=(-1)^n\,(2j_1+3j_3)\,BHL_n.$$

Theorem 3. *(Vajda's Identity) Let n, r and m be any integers. Then, the following identity is satisfied for the bihyperbolic generalized Fibonacci numbers*

$$BHW_{n+r}\;BHW_{n+m}-BHW_n\;BHW_{n+r+m}=(-1)^nAB\,(2j_1+3j_3)\,F_r\,F_m.$$

Proof: From the equation (2), we obtain the following equality

$$BHW_{n+r}\;BHW_{n+m}-BHW_n\;BHW_{n+r+m}=\frac{1}{(\alpha-\beta)^2}AB\,\hat{\alpha}\,\hat{\beta}(\alpha\beta)^n\left(-\alpha^m\beta^r-\alpha^r\beta^m+\alpha^{r+m}+\beta^{r+m}\right). \tag{6}$$

Then, the equation (6) is rewritten as

$$BHW_{n+r}\;BHW_{n+m}-BHW_n\;BHW_{n+r+m}=AB\;\left(\hat{\alpha}\hat{\beta}\right)(\alpha\beta)^n\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right).$$

Using the fact that $\hat{\alpha}\hat{\beta}=\ \hat{\beta}\hat{\alpha}=2j_1+3j_3$ and $\alpha\beta=-1$ we can deduce that

$$BHW_{n+r}\;BHW_{n+m}-BHW_n\;BHW_{n+r+m}=(-1)^nAB\;(2j_1+3j_3)\,F_m\,F_r.$$

□

Corollary 3. ***i)*** *For any integers n, m, r and $W_n=F_n,\ F_0=0$ and $F_1=1$, Vajda identity for bihyperbolic Fibonacci numbers is given by*

$$BHF_{n+r}\;BHF_{n+m}-BHF_n\;BHF_{n+r+m}=(-1)^n\,(2j_1+3j_3)\,F_r\,F_m.$$

ii) *For any integers n, m, r and $W_n=L_n,\ L_0=2,\ L_1=1$, Vajda identity for bihyperbolic Lucas numbers is given by*

$$BHL_{n+r}\;BHL_{n+m}-BHL_n\;BHL_{n+r+m}=5(-1)^{n+1}\,(2j_1+3j_3)\,F_r\,F_m.$$

4 Some generating functions for the bihyperbolic generalized Fibonacci numbers

In this section, we will give generating function for the bihyperbolic generalized Fibonacci numbers. Then, the exponential and Poisson generating functions will be obtained for the bihyperbolic generalized Fibonacci numbers. Also, special cases will be shown.

Theorem 4. *The generating function for bihyperbolic generalized Fibonacci numbers is*

$$G\,(BHW_n;x)=\frac{(W_0+W_1j_1+W_2j_2+W_3j_3)+[(W_1-W_0)+(W_2-W_1)\,j_1+(W_3-W_2)\,j_2+(W_4-W_3)\,j_3]\,x}{1-x-x^2}.$$

Proof: Let

$$G\left(BHW_n;x\right)=BHW_0+BHW_1\,x+BHW_2\,x^2+...+BHW_n\,x^n+...$$

be the generating function of the bihyperbolic generalized Fibonacci numbers. Then, one can write the following equalities

$$\begin{aligned} xG\left(BHW_n;x\right)&=BHW_0\,x+BHW_1\,x^2+BHW_2\,x^3+...+BHW_{n-1}\,x^n+...\\ x^2G\left(BHW_n;x\right)&=BHW_0\,x^2+BHW_1\,x^3+BHW_2\,x^4+...+BHW_{n-2}\,x^n+...\end{aligned}$$

If we consider the recurrence relation $BHW_n=BHW_{n-1}+BHW_{n-2}$, we have

$$\begin{aligned} G\left(BHW_n;x\right)-xG\left(BHW_n;x\right)-x^2\left(BHW_n;x\right)&=BHW_0+\left(BHW_1-BHW_0\right)x+\left(BHW_2-BHW_1-BHW_0\right)x^2+...\\ \left(1-x-x^2\right)\left(BHW_n;x\right)&=BHW_0+\left(BHW_1-BHW_0\right)x\\ \left(BHW_n;x\right)&=\tfrac{BHW_0+\left(BHW_1-BHW_0\right)x}{1-x-x^2}.\end{aligned}$$

Substituting the first two elements of bihyperbolic generalized Fibonacci numbers into the above equation yields

$$G\left(BHW_n;x\right)=\frac{\left(W_0+W_1j_1+W_2j_2+W_3j_3\right)+\left[\left(W_1-W_0\right)+\left(W_2-W_1\right)j_1+\left(W_3-W_2\right)j_2+\left(W_4-W_3\right)j_3\right]x}{1-x-x^2}$$

where

$$\begin{aligned} BHW_0&=W_0+W_1j_1+W_2j_2+W_3j_3=a+bj_1+\left(a+b\right)j_2+\left(a+2b\right)j_3,\\ BHW_1&=W_1+W_2j_1+W_3j_2+W_4j_3=b+\left(a+b\right)j_1+\left(a+2b\right)j_2+\left(2a+3b\right)j_3.\end{aligned}$$

□

Corollary 4. *Generating functions for bihyperbolic Fibonacci and bihyperbolic Lucas numbers are*

$$G\left(BHF_n;x\right)=\frac{j_1+j_2+2j_3+\left(1+j_2+j_3\right)x}{1-x-x^2},\ \text{(see [4])}$$

and

$$G\left(BHL_n;x\right)=\frac{2+j_1+3j_2+4j_3+\left(-1+2j_1+j_2+3j_3\right)x}{1-x-x^2}.$$

Theorem 5. *The exponential generating function for bihyperbolic generalized Fibonacci numbers is*

$$\sum_{n=0}^{\infty}BHW_n\frac{x^n}{n!}=\frac{A\hat{\alpha}e^{\alpha x}-B\hat{\beta}e^{\beta x}}{\alpha-\beta}.$$

Proof: Using the equation (2) and Maclaurin expansion for the exponential function, we get

$$\begin{aligned} \sum_{n=0}^{\infty}BHW_n\tfrac{x^n}{n!}&=\sum_{n=0}^{\infty}\left(\tfrac{A\hat{\alpha}\alpha^n-B\hat{\beta}\beta^n}{\alpha-\beta}\right)\tfrac{x^n}{n!}\\ &=\tfrac{A}{\alpha-\beta}\hat{\alpha}\sum_{n=0}^{\infty}\tfrac{(\alpha x)^n}{n!}-\tfrac{B}{\alpha-\beta}\hat{\beta}\sum_{n=0}^{\infty}\tfrac{(\beta x)^n}{n!}\\ &=\tfrac{A\hat{\alpha}e^{\alpha x}-B\hat{\beta}e^{\beta x}}{\alpha-\beta}.\end{aligned}$$

□

Corollary 5. *Exponential generating functions for bihyperbolic Fibonacci and bihyperbolic Lucas numbers are*

$$\sum_{n=0}^{\infty}BHF_n\frac{x^n}{n!}=\frac{\hat{\alpha}e^{\alpha x}-\hat{\beta}e^{\beta x}}{\alpha-\beta}$$

and

$$\sum_{n=0}^{\infty}BHL_n\frac{x^n}{n!}=\hat{\alpha}e^{\alpha x}+\hat{\beta}e^{\beta x}$$

respectively.

Theorem 6. *The Poisson generating function for bihyperbolic generalized Fibonacci numbers is*

$$\sum_{n=0}^{\infty}BHW_n\frac{e^{-x}x^n}{n!}=\frac{A\hat{\alpha}e^{(\alpha-1)x}-B\hat{\beta}e^{(\beta-1)x}}{\alpha-\beta}.$$

Proof: Taking into consideration the equation (2) and Maclaurin expansion for the Poisson function, we have

$$\begin{aligned}\sum_{n=0}^{\infty} BHW_n \frac{e^{-x}x^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{A\hat{\alpha}\alpha^n - B\hat{\beta}\beta^n}{\alpha-\beta}\right)\frac{e^{-x}x^n}{n!}\\ &= \frac{A\hat{\alpha}e^{(\alpha-1)x} - B\hat{\beta}e^{(\beta-1)x}}{\alpha-\beta}.\end{aligned}$$

This completes the proof. □

Corollary 6. *Poisson generating functions for bihyperbolic Fibonacci and bihyperbolic Lucas numbers are*

$$\sum_{n=0}^{\infty} BHF_n \frac{e^{-x}x^n}{n!} = \frac{\hat{\alpha}e^{(\alpha-1)x} - \hat{\beta}e^{(\beta-1)x}}{\alpha-\beta}$$

and

$$\sum_{n=0}^{\infty} BHL_n \frac{e^{-x}x^n}{n!} = \hat{\alpha}e^{(\alpha-1)x} + \hat{\beta}e^{(\beta-1)x}$$

respectively.

5 Conclusion

The fact that Fibonacci numbers have applications in many fields such as music, economics, optics and computer programming has increased the interest of researchers in the literature. For this purpose, obtaining Fibonacci identities has become an important goal. In this study, we have introduced Melham, Gelin-Cesaro, and Vajda identities for bihyperbolic generalized Fibonacci numbers. We have presented the generating function which helps us to obtain generalized Fibonacci numbers. Also, we have found the exponential and the Poisson generating functions. We have taken the Binet formula as a basis while making our proofs. Finally, we have expressed the correspondence of all theorems for bihyperbolic Fibonacci and bihyperbolic Lucas numbers.

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Emotion Classification based on Twitter Data

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Abstract: The purpose of this study is to obtain qualified information on social media using text mining methods. The tweets related to Coronavirus Disease 2019 (Covid19) hashtags are investigated and collected from Twitter. The Twitter based data is filtered and pre-processed to clean or to remove unnecessary information including punctuation marks, numbers and symbols, etc. Based on the words used in the tweets, Covid19 cases are classified into two classes as positive and negative cases. Moreover, a sentiment analysis is conducted to reveal how people are affected by this epidemic and to investigate the emotional changes of the people more clearly in this process.

Keywords: Emotion analysis, Sentiment classification, Text mining.

1 Introduction

Covid-19 started to spread to humans from Wuhan, China, in December 2019. This epidemic has become a global epidemic, affecting many countries and nations. Soon after, the World Health Organization (WHO) declared a pandemic on March 11, 2020 (WHO, 2020).

Humans have experienced three corona virus-related pandemics in this century: SARS in 2003, MERS in 2012, and Covid19 in 2019. Covid19 can be spread as small liquid particles when an infected person coughs, sneezes, talks or breathes from their mouth or nose. These particles range from larger respiratory droplets to smaller aerosols (WHO). The epidemic spread day by day in the presence of these conditions, and managed to affect the whole world and humanity. Public health authorities around the world have implemented quarantine with special measures to reduce physical contact, as the virus mainly spreads between individuals in close physical contact.

As of 28 May 2021, a total of 168,599,045 global cases of Covid19 were reported to the WHO, a total of 3,507,477 deaths. As of 26 May 2021, a total of 1,546,316,352 doses of vaccine have been administered (WHO, 2021). The global pandemic has aroused great interest among the scientific community in this period when habits have changed, while vaccine, treatment, emergency care studies related to Covid19 continue, as well as people's economic conditions, remote work, socio-economic factors, global environmental change, e-learning difficulties were tried to be improved [1]-[3].

These restrictions, implemented in conjunction with the Covid19 pandemic, have caused significant disruption to individuals globally and to most (if not all) of the world's populations; it was a process that complicated the previously used aspects of daily life. For many people, daily life changed abruptly and dramatically, and "normal" lifestyles as they were known were differentiated and suspended [4]. People's social media posts have also increased due to the constantly changing life and psychological conditions. It was widely discussed, including on social media platforms like Twitter, and millions of Tweets were posted on Twitter.

Analysis of Twitter data allows the assessment of public knowledge, personal experiences, and fears or concerns during rapidly evolving epidemics such as Covid19. One of the main advantages of social media platforms like Twitter for research is that they are used as a collaborative network that makes it possible to research groups or communities of people united by common interests rather than individual profiles or personalities. It is achieved by the widespread use of hashtags, shares, and retweets that form a complex network [5, 6]. Social media networks that contribute big data are of great importance for researches.

This study is organized as follows. Section two describes briefly the methodology and the background of related works in this field. Section three provides the details of datasets used in the analysis. Section four presents the results achieved on Twitter data. Next, the section discusses them. Finally, section six provides the conclusion of the whole work.

1.1 Text mining background

1.1.1 Text mining: Data mining is the process of extracting previously unknown valid patterns and information that can be used in various application areas from large data sets that have become widespread [7]. Although the history of data mining goes back decades, its usability has increased in the last 15 years. Data mining uses both algorithms and methods used by statistical science and machine learning algorithms, which are a component of artificial intelligence. There are basically three different models of data mining. These are classification, clustering and association rule mining algorithms. All three models are used both together and separately in different fields in practice [8, 9].

Data mining is an interdisciplinary field that uses information retrieval, machine learning, statistics and computational linguistics [10]. Text mining is actually similar to data mining. However, in text mining, unstructured text data is used instead of structured data. In text mining, first of all, it is necessary to organize and manipulate the data. Since, both qualitative and quantitative analyzes can be made on the data that is easy to analyze.

The purpose of text mining is to collect unstructured text, pre-process it, categorize, cluster and label text; summarizing datasets, creating taxonomies and obtaining information about word frequencies and relationships between data entities [11, 12]. It consists of applying clustering or classification algorithms for the discovery of relationships or patterns of words and finally visualizing them [13].

Text mining discovers new information by extracting previously unidentified or secret information from textual data using different techniques. While text mining is a multidisciplinary field related to information retrieval, text analysis, information extraction..., etc., its steps are [14]:

- Gathering information from unstructured data.
- Transform this obtained information into structured data.
- Identify the model of the structured data.
- Analyze the model.
- Extract valuable information and store it in the database .

1.1.2 Sentiment analysis: Sentiment analysis is a growing field of Natural Language Processing (NLP), with research ranging from document-level classification to learning to the polarity of words and expressions. It basically classifies emotions as positive and negative or neutral [15, 16]. To deal with unstructured text data, traditional NLP methods, namely information retrieval and information extraction, are emerging. To get an idea of the extracted text, it has become a popular method in recent years, leading to sentiment analysis, an expanded field of NLP research [17]. Sentiment analysis is not a single problem; instead, it is a multifaceted problem [18]. Since opinion mining texts can be obtained in different formats from various sources, various steps are needed to detect emotions from these complex data. Sentiment analysis is the procedure for categorizing opinions expressed on a particular object. Recent advances in sentiment analysis and computational linguistics, in general allow us to perform more advanced tasks such as sentiment detection in documents. To detect emotions, researchers use commonly known algorithms built for emotion analysis. Three main approaches are used to detect emotion from text [19]:

- Lexicon-based methods—the most intuitive approach. The main goal is to find patterns similar to emotion keywords and match them. The first is to find the word that expresses the emotion in a sentence. This is usually done by tagging the words of a sentence and then weeding out the words Noun, Verb, Adjective, and Adverb (NAVA), which are the words with the most likely emotion. These words are then matched with a list of words that represent emotions according to a particular emotion pattern. The emotion that matches the keyword is considered the emotion of the particular sentence. Different approaches can be applied when it matches more than one emotion from the word list. In some keyword dictionaries, each word has a probability score for each emotion, and the emotion with the highest score is chosen as the emotion of the word. In some works, the first emotion that matches the word is chosen as the primary emotion of the word. The reference list or keyword dictionary of keywords differs by the researcher. There is no need for labeled training data as in supervised machine learning methods. Sentences are analyzed with natural language processing tools and methods, semantic inferences are made by determining the emotional terms in the sentences. There are several approaches to execute and construct an opinion lexicon given in Jain and Dandannavar ‘s study (2016)[19]. In the dictionary based approach: “A small set of opinion words is collected manually with known orientations. Then, synonyms and antonyms of these words are searched in corpora like WordNet or thesaurus and added to the set. The set gradually grows until no new words are found. This approach has a disadvantage that the strength of the sentiment classification depends on the size of the dictionary. As the size of the dictionary grows this approach becomes more erroneous.” In the Corpus based approach: “They depend on large corpora for syntactic and semantic patterns of opinion words. The words that are generated are context specific and may require a huge labelled dataset” [20].
- Machine Learning based methods—both supervised and unsupervised methods: they are used for emotion classification. It is generally preferred in cases where there is more data. Supervised methods use an annotated sentiment dataset that learns which features are most prominent to separate classes. The data set is divided into training and test sets. Naive Bayes classifier, Support Vector Machine, Artificial Neural Networks, MaxEntropy and Decision Tree are the most used algorithms.
- Hybrid based methods: they are combined methods defined to achieve maximum accuracy by using multiple methods together [19].

2 Application and results

2.0.1 Datasets: Our purpose in this study is to research the tweets including hashtags with Covid19 on Twitter and to reveal the emotions (moods) from these tweets. For this purpose, after obtaining the necessary permissions via Twitter, an account was created on Twitter’s API. Access was provided using the twitteR Library via the Rstudio Program with Twitter’s API parameters to receive tweets. First, a search was conducted using the $search_twitter()$ function in the TwitteR package to find the tweets posted under the hashtag Covid19 [21]. Only tweets other than English tweets and retweets were included in the research. For this purpose, tweets were searched in two stages and tweet data were recorded at different time of period. The process related to the data extraction is given in Fig. 1.

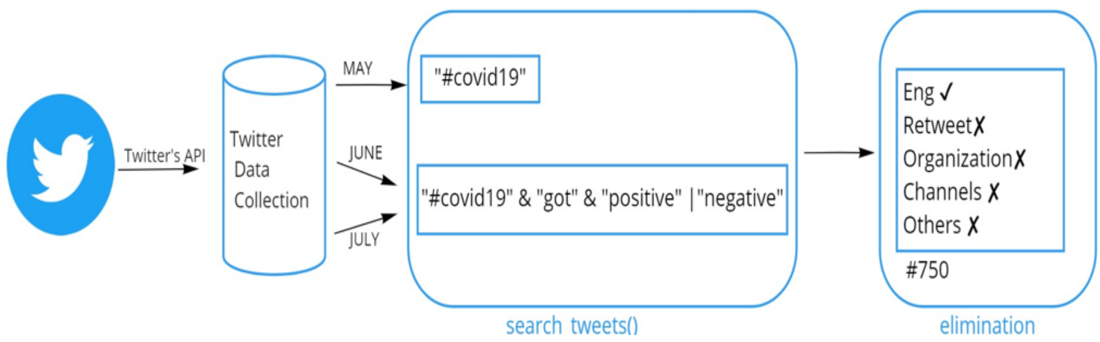

Fig. 1: Data Extraction

In Fig.1, it can be seen that, the tweets posted from TV channels, organizations, institutions, etc. are not considered in data extraction.

2.0.2 First stage: In this stage, the tweets with “#covid19” between dates 21-05-2021 and 30-05-2021 were examined. During this search, which was conducted for a total of 7 days, 100 tweets were recorded every day. It was detected that there was more than one tweet(s) belonging to the same user(s) in a total of 700 data. In addition, it was decided to remove these tweets belonging to news, magazines, TVs, popular organs that do not include individual Covid19 information. Hence they were not suitable for the analyses. In order to eliminate these problems, tweets belonging to popular media organs were removed as well. At the end of the cleanup phase, 545 tweets were taken out of 700 individual tweets.

2.0.3 Second stage: At this stage, more subjective expressions were added to the search function. In tweets related to Covid19, clearer expressions were sought to facilitate the detection of case situations. For this purpose, 205 more tweets were recorded in June and July. Among the search function, the keywords such as "got", "positive" or "negative" were used. After these records were examined in detail, the tweets and the repeating tweets of the same users were removed. Then, the number of data obtained in this stage was combined with the number of data obtained in the first stage. Accordingly, a set of 750 tweets data was created.

Next, these 750 tweets were examined very carefully to create a variable by using the information given in the tweets. This information of interest was to be “Covid+” or “Covid-”. After examining each tweet, two different classes were determined for the variable, y, as 0 for a negative case and 1 for positive. Accordingly, if a test result for Covid19 given in the tweet of any user or the people around the same user were positive, this variable was coded as 1 and otherwise 0. Suspicious cases were removed from the data set in order to carry out this coding phase properly. Classifications of thee Covid19 cases are given in Fig. 2.

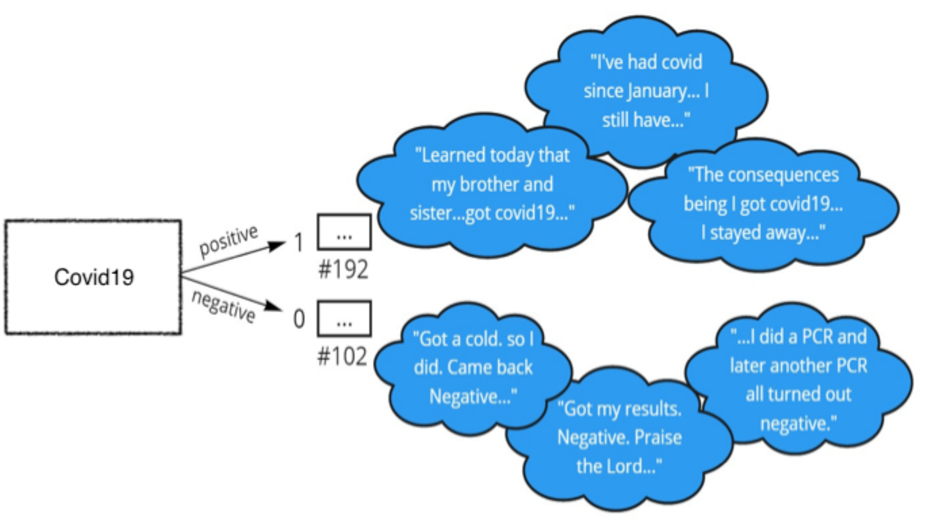

Fig. 2: Classification of covid19 case

As seen in Fig.2 there are 192 tweets coded as 1 and 102 tweets coded as 0. Also, some of the tweets to indicate how the process was continued are given with some contents as well. On the other hand, the user feedbacks or inputs were used for sentiment classification. For instance, the terms in the tweets were either positive or negative according to their meanings were summarized. For this purpose, the number of positive words which represents the variable x_1 and the variable x_2 which indicates the number of negative words in each tweet were filtered.

The *Bing* dictionary included in the tidytext library was used to classify the positive and negative words in the tweets. The Bing dictionary categorizes words as positive or negative in a binary way [22]. In Table 1, categories are given for the first 10 words.

Positive	Negative
abound	2-faces
abounds	abnormal
abundance	abolish
abundant	abominable
accessable	abominably9
accessible	abominate
acclaim	abomination
acclaimed	abortaborted
acclamation aborts ...	...

Table 1 Proposed positive and Negative Terms in the library

In some cases, the bing dictionary has its drawbacks; the word "Covid positive" should actually have a negative meaning, as it denotes being sick. In order to prevent such false classification, a better understanding of tweets are considered. For instance in a tweet given below, "my neighbour positive covid19(near 3 houses from me) and ... close contact with someone has covid ..."

Positive (x_1)	Negative (x_2)
salute	difficult
amazing	disabled
happiness	decline
excited	ridiculuous
enjoys	virus
rich	emergency
support	killed
...	...

Table 2 Classification for Tweets wrt. to the library

The bing library detects the word "positive" and assigns it as a positive sentiment although the complete meaning refers to the opposite. Some of the examples of the sentiments classified as positive and negative in our dataset given in Table 2.

y	x_1	x_2
1	0	2
1	1	2
1	2	4
1	0	2
1	1	1
1	0	2
0	1	1
1	1	1
0	0	0
0	1	0
0	2	1
...	...	...

Table 3 Coding ofTweets

In Table 3, the frequencies of x_1 and x_2 in each tweet and the variable for Covid19 case are summarized. The total number of rows of Table 3 includes 294 multivariate data.

2.1 Pre-processing

Since posted tweets usually contain plain sentences, emotions, emojis, various url addresses, etc., the texts have to go through a pre-processing step. In order to extract the information to be used for the desired purpose from these texts, unnecessary information including punctuation marks, numbers and symbols, etc. must be removed. Afterwards the edited data is transferred to the computer and the texts are converted into numbers and the processing steps are performed [23, 24]. At this stage of the analyses, using the tm library and $tm_{map()}$ function via R Studio;

- all words to lowercase are converted, and
- punctuation marks,
- symbols adjacent to the word,
- numbers,
- url information,
- extra spaces at the beginning/at the end/within the sentence,
- emoji,
- english stop words,
- removal of slang and non-meaning words

are removed. In addition, when the text files are re-examined, the special characters or words were purified from the text by using a different function.

After all these operations were performed, the term document matrix was created with the help of the Termdocumentmatrix() function for words contained in the corpus object. An example of Document Term Matrix is Given in Fig.3. A term document matrix of 2779 rows and 294 columns were created. Here, each text is converted into a numerical value in order to generate the complete document matrix, so-called a vector space model. If a term is included in the document, the frequency of its occurrences changes the value in the vector as well. Hence, a value in a vector expresses the frequency of this term observed in the text.

$$
\begin{array}{c|ccccc}
 & t_1 & t_2 & t_3 & \dots & t_j \\
\hline
D_1 & x_{11} & x_{12} & x_{13} & \cdots & x_{1j} \\
D_2 & x_{21} & x_{22} & x_{23} & \cdots & x_{2j} \\
D_3 & & & & & \\
\vdots & & & & & \\
D_j & x_{j1} & x_{j2} & x_{j3} & \cdots & x_{jj}
\end{array}
$$

Fig. 3: Document term matrix

It is important to weigh each of the word in the documents for constructing the vector space model. Vector weighting affects the success of classification. Three most commonly used methods are Term Frequency (TF), Reverse Document Frequency (TDF) and TF-IDF method for weighting [25].

2.1.1 Term Frequency (TF) method: In this method, more words in the text are thought to be more important. The calculation of the TF weighting method is as follows:

$$TD_{ij} = \frac{n_{ij}}{d_i}$$

where d_i is i^{th} document and n_{ij} is the frequency of passage of the w_j word in d_i.

2.1.2 Inverse Term Frequency (IDF) method: In this method, fewer words in the text are thought to be distinctive [16].

$$IDF_j = log\frac{n}{n_i}$$

Where n_i is the number of documents contains the w_j word and n is the number of documents in a set of documents.

2.1.3 TF-IDF method: This method combined TF and IDF methods to weigh the terms [24].

$$x_{ij} = TF_{if} \times IDF_j$$

Here x_{ij} is the TF-IDF weight of the w_j word in di document. After the weighting process, the document term matrix is generated as shown in Fig. 3.

A word cloud is used to visually represent large documents (speeches, reports, tweets ..., etc.). Creating a word cloud, also called a word or tag cloud, which is a visual representation of text data. The more commonly the term appears in the text, the larger the word appears in the rendered image. It allows us to highlight the most frequently used keywords in a paragraph of text. For this purpose, wordcloud package was used to create a word cloud of Twitter data in Fig. 4 [26].

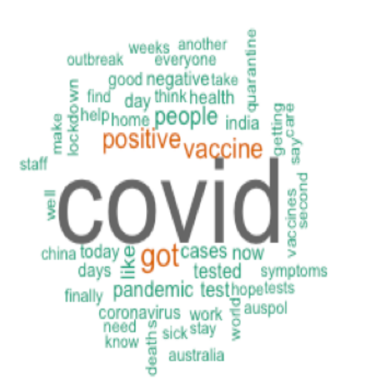

Fig. 4: Most occurred terms

As seen in Fig.4, the words are sized and highlighted individually regarding to its frequently as the weight.

3 Discussion

Nowadays, data such as consumer choices, opinions and suggestions are extensively produced on social media. Research in this paper extends Twitter data to explore the Covid19 cases and emotion classification. Results in this paper showed that dictionary based detection would be easily produce false classification whereas a human could produce a true sentiment of the words. Hence, it seems a hybrid detection method for sentiments might be more proper. Furthermore, different approaches can be applied when it matches more than two emotions (positive, negative, and neutral) from the list or different analysis can be done by determining new outline words. The data obtained from many different media sources can be used to analyze and model from related statistics as well.

4 Conclusion

Twitter is one of the most frequently used social media platforms for data analysis. Based on this study, the Covid19 cases were investigated and classified as positive and negative. Sentiment statements using the users' tweets have been examined by both dictionary and human based detection and classified into positive and negative. Finally, a word cloud is proposed to summary the users' commonly views. In this sense, as a data storage Twitter and R Software showed how large and rich they can be in terms of data investigation and visualization. Once the text is converted to a Term document matrix and determination of new variables from the text, several statistical modeling methods (machine learning models, deep learning models..., etc.) can be used for future studies.

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m-quasi-Einstein Manifolds and Concircular Vector Fields

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Abstract: In this paper, m- quasi-Einstein manifold $(M^n, g, \nabla f, \lambda)$ whose potential field ∇f is a concircular vector field are studied. In addition, some rigidity conditions for m- quasi-Einstein manifolds whose potential field is concircular are given.

Keywords: Concircular vector field, m- quasi Einstein manifold, Potential field, Ricci soliton.

1 Introduction and preliminaries

In the last years, very much attention has been given to Einstein metrics and their generalizations, for instance, Ricci solitons, almost Ricci solitons and quasi-Einstein metrics. A Riemannian manifold (M^n, g) is said to be m-quasi Einstein manifold if there exists a positive smooth function f so that the equation

$$Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g \tag{1}$$

is satisfied for some constant λ and it is denoted by $(M^n, g, \nabla f, \lambda)$. Here, $0 < m \leq \infty$ is an integer and Ric, ∇^2 and $\otimes$ stand for the Ricci tensor, the Hessian and tensorial product, respectively. An m- quasi-Einstein manifold $(M^n, g, \nabla f, \lambda)$ is called trivial if $\nabla f = 0$(the rigid case). Notice that when $m = \infty$ and λ is constant, then equation (1) reduces to a form associated with a gradient Ricci soliton [4, 5],[9]-[12] as well as when m is a positive integer it corresponds to m- quasi-Einstein metrics that are exactly those n- dimensional manifolds which are the base of an $(n+m)$- dimensional Einstein warped product $M^n \times_{e^{-\frac{f}{m}}} F^m$ [2, 3, 6]. Also in [6], the authors showed that a compact m-quasi Einstein metric with constant scalar curvature is trivial. In [2], the authors proved that when $\lambda n \geq R$ and $|\nabla f| \in L^1(M)$ in a complete non-compact m-quasi Einstein manifold, the manifold becomes Einstein.The 1-quasi-Einstein metrics that satisfy $\Delta e^{-f} + \lambda e^{-f} = 0$, are more commonly called static metrics, for more detail see [1, 8].

The goal of this paper is to study rigidity of an m- quasi-Einstein manifold $(M^n, g, \nabla f, \lambda)$ such that the potential field ∇f is concircular. Concircular vector fields have interesting applications in physics as well, especially in general relativity. A vector field X on a Riemannian manifold (M^n, g) is said to be concircular vector field if it satisfies

$$\nabla_Y X = \mu Y \tag{2}$$

for any vector field Y tangent to M^n, where μ is a non-trivial function on M^n. When M is a compact manifold, then μ is a trivial function. Therefore, M must not be compact throughout the paper. In [7], Chen has shown that for a function f on a Riemannian manifold, ∇f is a concircular vector field if and only if

$$Hessf(X, Y) = \mu g(X, Y) \tag{3}$$

for $X, Y \in \chi(M)$. Equation (3) allows us to deduce the following equation

$$\nabla_{\nabla f} \nabla f = \mu \nabla f,$$

where μ is a non-trivial function on M^n. Then we have

$$\Delta f = n\mu.$$

A concircular vector field X is called a concurrent vector field if the function $\mu \equiv 1$ in equation (2). Now, we give some general formulas and then provide a result to be used in proving our main theorem. First, taking the trace of equation (1), we get

$$R + \Delta f - \frac{1}{m}|\nabla f|^2 = n\lambda. \tag{4}$$

Moreover, we consider m- quasi-Einstein manifold M^n $(n \geq 3)$ satisfying (1) with the potential field ∇f is concircular, then we have

$$Ric - \frac{1}{m} df \otimes df = (\lambda - \mu)g. \tag{5}$$

Recall the following general formulas, see [10] for the proofs.

Lemma 1. *[10] For a function f in a Riemannian manifold, we have*

$$2(div\nabla^2 f)(\nabla f) = \frac{1}{2}\Delta|\nabla f|^2 - |\nabla^2 f|^2 + Ric(\nabla f, \nabla f) + \langle \nabla f, \nabla\Delta f\rangle \tag{6}$$

and

$$div(\nabla^2 f) = Ric\nabla f + \nabla\Delta f. \tag{7}$$

In particular, we deduce from equation (6) and equation (7),

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + Ric(\nabla f, \nabla f) + \langle \nabla f, \nabla\Delta f\rangle.$$

Since $div(\varphi I)(\nabla f) = \langle \nabla\varphi, \nabla f\rangle$, where φ is a smooth function on M^n, we give the the following result by using Lemma 1.

Lemma 2. *Let $(M^n, g, \nabla f, \lambda)$ be an m- quasi-Einstein manifold with a non-trivial concircular potential field ∇f. Then we have*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 - Ric(\nabla f, \nabla f) + \frac{2n}{m}\mu|\nabla f|^2, \tag{8}$$

$$\frac{\nabla R}{2} = \frac{(m+n-1)}{m}Ric\nabla f - \frac{n-1}{m}\Big((1-n)\lambda + n\mu + R\Big)\nabla f, \tag{9}$$

$$\frac{\nabla R}{2} - \Big(\frac{n+1}{m}\Big)\mu\nabla f + \nabla\mu = 0. \tag{10}$$

Proof: Since $\Delta f = n\mu$, from equation (6) we have

$$\frac{1}{2}\Delta|\nabla f|^2 = 2(div\nabla^2 f)\nabla f + |\nabla^2 f|^2 - Ric(\nabla f, \nabla f) - n\langle \nabla f, \nabla\mu\rangle. \tag{11}$$

Taking the divergence of equation (1), we obtain

$$divRic + div\nabla^2 f - \frac{n}{m}\mu\nabla f - \frac{1}{m}\mu\nabla f = 0. \tag{12}$$

Using the contracted second Bianchi identity $\nabla R = 2divRic$ in this equation, we get

$$2div\nabla^2 f(\nabla f) + \langle \nabla R, \nabla f\rangle - \frac{2n}{m}\mu|\nabla f|^2 - \frac{2}{m}\mu|\nabla f|^2 = 0. \tag{13}$$

Now, by taking the covariant derivative of equation (4) yields

$$\nabla R + n\nabla\mu - \frac{2}{m}\mu\nabla f = 0. \tag{14}$$

Combining the above equation with the equations (11) and (13), we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 &= \langle n\nabla\mu - \frac{2}{m}\mu\nabla f, \nabla f\rangle + \frac{2n}{m}\mu|\nabla f|^2 + \frac{2}{m}\mu|\nabla f|^2 \\ &+ |\nabla^2 f|^2 - Ric(\nabla f, \nabla f) - n\langle \nabla f, \nabla\mu\rangle \\ &= |\nabla^2 f|^2 - Ric(\nabla f, \nabla f) + \frac{2n}{m}\mu|\nabla f|^2\end{aligned}$$

which gives the first statement.

For the next statement, by using contracted second Bianchi identity, equations (7) and (12), we get

$$\begin{aligned}\nabla R &= 2divRic \\ &= -2div(\nabla^2 f) + 2\Big(\frac{n+1}{m}\Big)\mu\nabla f \\ &= -2Ric\nabla f - 2n\nabla\mu + 2\Big(\frac{n+1}{m}\Big)\mu\nabla f.\end{aligned}$$

From the above equation and equation (14), we infer

$$\nabla R = 2Ric\nabla f - 2\Big(\frac{n-1}{m}\Big)\mu\nabla f. \tag{15}$$

Using the $(1, 1)$- tensorial notation of the fundamental equation, we have

$$\mu\nabla f = \lambda\nabla f + \frac{1}{m}|\nabla f|^2\nabla f - Ric\nabla f.$$

Plugging this into equation (15) we get

$$\begin{aligned}\nabla R &= 2Ric\nabla f - 2\Big(\frac{n-1}{m}\Big)(\lambda\nabla f + \frac{1}{m}|\nabla f|^2\nabla f - Ric\nabla f) \\ &= \frac{2}{m}(m+n-1)Ric\nabla f - 2\Big(\frac{n-1}{m}\Big)\Big((1-n)\lambda + n\mu + R\Big)\nabla f\end{aligned}$$

which is equation (9). Next, by taking the divergence of equation (5), we get

$$divRic - \frac{1}{m}(\Delta f\nabla f) - \frac{1}{m}\nabla_{\nabla f}\nabla f = -\nabla\mu.$$

If we use one more Schur's lemma and notice that $R = n(\lambda - \mu) + \frac{1}{m}|\nabla f|^2$, we have

$$\frac{\nabla R}{2} - \frac{n}{m}\mu\nabla f - \frac{1}{m}\mu\nabla f + \nabla\mu = 0$$

so we arrive at the last statement. □

Remark 1. *When $|\nabla f|^2$ is constant and ∇f is a non-trivial concircular vector field, we deduce $\nabla f = 0$. Then by using equation (8) in Lemma 2, we obtain $Ric(\nabla f, \nabla f) = 0$. Thus M is a Ricci flat manifold.*

2 Rigidity results for $m-$quasi-Einstein manifolds

In this section, we first present our rigidity result when the potential vector field is concircular.

Theorem 1. *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be a complete m-quasi-Einstein manifold with the concircular vector field ∇f so that $\nabla_{\nabla f}\nabla f = \mu\nabla f$. Then M^n is an Einstein manifold, if one of the following conditions holds.*

(i) The function μ is equal to the function $|\nabla f|^2$.
(ii) $\langle\nabla\mu, \nabla f\rangle \neq 0$.
(iii) $\langle\nabla R, \nabla f\rangle = 0$.

Proof: Since ∇f is a concircular vector field and $\mu = |\nabla f|^2$, we have

$$\nabla^2 f(\nabla f, \nabla f) = |\nabla f|^2 g(\nabla f, \nabla f)$$

and

$$(df \otimes df)(\nabla f, \nabla f) = |\nabla f|^2 g(\nabla f, \nabla f).$$

Plugging the above equations into equation (1), we obtain

$$Ric = \Big(\lambda + \Big(\frac{m+1}{m}\Big)\mu\Big)g.$$

Thus, M^n is an Einstein manifold, which gives the first assertion.

For (ii), we begin with using the equation $div\nabla^2 f = div(\mu g) = \nabla\mu$. Hence, we might use equation (7) to write

$$Ric\nabla f = (1-n)\nabla\mu. \tag{16}$$

We now combine $Ric\nabla f = \lambda\nabla f + \frac{1}{m}|\nabla f|^2\nabla f - \mu\nabla f$ with equation (16) to write

$$(1-n)\nabla\mu = \Big(\lambda + \frac{1}{m}|\nabla f|^2 - \mu\Big)\nabla f. \tag{17}$$

From equations (16) and (17), we obtain

$$(1-n)\langle\nabla\mu, \nabla f\rangle = Ric(\nabla f, \nabla f)$$

and

$$(1-n)\langle\nabla\mu, \nabla f\rangle = \Big(\lambda + \frac{1}{m}|\nabla f|^2 - \mu\Big)|\nabla f|^2,$$

respectively. Hence, if $\langle\nabla\mu, \nabla f\rangle \neq 0$, then we have

$$Ric = \Big(\lambda + \frac{1}{m}|\nabla f|^2 - \mu\Big)g,$$

which shows that M^n is an Einstein manifold.
For (iii), using equation (10) in Lemma 2, we have

$$\langle\nabla R, \nabla f\rangle = \frac{2(n+1)}{m}\mu|\nabla f|^2 - 2\langle\nabla\mu, \nabla f\rangle.$$

Since $\langle\nabla R, \nabla f\rangle = 0$, then we obtain

$$\nabla\mu = \frac{n+1}{m}\mu\nabla f.$$

From equation (16), we have

$$Ric(\nabla f, \nabla f) = (1-n)\langle\frac{n+1}{m}\mu\nabla f, \nabla f\rangle.$$

Also we get

$$Ric = \frac{(1-n)(n+1)}{m}\mu g.$$

Thus, M^n is an Einstein manifold. □

Proposition 1. *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be an m- quasi- Einstein manifold with the potential vector field ∇f is concurrent. Then the scalar curvature $R = n(\lambda - 1) + \frac{n}{m}|\nabla f|^2$ is constant.*

Proof: Since the potential vector field ∇f is concurrent, we arrive $\nabla^2 f = g$. This shows that

$$Ric = (\lambda - 1)g + \frac{1}{m}df \otimes df$$

and

$$Ric(\nabla f, \nabla f) = (\lambda - 1)g(\nabla f, \nabla f) + \frac{1}{m}|\nabla f|^2 g(\nabla f, \nabla f)$$

which gives

$$Ric = \Big(\lambda - 1 + \frac{1}{m}|\nabla f|^2\Big)g.$$

Since $n \geq 3$, we have from Schur's Lemma that $R = n(\lambda - 1) + \frac{n}{m}|\nabla f|^2$ is constant. □

Remark 2. *If M^n is compact and ∇f is non-trivial concircular vector field, then we use Stokes' formula to deduce that*

$$0 = \int_M div\nabla f = n\mu vol(M),$$

which shows that $\mu = 0$. Consequently, ∇f is a Killing vector field, i.e. $L_{\nabla f}g = 0$. From here, $\frac{1}{2}\langle\nabla|\nabla f|^2, \nabla f\rangle = 0$ implies that either $\nabla f = 0$ or $|\nabla f|^2$ is a constant.

Remark 3. *Let $(M^n, g, \nabla f, \lambda)$ be a non-trivial m-quasi-Einstein manifold with $n \geq 3$. If $|\nabla f|^2$ is constant, then $R = n\lambda + \frac{1}{m}|\nabla f|^2$ is constant. Hence, M^n is an Einstein manifold. Conversely, the constant scalar curvature does not require $|\nabla f|^2$ to be constant. In this spirit, we give the following proposition.*

Proposition 2. *Let $(M^n, g, \nabla f, \lambda)$ $(n \geq 3)$ be a non-trivial m-quasi-Einstein manifold with the constant scalar curvature. If $Ric(\nabla f)R = 0$, then the potential function ∇f is concircular.*

Proof: We see that $Ric(\nabla f)R = 0$ implies that $Ric(\nabla f, \nabla R) = 0$. From equation (1), we also have

$$Ric(\nabla f, \nabla R) + \nabla^2 f(\nabla f, \nabla R) - \frac{1}{m} df \otimes df(\nabla f, \nabla R) = \lambda g(\nabla f, \nabla R).$$

Since $Ric(\nabla f, \nabla R) = 0$ in the above equation, by using the $(1, 1)$-tensorial notation, we get

$$\nabla_{\nabla f} \nabla f - \frac{1}{m} |\nabla f|^2 \nabla f = \lambda \nabla f.$$

Thus we have $\mu = \lambda + \frac{1}{m} |\nabla f|^2$ is constant. $\square$

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Gould Integral on Signed Measurable Spaces for Functions Valued in Quasi-Normed Spaces

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Abstract: Gould defined his integral for real functions with respect to a finitely additive vector measure taking values in a Banach spaces in 1965. Since this year, different generalizations and topics on Gould integrability are introduced and studied.
In this paper, the initial purpose is to extend the concept of Gould's integral to quasi-normed spaces valued functions with respect to a signed measurable space. We will discuss also about the class of Gould m-integrable functions and some properties of this, emphasizing some properties of Gould integral on $\sigma-$ additive, signed measurable spaces and a generalization of Lebesgue type theorem in case of on finite, regular, $\sigma-$ additive, signed measurable space.

Keywords: Gould integral, Quasi-normed spaces, Signed measurable spaces, Totally measurable functions.

1 Introduction

Gould introduced an integral for real functions with respect to finite additive measures taking values in a Banach space and studied several properties of this kind of integral in his paper titled "On integration of vector - valued measures" (Proc.London Math.Soc 15, 193-225 (1965)). Gould's ideas on this integral have attracted the attention of other mathematicians, and since then many research articles have been done on the Gould integral. An interesting research of recent years is [1], where authors have presented some results of Gould integrability on finitely purely atomic measure spaces, such as a Lebesgue type theorem of convergence and comparative results among Gould integrability, Choquet integrability and total measurability.
In this paper, the initial purpose is to extend the concept of Gould's integral to quasi-normed spaces valued functions with respect to a signed measurable space. The structure of the paper is as follows: In section 2 we give some preliminaries, among which also an example on the representation of the Gould integral for simple functions.
One of the generalizations of this paper is the integration according to a signed measure (defined in [4] definition 10.1). Section 3 contains some results on regular, $\sigma-$ additive, signed measures and another Lebesgue type theorem but in case of quasi-normed valued functions and finite regular, $\sigma-$ additive, signed measurable spaces.
It is noticed that, we can define the concept of Bochner integral ([2] Definition 22.2.1) also for quasi-Banach space (define in [5], definition 1.2) valued functions, and we can see easily that some properties of Bochner m-integrable functions class that are proved for normed space valued functions holds. Section 4 contains some results of Gould integral on $\sigma-$ additive, signed measurable spaces such as: a limit theorem in case of quasi-Banach valued functions and a comparative result among Gould integrability and Bochner integrability. The last section contains the conclusions.

2 Preliminaries

Let T be a nonempty set, $\mathcal{P}(T)$ the family of all subsets of T and Σ an σ - algebra of subsets of T. A partition of T is a finite family $P=(A_i)_{i=1}^n$ in Σ such that $A_i \cap A_j$=$\emptyset$, i $\neq$j and $\cup_{i=1}^n A_i$ = T.

Definition 1 ([1] Definition 3.8). *] (i) If $P=(A_i)_{i=1}^n$, $P'=(A_j)_{j=1}^m$ are two partitions of T, then P' is said to be finer than P (denoted by $P \leq P'$ or $P' \geq P$) if for every $j \in \{1,2,...,m\}$ there exists $i_j \in \{1,2,...,n\}$ so that $A_j \subseteq A_{i_j}$.*
(ii) The common refinement of two partitions $P=(A_i)_{i=1}^n$ and $P'=(A_j)_{j=1}^m$ is called the partition $P \wedge P' = (A_i \cap A_j)_{i,j=1}^{n,m}$.

Definition 2 ([1] Definition 2.1). *The set function $m:\Sigma \to [0,+\infty]$ with $m(\emptyset)=0$ is said to be:*
(i) monotone measure if $m(A) \leq m(B)$ for every $A, B \in \Sigma$ with $A \subseteq B$.
(ii) subadditive measure if $m(A \cup B) \leq m(A)+m(B)$ for every $A, B \in \Sigma$.
(iii) σ - subadditive measure if $m(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m(A_n)$, for every $(A_n)_{n \in N} \subset \Sigma$, so that $\cup_{n=1}^{\infty} A_n \in \Sigma$.
(iv) finitely additive measure if $m(A \cup B) = m(A)+m(B)$, for every $A, B \in \Sigma$, with $A \cap B = \emptyset$.
(v) σ- additive measure if $m(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$, for every $(A_n)_{n \in N} \subset \Sigma$, so that $\cup_{n=1}^{\infty} A_n \in \Sigma$ and $A_i \cap A_j = \emptyset$ for $i \neq j$; $i, j \in \mathbb{N}$.

(vi) null-additive measure if $m(A \cup B) = m(A)$, for every $A, B \in \Sigma$ with $m(B) = 0$.
(vii) σ - null - additive measure if $m(\cup_{n \in N} A_n) = 0$ as soon as $A_n \in \Sigma$ and $m(A_n) = 0$ for all $n \in N$.

Definition 3 ([4] Definition 10.1)**.** *The set function m with $m(\emptyset) = 0$ is said to be a signed measure if*
1. For every $A \in \Sigma$, $m(A) \neq -\infty$ or for every $A \in \Sigma$, $m(A) \neq +\infty$.
2.(σ - additivity property) For every sequence of sets $(A_n)_{n \in N}$ in Σ such that, $A_{n_1} \cap A_{n_2} = \emptyset$ if $n_1 \neq n_2$, $m(\cup_{n \in N} A_n) = \sum_{n \in N} m(A_n)$.

Definition 4 ([1] Definition 3.1)**.** *Let $m : \Sigma \to [0, +\infty]$ be an arbitrary set of functions, with $m(\emptyset) = 0$.*
(i) A set $A \in \Sigma$ is said to be an atom of m if $m(A) > 0$ and for every $B \in \Sigma$, with $B \subset A$, we have $m(B) = 0$ or $m(A \setminus B) = 0$.
(ii) m is said to be finitely purely atomic (and T a finitely purely atomic space) if there is a finite disjoint family $(A_i)_{i=1}^n \subset \Sigma$ of atoms of m so that $T = \cup_{i=1}^n A_i$.

Definition 5 ([5] Definition 1.1)**.** *Let X be a vector space. A function $\| \, . \, \|: X \to [0, +\infty)$ is said to be quasi-norm on X if the following conditions hold:*
(i) For every $x \in X$, $\| x \| \geq 0$ and $\| x \| = 0 \Leftrightarrow x = 0$.
(ii)For every $x \in X$ and for every $\lambda \in R$, $\| \lambda x \| = |\lambda| \, \| x \|$.
(iii) For every $x, y \in X$, $\| x + y \| \leq K(\| x \| + \| y \|)$ where $K \geq 1$ is a constant independent of variables x and y.

The smallest of constant K, such that the above conditions hold, is called the modulus of concavity of quasi-norm $\| \, . \, \|$.

If the vector space X is equipment with a quasi - norm $\| \, . \, \|$, then $(X, \| \, . \, \|)$ is called quasi - normed space. A quasi-normed space X is called a quasi-Banach space if it is complete, which means that a sequence $(x_n)_{n \in \mathbb{N}}$ in X is convergent if and only if $\| x_n - x_m \| \to 0$ as $m, n \to +\infty$.

Let $m : \Sigma \to [-\infty, +\infty]$ be an arbitrary set functions, with $m(\emptyset) = 0$. In the same way to Definition 3.9 to [1], we can give the following definition.

Definition 6. *A vector function $f : T \to X$ is said to be:*
(i) m-totally-measurable (on T) if for every $\varepsilon > 0$, there exist a partition of T, $(A_i)_{i=0}^n \subset \Sigma$, with $\{A_1, A_2, ..., A_n\} \subset \Sigma \setminus \{\emptyset\}$, such that the following two conditions hold:
(1) $|m|(A_0) = sup\{\sum_{j=1}^m |m(A_j)|\} < \varepsilon$; where $(A_j)_{j=1}^m$ is a partition of A_0 and superior is extended over all finite partitions of set A_0.
$|m|$ is called absolute variation of m.
(2) $sup_{t,s \in A_i} \| f(t) - f(s) \| < \varepsilon$, for all $i \in \{1, 2, ..., n\}$.
(ii) The vector function f is called m-totally-measurable on $B \in \Sigma$ if the restriction $f|_B$ is m-totally-measurable on (B, Σ_B, m_B), where $\Sigma_B = \{A \cap B : A \in \Sigma\}$ and $m_B = m|_{\Sigma_B}$.

Recall that
([4] Definition 10.3) (i) A set $P \in \Sigma$ is called a positive set, according to signed measure m, if for every $P' \in \Sigma$ such that $P' \subseteq P$, $m(P') \geq 0$.
A set $N \in \Sigma$ is called a negative set, according to signed measure m, if for every $N' \in \Sigma$ such that $N' \subseteq N$, $m(N') \leq 0$.
A set $Q \in \Sigma$ is called a null set, according to signed measure m, if for every $Q' \in \Sigma$ such that $Q' \subseteq Q$, $m(Q') = 0$.
([4], definition 10.4, theorem 10.3) (ii) from Hanh decomposition of signed measure m, we can write $m = m^+ - m^-$ where $\forall A \in \Sigma$, $m^+(A) = m(A \cap P)$, $m^-(A) = -m(A \cap N)$, P and N are respectively positive and negative set of m and $P \cup N = T$, $P \cap N = \emptyset$.
([4], definition 10.6, theorem 10.5) (iii) $m^+, m^-, |m| = m^+ + m^-$ are $\sigma-$ additive, monotone , non negative measures on T.

In what follows, $m : \Sigma \to [-\infty, +\infty]$ is a signed measure with $m(T) \neq 0$.
For an arbitrary vector function $f : T \to X$, where $(X, \| \, . \, \|)$ is a quasi-normed space, $\sigma(P)$ denotes the sum

$$\sum_{i=1}^n f(t_i) m(A_i) = \sum_{i=1}^n f(t_i) m^+(A_i) - \sum_{i=1}^n f(t_i) m^-(A_i) = \sigma^+(P) - \sigma^-(P)$$

where $\sigma^+(P) = \sum_{i=1}^n f(t_i) m^+(A_i)$ and $\sigma^-(P) = \sum_{i=1}^n f(t_i) m^-(A_i)$.
The sums $\sigma(P), \sigma^+(P), \sigma^-(P)$ are written for every partition $P = (A_i)_{i=1}^n$ of T and for every $t_i \in A_i, i \in \{1, 2, ..., n\}$.

In the same way with the definition of Gould integrable function in case of positive measure ([1] definition 4.1) we can define this concept also in case of signed measure.

Definition 7. *(a) A vector function $f : T \to X$ is said to be Gould m^+- integrable (m^-− integrable) on T, if the net $(\sigma^+(P))_{P \in (\mathcal{P}, \leq)}$, where $\mathcal{P}$ is denoted the collection of all partition of T ordered by the relation $\leq$ in definition 1, is convergent in X ($(\sigma^-(P))_{P \in (\mathcal{P}, \leq)}$ is convergent in X).*
Usually the Gould integral denoted (G) $\int_T f dm$, where m is a positive measure on T and (G) $\int_T f dm = lim(\sigma(P))_{P \in (\mathcal{P}, \leq)}$.
(b) A vector function $f : T \to X$ is said to be Gould $m-$ integrable on T if it is both Gould m^+- integrable and Gould m^-− integrable on T, and $(G) \int_T f dm = (G) \int_T f dm^+ - (G) \int_T f dm^-$.
(c) For every $B \in \Sigma$ the vector function $f : T \to X$ is said to be Gould $m-$ integrable on B if restriction $f_{|_B}$ of f to B is Gould $m-$ integrable on (B, Σ_B, m_B).

Remark 1. *If a vector function $f : T \to X$ is Gould $m-$ integrable, then f is also Gould $|m|-$ integrable.*
The proof is immediately from the equality $|m| = m^+ + m^-$; the conditions of quasi-norm and definition 7 (a), (b).

Remark 2. *By definition 7, the equality $\sigma(P) = \sigma^+(P) - \sigma^-(P)$ and Remark 4.2 in [1], the following statements hold:*
(a) The vector function f is Gould $m-$ integrable on T if and only if there exist $\alpha \in X$ such that for every $\varepsilon > 0$, there exist a partition P_ε of T, so that for every other partition of T, $P = (A_i)_{i=1}^n$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i \in \{1, 2, ..., n\}$, we have

$\parallel \sigma(P) - \alpha \parallel < \varepsilon$.
(b) Let $B, C \in \Sigma$ satisfy $B \cap C = \emptyset$. If the vector function $f : T \to X$ is Gould $m-$ integrable on B and C, then the function f is Gould $m-$ integrable on $B \cup C$ and $\int_{B \cup C} f dm = \int_B f dm + \int_C f dm$.

Proof: (a) Is immediately from definition 7 and Remark 4.2 in [1].

(b) From definition 7 (b), we can write:

$$(G) \int_{B \cup C} f dm = (G) \int_{B \cup C} f dm^+ - (G) \int_{B \cup C} f dm^-.$$

from Remark 4.2 in [1], we can write:

$$(G) \int_{B \cup C} f dm^+ = (G) \int_B f dm^+ + (G) \int_C f dm^+$$

and

$$(G) \int_{B \cup C} f dm^- = (G) \int_B f dm^- + (G) \int_C f dm^-$$

Thus,

$$(G) \int_{B \cup C} f dm = (G) \int_B f dm^+ + (G) \int_C f dm^+ - ((G) \int_B f dm^- + (G) \int_C f dm^-) =$$
$$((G) \int_B f dm^+ - (G) \int_B f dm^-) + ((G) \int_C f dm^+ - (G) \int_C f dm^-) = (G) \int_B f dm + (G) \int_C f dm.$$

□

Example 1. *Let (T, Σ, m) be a finite measurable space and $(X, \| \, . \, \|)$ be a quasi-norm space. If $\varphi : T \to X$ is a simple function, $\varphi(t) = \sum_{i=1}^n a_i \chi_{A_i}$, then the function f is Gould $m-$ integrable on T and $(G) \int_T \varphi dm = \sum_{i=1}^n a_i m(A_i)$.*

Proof: First, let prove the above proposition in m^+ (m^-) case.
From definition of simple function, $\varphi(t) = \sum_{i=1}^n a_i \chi_{A_i}$, where $a_i \in X$ and $A_i \in \Sigma$ for every $i \in \{1, 2, ..., n\}$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^n A_i = T$. So, the collection of sets $P = (A_i)_{i=1}^n$ is a partition of T.
If $P' = \{B_j\}_{j=1}^l$ is another partition of T, then $\sigma^+(P') = \sum_{j=1}^l \varphi(t_j) m^+(B_j)$, where $t_j \in B_j, j \in \{1, 2, ..., l\}$.
On the other hand, from finitely additivity property, we can write:

$$m^+(B_j) = \sum_{i=1}^n m^+(B_j \cap A_i), \text{for every } j \in \{1, 2, .., l\}.$$

So, the following equalities hold:

$$\sigma^+(P') = \sum_{j=1}^l \varphi(t_j) (\sum_{i=1}^n m^+(B_j \cap A_i)) = \sum_{j=1}^l (\sum_{i=1}^n \varphi(t_j) m^+(B_j \cap A_i)) = \sum_{i=1}^n (\sum_{j=1}^l a_i m^+(B_j \cap A_i)) =$$
$$\sum_{i=1}^n a_i (\sum_{j=1}^l m^+(B_j \cap A_i)) = \sum_{i=1}^n a_i m^+(A_i).$$

Therefore, $\sigma^+(P)_{P \in (\mathcal{P}, \leq)}$ converges to $\sum_{i=1}^n a_i m^+(A_i)$.
In the same way, we prove that $\sigma^-(P)_{P \in (\mathcal{P}, \leq)}$ converges to $\sum_{i=1}^n a_i m^-(A_i)$, and from definition 7 (b) we conclude that $(G) \int_T f dm = \sum_{i=1}^n a_i m^+(A_i) - \sum_{i=1}^n a_i m^-(A_i) = \sum_{i=1}^n a_i m(A_i)$. □

3 Lebesgue type's Theorem on regular $\sigma-$ additive signed measurable spaces

Let $\mathcal{B}$ be the Borel's σ- algebra in real line $\mathbb{R}$ and the set function $m : \mathcal{B} \to \mathbb{R}$.

Definition 8 ([1] Definition 3.5)**.** *The set function $m : \mathcal{B} \to [0, +\infty)$ is called regular if for each set $A \in \mathcal{B}$ and each $\varepsilon > 0$, there exist $K \in \mathcal{K}$ and $D \in \tau$ such that $K \subseteq A \subseteq D$ and $m(D \setminus K) < \varepsilon$. (Notice that, $\mathcal{K}$ is denoted the net of compact sets and τ is denoted the class of open sets).*

So, we can give a definition for regular signed measure.

Definition 9. *The signed measure $m : \mathcal{B} \to (-\infty, +\infty)$ is called regular if set functions m^+ and m^- are both regulars.*

Remark 3. *The signed measure $m : \mathcal{B} \to (-\infty, +\infty)$ is regular if and only if the measure $|m|$ is regular.*

Proof: Suppose that m is a regular signed measure. So, the measures m^+ and m^- are both regulars. Thus, for every $\frac{\varepsilon}{2} > 0$ and for every $A \in \mathcal{B}$ exists $K_1, K_2 \in \mathcal{K}$, $D_1, D_2 \in \tau$ such that

$$K_1 \subset A \subset D_1 \text{ and } m^+(D_1 \setminus K_1) < \tfrac{\varepsilon}{2}$$
$$K_2 \subset A \subset D_2 \text{ and } m^-(D_2 \setminus K_2) < \tfrac{\varepsilon}{2}$$

Take $K = K_1 \cup K_2 \in \mathcal{K}$ and $D = D_1 \cap D_2 \in \tau$. We can see that:

$$K \subset A \subset D \text{ and } D \setminus K = (D_1 \cap D_2) \setminus (K_1 \cup K_2) = (D_1 \cap D_2) \cap (K_1^c \cap K_2^c) = (D_1 \setminus K_1) \cap (D_2 \setminus K_2).$$

Thus, from above inequality, we conclude that:

$$|m|(D \setminus K) = |m|((D_1 \setminus K_1) \cap (D_2 \setminus K_2)) = m^+((D_1 \setminus K_1) \cap (D_2 \setminus K_2)) + m^-((D_1 \setminus K_1) \cap (D_2 \setminus K_2))$$
$$\leq m^+(D_1 \setminus K_1) + m^-(D_2 \setminus K_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ (because } m^+ \text{ and } m^- \text{ are monotone measures).}$$

So, $|m|$ is a regular measure. Conversely, suppose that $|m|$ is a regular measure. Let proof that $m : \mathcal{B} \to (-\infty, +\infty)$ is a regular, signed measure.
From definition 8 we can see easily that, for every $\varepsilon > 0$ exists $K \in \mathcal{K}$, $D \in \tau$ such that $K \subseteq A \subseteq D$, $|m|(D \setminus K) < \varepsilon$ and $m^+(D \setminus K) \leq |m|(D \setminus K) < \varepsilon$, $m^-(D \setminus K) \leq |m|(D \setminus K) < \varepsilon$. So, m^+ and m^- are both regular measures. From definition 9, we conclude that m is a regular signed measure. □

From Theorem 4.6 in [3], we have:
Let $m : \mathcal{B} \to [0, +\infty)$ be a regular null-additive monotone set function. If $A \in \mathcal{B}$ is an atom of m, then exist a single point $a \in A$ such that $m(A) = m\{a\}$ and $m(A \setminus \{a\}) = 0$.

So, we can formulate this Theorem:

Theorem 1. *Let $m : \mathcal{B} \to (-\infty, +\infty)$ be a $\sigma-$ additive regular signed measure. If $A \in \mathcal{B}$ is an atom of $|m|$, then exist a single point $a \in A$ such that $m(A \setminus \{a\}) = 0$.*

Proof: The proof is immediately from Theorem 4.6 in [3], Remark 3 and the fact that if a measure is $\sigma-$ additive then it is null-additive. So, exist an only point $a \in A$ such that $|m|(A \setminus \{a\}) = 0 \Leftrightarrow m^+(A \setminus \{a\}) = 0$ and $m^-(A \setminus \{a\}) = 0$.
Thus, $m(A \setminus \{a\}) = m^+(A \setminus \{a\}) - m^-(A \setminus \{a\}) = 0$. □

From the above Theorem and Theorem 4.4 of [1], we can formulate this Theorem:

Theorem 2. *Let $m : \mathcal{B} \to (-\infty, +\infty)$ be a $\sigma-$ additive regular signed measure and $f : T \to X$ be a whatever vector function. If $A \in \mathcal{B}$ is an atom of $|m|$, then f is m-integrable on A and $\int_A f dm = f(a)m(A)$, where $a \in A$ is the single point resulting by Theorem 1.*

The proof is similar with Theorem 4.4 in [1], replacing m with $|m|$ and using equality $|m| = m^+ + m^-$.

Remark 4. *Let $\mathcal{A}$ be a whatever $\sigma-$ algebra on T. If $A \in \mathcal{A}$ is an atom of $|m|$, then A is an atom of m^+ or m^-.*

Proof: Since $A \in \mathcal{A}$ is an atom of $|m|$, we have: $|m|(A) > 0$ and $\forall B \subset A, |m|(B) = 0$ or $|m|(A \setminus B) = 0$.
It is clear that $|m|(A) > 0 \Leftrightarrow m^+(A) + m^-(A) > 0 \Leftrightarrow m^+(A) > 0$ or $m^-(A) > 0$.
So, we can write:
If $|m|(B) = 0$, then $m^+(B) = 0$ and $m^-(B) = 0$. (If $|m|(A \setminus B) = 0$, then $m^+(A \setminus B) = 0$ and $m^-(A \setminus B) = 0$).
If $m^+(A) > 0$ and $m^+(B) = 0$ or $m^+(A \setminus B) = 0$, then A is an atom of m^+. (If $m^-(A) > 0$ and $m^-(B) = 0$ or $m^-(A \setminus B) = 0$, then A is an atom of m^-.) □

Corollary 1. *If $A \in \mathcal{A}$ is an atom of $|m|$ and both $m^+(A) > 0$, $m^-(A) > 0$, then A is an atom of m^+ and m^-.*

Theorem 3. *(Lebesgue type)*
Let $f, f_n : \mathbb{R} \to X$ be vector functions on finite, regular, $\sigma-$ additive, signed measurable space $(\mathbb{R}, m)$ and $A \in \mathcal{B}$ an atom of $|m|$. If $\lim_{n\to+\infty} f_n(a) = f(a)$, where $a \in A$ is the single point like as Theorem 1, then $\lim_{n\to+\infty}(G)\int_A f_n dm = (G)\int_A f dm$.

Proof: We have that the set $A \in \mathcal{A}$ is an atom of $|m|$. So, the set A is an atom m^+ or m^- (or both m^+ and m^- when both $m^+(A) > 0$ and $m^-(A) > 0$).
For every natural number n, $(G)\int_A f_n dm = (G)\int_A f_n dm^+ - (G)\int_A f_n dm^-$.
If $m^+(A) = 0$ or $m^-(A) = 0$ (that corresponds with the case when the set A is an atom m^+ or m^-), then we can see easily that $(G)\int_A f_n dm^+ = (G)\int_A f dm^+ = 0$ or $(G)\int_A f_n dm^- = (G)\int_A f dm^- = 0$.
Thus, for every natural number n,

$$(G)\int_A f_n dm = -(G)\int_A f_n dm^- \text{ and } (G)\int_A f dm = -(G)\int_A f dm^-$$
$$\text{(or } (G)\int_A f_n dm = (G)\int_A f_n dm^+ \text{ and } (G)\int_A f dm = (G)\int_A f dm^+\text{).}$$

For every $n \geq n_0$ and $m^+(A) > 0$, we can see that:

$$\| (G)\int_A f_n dm^+ - (G)\int_A f dm^+ \| = \| f_n(a)m^+(A) - f(a)m^+(A) \| = m^+(A) \| f_n(a) - f(a) \| < \varepsilon$$

(because $\lim_{n\to+\infty} f_n(a) = f(a)$).
So, we conclude that:
If $m^+(A) > 0$ and $m^-(A) = 0$, then $\lim_{n\to+\infty}(G)\int_A f_n dm = (G)\int_A f dm \Leftrightarrow \lim_{n\to+\infty}(G)\int_A f_n dm^+ = (G)\int_A f dm^+$.

Thus, the above equalities, imply that $\lim_{n\to+\infty}(G)\int_A f_n dm = (G)\int_A f dm$.
If $m^-(A)>0$ and $m^+(A)=0$, then we can use the same way.
If A is an atom both for m^+ and m^- (so, $m^+(A)>0$ and $m^-(A)>0$), then

$$\| (G)\int_A f_n dm - (G)\int_A f dm \| = \| (G)\int_A f_n dm^+ - (G)\int_A f_n dm^- - ((G)\int_A f dm^+ - (G)\int_A f dm^-) \| \\ \le K \| (G)\int_A f_n dm^+ - (G)\int_A f dm^+ \| + K \| (G)\int_A f_n dm^- - (G)\int_A f dm^- \| = \\ K \| f_n(a)m^+(A) - f(a)m^+(A) \| + K \| f_n(a)m^-(A) - f(a)m^-(A) \| \text{ (1).}$$

Since $\lim_{n\to+\infty} f_n(a) = f(a)$, for every $\frac{\varepsilon}{2K|m|(A)} > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$ we have $\| f_n(a) - f(a) \| < \frac{\varepsilon}{2K|m|(A)}$.
This implies $\| f_n(a)m^+(A) - f(a)m^+(A) \| < \frac{\varepsilon}{2K}$ and $\| f_n(a)m^-(A) - f(a)m^-(A) \| < \frac{\varepsilon}{2K}$.
Finally, from (1) we conclude that $\lim_{n\to+\infty}(G)\int_A f_n dm = (G)\int_A f dm$. □

Remark 5. *If $A \in \mathcal{B}$ is an atom of $|m|$ and $m(A) \neq 0$, then in conditions of Theorem 2, we can say:*
If $\lim_{n\to+\infty}(G)\int_A f_n dm = (G)\int_A f dm$, then $\lim_{n\to+\infty} f_n(a) = f(a)$ (So, it is true the inverse of Lebesgue type's Theorem).

The proof is immediately from equality $\| f_n(a)m(A) - f(a)m(A) \| = |m(A)| \, \| f_n(a) - f(a) \|$ and the fact that $|m(A)| > 0$.

4 Some properties of Gould Integral on $\sigma-$ additive signed measurable spaces

Let (T,Σ,m) be a $\sigma-$ additive, signed measurable space and X a quasi-normed space.
We note that the following proposition hold.

Proposition 1. *Every Gould $m-$ integrable function on T is m - totally - measurable function on T.*

Proof: Since the vector function f is Gould $m-$ integrable function on T, then exists $x \in X$ such that for every $\varepsilon > 0$, there exist a partition P_ε of T, so that for every other partition of T, $P = (A_i)_{i=1}^n$, with $P \ge P_\varepsilon$ and every choice of points $t_i \in A_i, i \in \{1,2,...,n\}$, we have $\| \sigma(P) - x \| = \| \sum_{i=1}^n f(t_i)m(A_i) - x \| < \varepsilon$.
For every $A \subset T$, the collection $P = (A \cap A_i)_{i=1}^n$ is a partition of A and $\| \sum_{i=1}^n f(t_i)m(A\cap A_i) - x + \sum_{i=1}^n f(t_i)m(A \setminus A_i) \| =$ $\| \sum_{i=1}^n f(t_i)m(A_i) - x \| < \varepsilon$. Thus, $\| \sum_{i=1}^n f(t_i)m(A\cap A_i) - x' \| < \varepsilon$, where $x' = x - \sum_{i=1}^n f(t_i)m(A \setminus A_i) \in X$.
The last inequality holds for every partition $P' \ge P$ of A, that is equivalente with f is Gould $m-$ integrable function on A.
For every $t, s \in A_i$ and for every $P' \ge P$ of A, we can write:

$$\| \sum_{i=1}^n (f(t) - f(s))m(A_i) \| \le K \| \sum_{i=1}^n f(t)m(A_i) - x \| + K \| \sum_{i=1}^n f(s)m(A_i) - x \| \le 2K\varepsilon = \varepsilon' \text{ (2).}$$

On the other hand, $\| (f(t)-f(s))m(A_{i_0}) \| \le K \| \sum_{i=1}^n (f(t)-f(s))m(A_i) \| + K \| \sum_{i=1, i\neq i_0}^n (f(t)-f(s))m(A_i) \|$.
Since f is Gould $m-$ integrable on $T \setminus A_{i_0}$, then from (2) we have $\| \sum_{i=1,i\neq i_0}^n (f(t)-f(s))m(A_i) \| < \varepsilon'$.
So, $\| (f(t)-f(s))m(A_{i_0}) \| \le 2K\varepsilon' = \varepsilon"$ for every $i_0 \in \{1,2,...,n\}$.
This implies that, for A_{i_0} with $m(A_{i_0}) \neq 0$ and $\forall t, s \in A_{i_0}$ we have $\| (f(t)-f(s))m(A_{i_0}) \| \le \varepsilon"$.
Thus, $sup_{t,s\in A_i} \| (f(t)-f(s)) \| < \frac{\varepsilon"}{|m(A_i)|} = \varepsilon^\star, \forall i \in \{1,2,...,n\}$ and $m(A_i) \neq 0$.
Suppose we have ordered the set in this way: first, sets with measure zero and then others. Let denote $B_0 = \cup_{i=1}^{n_0} A_i$ with $m(A_i) = 0, \forall i \in \{1,2,...,n_0\}$. So, $m(B_0) = 0$ that imply $|m|(B_0) = 0 < \varepsilon$ (If every set in partition P is not with measure zero, then denote $B_0 = \emptyset$). For every $i \in \{n_0+1,...,n\}$, denote $B_{i-n_0} = A_i$.
Thus, we built a partition $P_1 = (B_j)_{j=0}^{n-n_0}$ of T such that

$$|m|(B_0) = 0 \text{ and } sup_{t,s\in B_j} \| (f(t)-f(s)) \| < \frac{\varepsilon"}{m(B_j)} = \varepsilon^\star, \forall j \in \{1,2,...,n-n_0\}.$$

This complete the proof. □

We can prove easily from definition of Gould Integral, that hold the following properties.

Proposition 2. *Let $f, g : T \to X$ be Gould $m-$ integrable functions on T and c a real constant.*
a) The set function cf is Gould $m-$ integrable and $(G)\int_T cf dm = c\int_T f dm$.
b) The set functions f + g and f-g are Gould $m-$ integrable functions on T and $(G)\int_T (f \pm g) dm = (G)\int_T f dm \pm (G)\int_T g dm$.

An interesting result is presented in the following.

Theorem 4. *Let $(f_n : T \to X)_{n\in\mathbb{N}}$ be a Gould $m-$ integrable functions sequence, $(X, \| . \|)$ a quasi - Banach space and (T,Σ,m) a $\sigma-$ additive signed measurable space with $|m|(T) < +\infty$.*
If the functions sequence (f_n) converge uniformly to function $f : T \to X$ acoording to $|m|$, then the function f is also a Gould $m-$ integrable function and $(G)\int_T f dm = lim_{n\to+\infty}(G)\int_T f_n dm$.

Proof: It suffices to treat the case when the measure m is non-negative and to use definition 7 (b). For every $n \in \mathbb{N}$ denote $x_n = (G)\int_T f_n dm$.
From remark 2 (a) we have:
$\forall \varepsilon > 0$ exists a partition $P = (A_i^{(n)})_{i=1}^{l(n)}$ of T such that, for every partition $P' = (B_i^{(n)})_{i=1}^{l'(n)}$ of T with $P' \ge P$ the following inequality hold

$$\| \sum_{i=1}^{l'(n)} f_n(t_i)m(B_i^{(n)}) - x_n \| < \frac{\varepsilon}{K^2}\text{ , for every } t_i \in B_i^{(n)} \text{ (3).}$$

Since the functions sequence (f_n) converge uniformly to function $f : T \to X$ according to $|m|$, then for every $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ holds the inequality (4) $\| f_n(t) - f(t) \| < \frac{\varepsilon}{2K^{n+3}}$, for every $t \in T$, (the index n_0 depend on only from ε).
First, we conclude that the sequence x_n converge in X.
For every $p, n \in \mathbb{N}$, let $P' = \{B_i^{(n)}\}_{i=1}^{l'(n)}$ be a partition of T such that inequality (3) holds for function f_n and $P" = \{B_i^{(p)}\}_{i=1}^{l'(p)}$ be a partition of T such that inequality (3) holds for function f_p. Denote $P^\star = \{C_i^{(p,n)}\}_{i=1}^{l(p,n)}$ the common refinement of two partitions P' and $P"$.
We can see that:

$$\| x_n - x_p \| \leq K \| \sum_{i=1}^{l(p,n)} f_n(t_i)m(C_i^{(p,n)}) - x_n \|$$
$$+K(K \| \sum_{i=1}^{l(p,n)} f_p(t_i)m(C_i^{(p,n)}) - x_p \| + K \| \sum_{i=1}^{l(p,n)} f_n(t_i)m(C_i^{(p,n)}) - \sum_{i=1}^{l(p,n)} f_p(t_i)m(C_i^{(p,n)}) \|)$$
$$= K \| \sum_{i=1}^{l(p,n)} f_n(t_i)m(C_i^{(p,n)}) - x_n \| + K^2 \| \sum_{i=1}^{l(p,n)} f_p(t_i)m(C_i^{(p,n)}) - x_p \| + K^2 \| \sum_{i=1}^{l(p,n)} (f_n(t_i) - f_p(t_i))m(C_i^{(p,n)}) \| \text{ (5)}$$

The partition $P^\star$ is finer than both P' and $P"$. So, the inequality (3) holds both for f_n and f_p functions.
Thus, we can write:

$$\| \sum_{i=1}^{l(p,n)} (f_n(t_i) - f_p(t_i)) \cdot m(C_i^{(p,n)}) \| = \| \sum_{i=1}^{l(p,n)} (f_n - f_p)(t_i) \cdot m(C_i^{(p,n)}) \| \leq K \cdot m(C_1^{(p,n)}) \cdot \| (f_n - f_p)(t_1) \|$$
$$+K \cdot \| \sum_{i=2}^{l(p,n)} (f_n - f_p)(t_i) \cdot m(C_i^{(p,n)}) \| \leq ... \leq K \cdot m(C_1^{(p,n)}) \cdot \| (f_n - f_p)(t_1) \| + K^2 \cdot m(C_2^{(p,n)}) \cdot \| (f_n - f_p)(t_2) \|$$
$$+K^3 \cdot m(C_3^{(p,n)}) \cdot \| (f_n - f_p)(t_3) \| ... + K^{l(p,n)} \cdot m(C_{l(p,n)}^{(p,n)}) \cdot \| (f_n - f_p)(t_{l(p,n)}) \| \text{ (6).}$$

From equality (4) for every $p > n > n_0$ we have:

$$\| (f_n - f_p)(t_i) \| \leq K(\| (f_n - f)(t_i) \| + \| (f_p - f)(t_i) \|) < K \cdot (\frac{\varepsilon}{2K^{n+3}} + \frac{\varepsilon}{2K^{p+3}}) < \frac{\varepsilon}{K^{n+2}}.$$

Thus,

$$\| \sum_{i=1}^{l(p,n)} (f_n - f_p)(t_i) \cdot m(C_i^{(p,n)}) \| \leq K \cdot m(C_1^{(p,n)}) \cdot \frac{\varepsilon}{K^3} + K^2 \cdot m(C_2^{(p,n)}) \cdot \frac{\varepsilon}{K^4} + ... + K^{l(p,n)} \cdot m(C_{l(p,n)}^{(p,n)}) \cdot \frac{\varepsilon}{K^{l(p,n)+2}} =$$
$$\frac{\varepsilon}{K^2} \cdot (m(C_1^{(p,n)}) + m(C_2^{(p,n)}) + m(C_{l(p,n)}^{(p,n)})) = \frac{\varepsilon}{K^2} \cdot m(T) \text{ (7).}$$

So, from inequalities (3) (5) and (7) we have:

$$\| x_n - x_p \| \leq K \cdot \frac{\varepsilon}{K^2} + K^2 \cdot \frac{\varepsilon}{K^2} + K^2 \cdot \frac{\varepsilon}{K^2} \cdot m(T) = \frac{\varepsilon}{K} + \varepsilon + \varepsilon \cdot m(T) \leq 2\varepsilon + m(T) \cdot \varepsilon = \varepsilon'.$$

So, we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence on X and since X is a Banach space, then the $(x_n)_{n\in\mathbb{N}}$ sequence converges to a point $x \in X$.
Let proof that the set function f is Gould m - integrable on T and $(G)\int_T f dm = x = lim_{n\to+\infty} x_n = lim_{n\to+\infty}(G)\int_T f_n dm.$
For every $n \geq n_0$ (n_0 is such that for $n \geq n_0$ the inequality (4) above holds), partition $P = \{A_i^n\}_{i=1}^{l(n)}$ of T (such that the inequality (3) holds) and $\forall t \in A_i^n$ and we have:

$$\| \sum_{i=1}^{l(n)} f(t_i)m(A_i^n) - x \| \leq$$

$$K \| \sum_{i=1}^{l} f(t_i)m(A_i^n) - \sum_{i=1}^{l(n)} f_n(t_i)m(A_i^n) \| + K^2 \| \sum_{i=1}^{l(n)} f_n(t_i)m(A_i^n) - x_n \| + K^2 \| x_n - x \| <$$

$$< K \cdot \frac{\varepsilon}{2K^{n+3}} \cdot m(T) + K^2 \cdot \frac{\varepsilon}{K^2} + K^2 \| x_n - x \| .$$

Since the $(x_n)_{n\in\mathbb{N}}$ sequence converge in a point $x \in X$, then for every $\frac{\varepsilon}{K^2}$, $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$ we have $\| x_n - x \| < \frac{\varepsilon}{K^2}$.
So, for every $n \geq max(n_0, n_1)$ we have

$$\| \sum_{i=1}^{l(n)} f(t_i)m(A_i^n) - x \| < K \cdot \frac{\varepsilon}{2K^{n+3}} \cdot m(T) + K^2 \cdot \frac{\varepsilon}{K^2} + K^2 \| x_n - x \| < \varepsilon + \varepsilon + \varepsilon = 2\varepsilon + m(T) \cdot \varepsilon = \varepsilon'.$$

Since, for every $n \in \mathbb{N}$ the set function f_n is Gould $m-$ integrable, then the inequality (3) holds for every partition finer than P. This finish the proof. □

From Egoroff's Theorem ([4] Theorem 9.4 i)):
If (T,m) is a finite measurable space (where m is a non-negative measure) and $(f_n : T \to \mathbb{R})_{n\in\mathbb{N}}$ is a real functions sequence that converges point wise to real function $f : T \to R$, then the sequence f_n converges uniformly to real function f.
The theorem's proof is not depending on the properties of absolute value in $\mathbb{R}$ so, exactly the same way as in this case, we can prove that:
If the vector functions sequence $(f_n : T \to X)_{n\in\mathbb{N}}$ converges point wise to vector function $f : T \to X$ and (T,m) is a finite measurable space (where m is a non-negative measure) then the vector functions sequence $(f_n : T \to X)_{n\in\mathbb{N}}$ converges uniformly to vector function f.

Recall that, if X is a Banach space and (T,m) is a finite measurable space (where m is a non-negative measure) ([2] Definition 22.2.1, Definition 22.2.3), then:

1. A function $f : T \to X$ m-measurable is Bochner $m-$ integrable if there exists a sequence of simple functions φ_n converging to function f point wise and satisfying $\int_T \parallel \varphi_n(t) - \varphi_m(t) \parallel dm \to 0$ as $m, n \to \infty$.
2. Every simple function $\varphi : T \to X$ is Bochner $m-$ integrable (Gould m-intgrable) and $(B)\int_T \varphi dm = (G)\int_T \varphi dm$.
If the function f is Bochner $m-$ integrable define $(B)\int_T f dm = lim_{n\to+\infty}(B)\int_T \varphi_n dm = lim_{n\to+\infty}(G)\int_T \varphi_n dm$.

The definitions 22.2.1 and 22.2.3 [2] we can write also for quasi-normed space valued functions, and we can see easily that some properties of Bochner $m-$ integrable functions class that are proved for normed space valued functions holds.

Corollary 2. *Let X is a quasi-Banach space and (T,m) is a finite, $\sigma-$ additive, singed measurable space. Every function $f : T \to X$ $m-$ measurable, Bochner $m-$ integrable is also Gould $m-$ integrable and $(G)\int_T f dm = (B)\int_T f dm$.*

Proof: According to definition of Bochner integrable function on T ; Theorem 4 and definition 22.2.3 [2], where m is a positive σ - additive measure, φ_n is a simple functions sequence that converges to f, $(X, \parallel . \parallel)$ is a quasi-normed space and definition 7 (b) we conclude that every Bochner integrable functions $f : T \to X$ is also Gould $m-$ integrable on T.
Now, from equality in Theorem 4, we conclude that $(G)\int_T f dm = (B)\int_T f dm$. □

5 Conclusions

1.The Lebesgue type Theorem on regular, null-additive, monotone, measurable spaces ([1] corollary 4.7) can be generalized even in the case of regular, $\sigma-$ additive, signed measurable spaces (theorem 3).
2.The collection of Gould $m-$ integrable vector functions on $\sigma-$ additive, signed measurable spaces is a vectorial subspace of $m-$ totally measurable vector functions. In case of quasi-Banach vector valued functions on finite, $\sigma-$ additive, singed measurable space, this collection contains the Bochner $m-$ integrable vector functions collection.

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On q-Bernardi Integral Operator and a Subclass of Harmonic Mappings

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Abstract: The main object of the study is to examine the closure properties of a subclass of harmonic functions $f = h + \overline{g}$ where h and g are analytic functions in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ under the $q-$Bernardi-Libera-Livingston integral operator denoted by $\mathbb{L}_{q,c}(f)$ $(c > -1, q \in (0,1))$ defined as $\mathbb{L}_{q,c}(f) := \mathbb{L}_{q,c}(h) + \overline{\mathbb{L}_{q,c}(g)}$ where

$$\mathbb{L}_{q,c}(h) = \frac{[c+1]_q}{z^c} \int_0^z \xi^{c-1} h(\xi) d_q\xi,$$

$$\mathbb{L}_{q,c}(g) = \frac{[c+1]_q}{z^c} \int_0^z \xi^{c-1} g(\xi) d_q\xi.$$

Keywords: Convolution, Harmonic univalent, q-integral operator.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m,$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$.
Let the complex-valued function $f(z) = u(x,y) + iv(x,y)$ be continuous in $\mathcal{U}$. Then, f is said to be harmonic in $\mathcal{U}$, if both $u(x,y)$ and $v(x,y)$ are real harmonic functions in $\mathcal{U}$. Suppose that, there exist functions K and L analytic in $\mathcal{U}$ such that $u = \mathrm{Re}K$ and $v = \mathrm{Im}L$. Hence,

$$\begin{aligned} f &= u + iv = \mathrm{Re}K + i\mathrm{Im}L \\ &= \frac{K+\overline{K}}{2} + i\frac{L-\overline{L}}{2i} \\ &= \frac{K+L}{2} + \overline{\frac{K-L}{2}} \\ &= h + \overline{g}. \end{aligned}$$

We call h the analytic part and g the co-analytic part of f. To this end, without loss of generality, we may write the harmonic function $f = h + \overline{g}$ in the form

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m \text{ and } g(z) = \sum_{m=1}^{\infty} b_m z^m. \tag{1}$$

Let $\mathcal{H}$ denote the class of functions $f = h + \overline{g}$ which are harmonic in the open unit disk $\mathcal{U}$. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{U}$ is that $\left|h'(z)\right| > \left|g'(z)\right|$. Let $\mathcal{SH}$ denote the class of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in $\mathcal{U}$ for which $h(0) = h'(0) - 1 = 0 = g(0)$. One shows easily that the sense-preserving property implies that $|b_1| < 1$. Clunie and Sheil-Small [6] investigated the class $\mathcal{SH}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathcal{SH}$ and its subclasses. A comprehensive study for the theory of harmonic univalent functions may be found in Duren [7].
Quantum calculus is the traditional calculus without the use of limits. In 1909 and 1910, Jackson initiated in-depth study of q-calculus [10]-[12].

Let $q \in (0,1)$ and $\alpha \in \mathbb{C}$. The q-number $[\alpha]_q$ is

$$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}, \quad [0]_q = 0.$$

If $\alpha = k \in \mathbb{N}$, we obtain $[k]_q = 1 + q + q^2 + ... + q^{k-1}$. Also it is clear that $\lim\limits_{q\to 1^-} [k]_q = k$.
The q-derivative (difference) operator of a function $f \in \mathcal{A}$ is defined by Jackson [10] as follows:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & ;\ z \neq 0 \\ f'(0) & ;\ z = 0 \end{cases}.$$

One can easily show that when $q \to 1^-$, then $D_q f(z) \to f'(z)$. For a function $f \in A$, the $q-$derivative is

$$D_q f(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1}.$$

Note that such an operator plays an important role in the theory of hypergeometric series, quantum physics, sobolev spaces, geometric function theory; for details one can refer to [4, 8, 9, 16].
Also, Jackson [11] introduced the q-integral of a function $f \in \mathcal{A}$ as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{m=0}^{\infty} q^m f(zq^m)$$

provided the series converges.
Several researchers used the theory of analytic and harmonic univalent functions and q-calculus; for example Ahuja and Çetinkaya [1] and Jahangiri [15].
Recently, Ahuja et al. [2] defined the class $\mathcal{SH}_q$ consisting of $q-$harmonic functions in $\mathcal{U}$ as follows:

A harmonic function $f = h + \bar{g}$ defined by (1) is said to be $q-$harmonic, locally univalent and sense-preserving in $\mathcal{U}$ if and only if the second dilatation w_q satisfies the condition

$$|w_q(z)| = \left| \frac{D_q g(z)}{D_q h(z)} \right| < 1 \tag{2}$$

where $0 < q < 1$ and $z \in \mathcal{U}$. Note that as $q \to 1^-$, $\mathcal{SH}_q$ reduces to the family $\mathcal{SH}$.

Also, Ahuja et al. [2] defined $q-$Bernardi-Libera-Livingston integral operator denoted by $\mathbb{L}_{q,c}(f)$ $(c = 1, 2, ...$ and $q \in (0,1))$ as $\mathbb{L}_{q,c}(f) := \mathbb{L}_{q,c}(h) + \overline{\mathbb{L}_{q,c}(g)}$ where

$$\mathbb{L}_{q,c}(h) = \frac{[c+1]_q}{z^c} \int_0^z \xi^{c-1} h(\xi) d_q \xi, \tag{3}$$

$$\mathbb{L}_{q,c}(g) = \frac{[c+1]_q}{z^c} \int_0^z \xi^{c-1} g(\xi) d_q \xi.$$

Note that for $g(z) \equiv 0$ in (3), this operator reduces to the $q-$Bernardi integral operator [21] and for $q \to 1^-$, $g(z) \equiv 0$, and $c = 1$ in (3), we have the Libera integral operator [17]. For functions $f \in \mathcal{H}$ given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=1}^{\infty} \overline{b_m z^m}$$

and $F \in \mathcal{H}$ given by

$$F(z) = z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} \overline{B_m z^m},$$

the convolution of f and F defined by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \sum_{m=1}^{\infty} \overline{b_m B_m z^m}.$$

In 2009, Murugusundaramoorthy and Salagean [19] defined a subclass of $\mathcal{H}$ denoted by $\mathcal{S}_{\mathcal{H}}(F;\beta)$, for $0 \leq \beta < 1$ consisting of functions f of the form (1) that satisfy the condition

$$\frac{\partial}{\partial\theta} \left(\arg\left[(f * F)(z)\right]\right) > \beta, \ \left(0 \leq \theta < 2\pi,\ z = re^{i\theta}\right)$$

or equivalently

$$Re\left[\frac{z\left(h(z) * H(z)\right)' - \overline{z\left(g(z) * G(z)\right)'}}{h(z) * H(z) + \overline{g(z) * G(z)}}\right] \geq \beta$$

where $z \in \mathcal{U}$.

Also, let $\mathcal{V}_H(F;\beta) := \mathcal{S}_{\mathcal{H}}(F;\beta) \cap \mathcal{V}_H$ where $\mathcal{V}_H$ is the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [14] consisting of functions $f \in \mathcal{H}$ of the form (1) for which there exists a real number ϕ such that

$$\eta_m + (m-1)\phi \equiv \pi (\mathrm{mod}\, 2\pi), \;\; \delta_m + (m-1)\phi \equiv 0 (\mathrm{mod}\, 2\pi), \; m \geq 2 \tag{4}$$

where $\eta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$. Murugusundaramoorthy and Salagean [19] also stated that some of the function classes emerge from the function class $\mathcal{S}_{\mathcal{H}}(F;\beta)$ defined above. Indeed, if we specialize the function $F(z)$ we obtain various operators studied before. Such as
(i) Wright's generalized operator on harmonic functions (see [20]),
(ii) the Dziok-Srivastava operator on harmonic functions (see [3]),
(iii) the Carlson-Shaffer operator (see [5]),
(iv) the Ruscheweyh derivative operator on harmonic functions (see [18]),
(v) the Srivastava-Owa fractional derivative operator (see [23]),
(vi) the Salagean derivative operator for harmonic functions (see [13, 22]).
Suppose that $F(z)$ is of the form

$$F(z) = H(z) + \overline{G(z)} = z + \overline{z} + \sum_{m=2}^{\infty} C_m \left(z^m + \overline{z}^m\right),$$

where $C_m \geq 0$ for $m \geq 2$.
Theorem A. [19, Theo. 1.1] Let $f = h + \overline{g} \in \mathcal{V}_H$ be given by (1) with the property (4) and $0 \leq b_1 < \frac{1-\beta}{1+\beta}$, $0 \leq \beta < 1$. Then $f \in \mathcal{V}_H(F;\beta)$ if and only if the inequality

$$\sum_{m=2}^{\infty} \left(\frac{m-\beta}{1-\beta} |a_m| + \frac{m+\beta}{1-\beta} |b_m| \right) C_m \leq 1 - \frac{1+\beta}{1-\beta} b_1 \tag{5}$$

is satisfied.

2 Main result

Theorem 1. *Let $f \in \mathcal{V}_H(F;\beta)$. Then $\mathbb{L}_{q,c}(f) \in \mathcal{V}_H(F;\Psi(\beta))$ where*

$$\Psi(\beta) = \frac{(2+\beta)(c+2)_q(1-b_1) - 2(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(2+\beta)(c+2)_q(1+b_1) + (c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]} > \beta.$$

Proof: Since $f \in \mathcal{V}_H(F;\beta)$ we have

$$\frac{\sum_{m=2}^{\infty} \left(\frac{m-\beta}{1-\beta} |a_m| + \frac{m+\beta}{1-\beta} |b_m| \right) C_m}{1 - \frac{1+\beta}{1-\beta} b_1} \leq 1.$$

Using Theorem A, it suffices to prove that $\mathbb{L}_{q,c}(f) \in \mathcal{V}_H(F;\Psi(\beta))$ if and only if

$$\frac{\sum_{m=2}^{\infty} \left(\frac{m-\Psi(\beta)}{1-\beta} \frac{(c+1)_q}{(c+m)_q} |a_m| + \frac{m+\Psi(\beta)}{1-\beta} \frac{(c+1)_q}{(c+m)_q} |b_m| \right) C_m}{1 - \frac{1+\Psi(\beta)}{1-\Psi(\beta)} b_1} \leq 1. \tag{6}$$

Note that the inequality

$$\begin{aligned} & \frac{\sum_{m=2}^{\infty} \left(\frac{m-\Psi(\beta)}{1-\beta} \frac{(c+1)_q}{(c+m)_q} |a_m| + \frac{m+\Psi(\beta)}{1-\beta} \frac{(c+1)_q}{(c+m)_q} |b_m| \right) C_m}{1 - \frac{1+\Psi(\beta)}{1-\Psi(\beta)} b_1} \\ \leq \quad & \frac{\sum_{m=2}^{\infty} \left(\frac{m-\beta}{1-\beta} |a_m| + \frac{m+\beta}{1-\beta} |b_m| \right) C_m}{1 - \frac{1+\beta}{1-\beta} b_1} \end{aligned}$$

implies (6). It is sufficient to obtain $\Psi(\beta)$ such that

$$\frac{\frac{m-\Psi(\beta)}{1-\beta} \frac{(c+1)_q}{(c+m)_q}}{1 - \frac{1+\Psi(\beta)}{1-\Psi(\beta)} b_1} \leq \frac{\frac{m-\beta}{1-\beta}}{1 - \frac{1+\beta}{1-\beta} b_1} \tag{7}$$

and

$$\frac{\frac{m+\Psi(\beta)}{1-\beta} \frac{(c+1)_q}{(c+m)_q}}{1 - \frac{1+\Psi(\beta)}{1-\Psi(\beta)} b_1} \leq \frac{\frac{m+\beta}{1-\beta}}{1 - \frac{1+\beta}{1-\beta} b_1} \tag{8}$$

hold. Inequality (7) is equivalent to

$$\frac{m-\Psi(\beta)}{1-\Psi(\beta)-b_1-\Psi(\beta)b_1}\frac{(c+1)_q}{(c+m)_q}\leq\frac{m-\beta}{(1-\beta)-(1+\beta)b_1}$$

which implies

$$\Psi(\beta)\leq\frac{(m-\beta)(c+m)_q(1-b_1)-m(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(m-\beta)(c+m)_q(1+b_1)-(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}. \tag{9}$$

Inequality (8) is equivalent to

$$\frac{m+\Psi(\beta)}{1-\Psi(\beta)-b_1-\Psi(\beta)b_1}\frac{(c+1)_q}{(c+m)_q}\leq\frac{m+\beta}{(1-\beta)-(1+\beta)b_1}$$

which implies

$$\Psi(\beta)\leq\frac{(m+\beta)(c+m)_q(1-b_1)-m(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(m+\beta)(c+m)_q(1+b_1)+(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}. \tag{10}$$

From (9) and (10) we observe

$$\begin{aligned}&\frac{(m-\beta)(c+m)_q(1-b_1)-m(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(m-\beta)(c+m)_q(1+b_1)-(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}\\ >\;&\frac{(m+\beta)(c+m)_q(1-b_1)-m(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(m+\beta)(c+m)_q(1+b_1)+(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}\end{aligned}$$

or equivalently

$$\frac{2(c+1)_q\Lambda^2m(m-1)}{\left[(m-\beta)(c+m)_q(1+b_1)-(c+1)_q\Lambda\right]\left[(m+\beta)(c+m)_q(1+b_1)+(c+1)_q\Lambda\right]}>0,$$

where $\Lambda=\left[(1-\beta)-(1+\beta)b_1\right]$. Then

$$\Psi(\beta)\leq\frac{(m+\beta)(c+m)_q(1-b_1)-m(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(m+\beta)(c+m)_q(1+b_1)+(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}.$$

Let us consider the function $\Phi:[2,\infty]\to\mathbb{R}$

$$\Phi(x)=\frac{(x+\beta)(c+x)_q(1-b_1)-x(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(x+\beta)(c+x)_q(1+b_1)+(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}.$$

Since

$$\Phi'(x)=\frac{(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]\left[(1+b_1)x^2+2x(1-b_1)+2\beta+b_1-1\right]}{\left[(x+\beta)(c+x)_q(1+b_1)+(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]\right]^2}>0,$$

then $\Phi(x)$ is an increasing function. Then we choose

$$\Psi(\beta)=\Phi(2)=\frac{(2+\beta)(c+2)_q(1-b_1)-2(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}{(2+\beta)(c+2)_q(1+b_1)+(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}$$

which guarantees $\Psi(\beta)\leq\Phi(m)$, for all $m\geq 2$. Finally,

$$\Psi(\beta)-\beta\;=\;\frac{\left[(1-\beta)-(1+\beta)b_1\right](2+\beta)\left[(c+2)_q-(c+1)_q\right]}{(2+\beta)(c+2)_q(1+b_1)+(c+1)_q\left[(1-\beta)-(1+\beta)b_1\right]}>0.$$

Thus proof is complete. □

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The Generalized Derivative Higher Order Non-Linear Schrödinger Equation: Dark and Singular Optical Soliton Solutions

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Abstract: In this study, generalized derivative higher order non-linear Schrödinger equation (DNLSE) that models pulses propagation in optical fibers is examined. By performing the tanh-coth series expansion method, dark and singular soliton solutions are recovered. Optical solitons have been the topics of immense investigation in non-linear optics because of their possible significances in telecommunication and ultra-fast signal processing systems. In addition to the solution forms in question, the required conditions on the non-linearity and dispersion parameters for the existence of wave solutions are deduced. We also presented two-dimensional and three-dimensional graphical simulations based on some specific selections of parameters to better understand the solution forms and thus the physical phenomenon.

Keywords: Exact solutions, Generalized higher order derivative Non-linear Schrödinger equation, Soliton solutions.

1 Introduction

Nonlinear evolution equations (NLEEs) are partial differential equations (PDEs) that model many physical phenomena and can be observed in fluid mechanics, quantum physics, biology, chemical reactions, nonlinear-optics, etc.

Investigating optical solitons which is a special type of exact solution to non-linear Schrödinger type NLEEs has been a very popular research the area recently due to their application in telecommunications and very fast signal processing systems.

The one-dimensional nonlinear Schrödinger equation (NLSE) is the basic equation of physics for describing quantum mechanical behavior. It is also frequently called the Schrödinger wave equation, and is a PDE,

$$iu_t + 2\,|u|^2\,u + u_{xx} = 0 \tag{1}$$

for complex field $u(x,t)$. Eq. (1) models many nonlinearity effects in fiber, including but not limited to self-phase modulation, four-wave mixing, second harmonic generation, stimulated Raman scattering, optical solitons, ultrashort pulses, etc. It is a classical field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides [1] and to Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps, in the mean-field regime [2].

Optical solitons arise as a result of the delicate balance between dispersion and non-linear terms, depending on the refractive index.

One of such NLEEs is the generalized derivative non-linear Schrödinger equation (DNLS), which is a special case of the higher-order non-linear Schrödinger equation studied by Kondo et al. [2], in the form of

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u + i\alpha|u|^2u_x + i\beta u^2u_x^* + i\gamma u_{xxx} = 0 \tag{2}$$

where $u(x,t)$ denotes the pulse in the optical fiber. $u(x,t)$ is the complex waveform and * symbolizes the complex conjugate. First term means to time evolution, second term the group velocity dispersion (GVD), third, fourth, and fifth terms correspond to nonlinearity terms and the last term corresponds to the 4th order dispersion.

In his study, [1] Seadawy addressed equation (2) and constructed the bright, dark, and bright-dark type optical soliton solutions of the model by means of the ansatz method. In addition, stability analysis of the obtained solutions of the model was performed. The Lagrangian of the model and the invariant variational principle are also extracted [1]. There are also some very useful techniques in the literature including the solution techniques of NLEEs [3]-[5].

The study is organized as follows: In the second part of the study, the main lines of the tanh method that we will use in the article are described. In the subsequent section, we examined the application of the method in question to the DNLSE model. In the last section, the results and discussion were presented.

2 Tanh function expansion method

We will briefly describe the main steps of the $tanh$ function expansion method ([6]-[9]). This method assumes that the equation has traveling wave solutions in $tanh$ form:

Let us consider the evolution type equation in the form of

$$\Delta(u, u_t, u_x, u_{xx}....) = 0 \tag{3}$$

To construct the traveling wave type solutions of the Eq.(3), with the wave variable $\zeta = c(x - \lambda t)$, let us make an ansatz in the form

$$u(x,t) = U(\zeta) \tag{4}$$

where $U(\zeta)$ localized wave solution moves with velocity λ. Based on this ansatz,

$$\begin{aligned}
\frac{d}{dt} &= -c\lambda \frac{d}{d\zeta}, \\
\frac{d}{dx} &= c\frac{d}{d\zeta}, \\
\frac{d^2}{dx^2} &= c^2\frac{d^2}{d\zeta^2}, \\
\frac{d^3}{dx^3} &= c^3\frac{d^3}{d\zeta^3}, \ etc.
\end{aligned} \tag{5}$$

Then, Eq. (3) turns into an ordinary differential equation

$$Q(U, U', U'', U''', ...) = 0 \tag{6}$$

by help of (5).

If all terms in the ODE (6) contain derivatives with respect to ζ, then we get the simplified ODE equation by integrating the equation and choosing the integral constant as zero. With the new

$$Y = \tanh(\zeta) \tag{7}$$

variable change, we get the following various new derivative operators

$$\begin{aligned}
\frac{d}{d\zeta} &= (1 - Y^2)\frac{d}{dY}, \\
\frac{d^2}{d\zeta^2} &= (1 - Y^2)(-2Y\frac{d}{dY} + (1 - Y^2)\frac{d^2}{dY^2}), \ etc.
\end{aligned} \tag{8}$$

Let us consider following ansatz form

$$U(\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k \tag{9}$$

where M is a positive constant. Substituting Eq. (8) and Eq. (9) in Eq. (6) gives an equation containing the powers of Y.

To find the parameter M, the highest order linear and nonlinear terms in the resulting reduced ODE are balanced. Once M is found, the coefficients of the powers of Y are set to zero. This will lead to the algebraic equation containing a_k, $(k = 0...M), c$, and λ. After these parameters are found and (9) is used, the closed-form solution of the equation is obtained.

3 Dark and singular soliton solutions for DNLSE

In this section, we seek solutions that describe the propagations for pulses in optical fiber as governed in Eq. (2). Using two-wave transforms, $\zeta(x,t) = x - ct$ and $\eta(x,t) = \lambda(x + wt)$, we assume the solution of Eq. (2) is

$$u(x,t) = \exp(i\eta(x,t))v(\zeta(x,t)). \tag{10}$$

where the function v represents the pulse shape, and c is the velocity of the soliton. From the phase component, λ is the soliton frequency, and w is the soliton wave number.

Substituting Eq. (10) in Eq. (2) and separating complex part from real part, results in the following system

$$\text{Real part} \ : \ (1 - 6\gamma\lambda)v'' + (-2\lambda w - \lambda^2 + 2\lambda^3\gamma)v + (2 - 2\alpha\lambda + 2\beta\lambda)v^3 = 0, \tag{11}$$

$$\text{Imaginary part} \ : \ \gamma v'' + (\lambda - c - 3\lambda^2\gamma)v + \frac{1}{3}(\alpha + \beta)v^3 = 0. \tag{12}$$

If the coefficients of the real part of equation are equal to zero, one obtains:

$$\gamma = \frac{1}{6\lambda}, \ w = -\frac{\lambda}{3}, \ \alpha = \frac{1 + \beta\gamma}{\lambda}. \tag{13}$$

To solve the Eq. (12) for the unknown function $v(\zeta)$, we use the tanh-coth expansion method, i.e.,

$$v(\zeta) = \sum_{l=0}^{m} k_l \tanh^l(\mu\zeta). \tag{14}$$

Now, we consider the homogenous balance between the linear term v'' and the nonlinear term v^3 to get $m+2=3m$. Thus, $m=1$.
Accordingly, (14) is identified as

$$v(\zeta) = k_0 + k_1 tanh(\mu\zeta). \tag{15}$$

Substituting (15) together with (13) in Eq. (12) and using the relation $sech^2(\mu\zeta)$=$1 - tanh^2(\mu\zeta)$, produce a polynomial equation of $Y(\zeta) = tanh(\mu\zeta)$.
Then, collecting all coefficients of the same powers of $tanh(\mu\zeta)$ in the resulting equation and set them to zero,i.e.

$$\begin{aligned}
4\beta k_0^3\lambda - 6ck_0\lambda + 2k_0^3 + 3k_0\lambda^2 &= 0,\\
12\beta k_0^2 k_1\lambda - 6ck_1\lambda + 6k_0^2k_1 + 3k_1\lambda^2 - 2k_1\mu^2 &= 0,\\
4\beta k_1^3\lambda + 2k_1^3 + 2k_1\mu^2 &= 0,\\
12\beta k_0 k_1^2\lambda + 6k_0k_1^2 &= 0
\end{aligned}$$

yields the following results:

$$k_0 = 0,\ k_1 = \sqrt{-\frac{-6\lambda c + 3\lambda^2}{2+4\beta\lambda}}, \mu = \sqrt{-3\lambda c + \frac{3}{2}\lambda^2}. \tag{16}$$

or

$$k_0 = 0,\ k_1 = -\sqrt{-\frac{-6\lambda c + 3\lambda^2}{2+4\beta\lambda}}, \mu = -\sqrt{-3\lambda c + \frac{3}{2}\lambda^2}. \tag{17}$$

Therefore, the first and second dark soliton solutions of DNLSE are

$$u_1(x,t) = \sqrt{-\frac{-6\lambda c + 3\lambda^2}{2+4\beta\lambda}}\exp(i\lambda(x - \frac{\lambda}{3}t))\tanh\left(\sqrt{-3\lambda c + \frac{3}{2}\lambda^2}(x-ct)\right) \tag{18}$$

and

$$u_2(x,t) = -\sqrt{-\frac{-6\lambda c + 3\lambda^2}{2+4\beta\lambda}}\exp(i\lambda(x - \frac{\lambda}{3}t))\tanh\left(-\sqrt{-3\lambda c + \frac{3}{2}\lambda^2}(x-ct)\right). \tag{19}$$

These solutions are valid provided the constraint conditions $-6\lambda c + 3\lambda^2 > 0$ and $1 + 2\beta\lambda < 0$ remains valid at all times. Also, they are classified optical dark soliton solutions to the model. Next, if we replace the $tanh$ function by coth in (15), we get the same results and

$$u_3(x,t) = \sqrt{-\frac{-6\lambda c + 3\lambda^2}{2+4\beta\lambda}}\exp(i\lambda(x - \frac{\lambda}{3}t))\coth\left(\sqrt{-3\lambda c + \frac{3}{2}\lambda^2}(x-ct)\right) \tag{20}$$

and

$$u_4(x,t) = \sqrt{-\frac{-6\lambda c + 3\lambda^2}{2+4\beta\lambda}}\exp(i\lambda(x - \frac{\lambda}{3}t))\coth\left(\sqrt{-3\lambda c + \frac{3}{2}\lambda^2}(x-ct)\right) \tag{21}$$

These are classified as optical singular soliton solutions to the model. The existence condition of Eq.(20) singular soliton solution is the same as Eq.(18), and the existence condition of Eq.(21) singular soliton solution is the same as Eq.(19). These were put in the body of the article.

4 Results, numerical simulations and concluding remarks for DNLSE

In this section, we drew the graphs of the squares of the absolute values of the Eq. (18) dark optical soliton and Eq. (20) singular soliton solutions separately for the values of λ=1,β=-2, and c=-1, separately. As can be seen, in order for these solutions to be physically meaningful in the form of non-singular solutions, the arguments in the denominator must be non-zero.

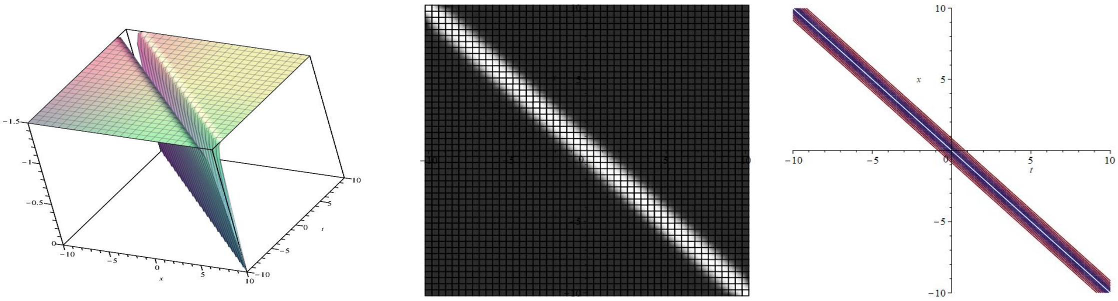

Fig. 1: 3D, density and countour plots of the solution (18).

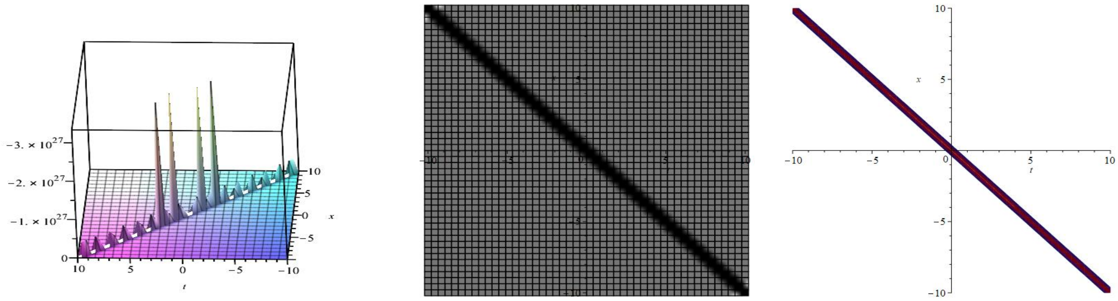

Fig. 2: 3D, density and countour plots of the solution (20).

Travelling wave solutions of the solitonic type, which have various properties of the DNLSE, were revealed by the well-known tanh-coth function series expansion technique. When the results obtained in the article were compared with the studies numbered [1, 2], it was concluded that some of them were structurally the same and some of them (especially singular optical soliton solutions) were new. It is obvious that the results obtained can be used in the description of the physical phenomenon modeling the event and in experimental studies. From the obtained results, we observed that the tanh-coth function method is effective, easy, robust, and can be applied to other NLEEs. We plan to obtain the group invariant solutions and conservation law forms of this model in further study.

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Dual-Mode Nonlinear Schrödinger's Equation: Modulation Instability Analysis and Optical Soliton Solutions

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Abstract: In this study, dual-mode nonlinear Schrödinger's equation (DMNLSE) that models the propagation of dual-mode pulses is examined. The standart nonlinear Schrödinger's equation (NLSE) models the amplification or absorption of pulses propagating in a single-mode optical fiber whereas the dual-mode phenomena represent the spread of two-moving waves simultaneously. By applying ansatz method, we obtain the distinct forms of the bright, dark and bright–dark solitary wave soliton solutions of the DMNLSE. By performing the modulation instability approach the stability analysis of the derived solutions are analyzed. We also depict two-dimensional and three-dimensional graphical simulations based on some specific selections of parameters to better view the solution forms and thus the physical phenomenon.

Keywords: Exact solutions, Dual-mode nonlinear Schrödinger's equation, Modulation instability, Solitary wave solutions.

1 Introduction

Nonlinear evolution equations (NLEEs) appear in the modeling of problems that arise in many fields of science and engineering. Experimental results have shown that the physical model is composed of the associated NLEEs and auxiliary conditions, which are the boundary or initial conditions under consideration. At this point, the construction of analytical solutions (especially solitonic solutions) with various physical properties, rather than a technique based on finding general solutions of nonlinear differential equations in general, has been a research topic that has been intensively studied in the last 20 years.

Solitons are self-reinforcing solitary waves (wavepacket or pulse wave) that propagate at a constant rate while maintaining their own shape. The concept of soliton in nonlinear optic field investigates the optical field, where the counterbalance between nonlinear and linear impacts is stable in the course of propagation. Videlicet, optical solitons appear from the delicate balance between group velocity dispersion (GVD) effect and non-linear effect (NLE) arising due to non-linear change in the refractive index [1]-[3].

The term dark soliton is a localized surface wave envelope that causes a temporary decrease in wave amplitude. Correspondingly, the term bright soliton is perceived that as a localized intensity peak above a continuous-wave background while a dark soliton is characterized as a localized intensity dip below a continuous-wave background. Moreover, bright soliton is thought to be in charge of amplifying ocean waves to rogue wave proportions. The terms bright and dark soliton are borrowed from optics, where they manifest as bright spots and dark shadows in optical fibers.

Lately, dual-mode type NLEEs have captivated various researches in the nonlinear optic. The reason is that those equations investigate the contemporaneous wave interplays on dual-mode.

Any non-linear partial differential equation (NLPDE)

$$u_\eta + A(u, u_\zeta ...) + B(u_{\zeta\zeta}, u_{\zeta\zeta\zeta} ...) = 0,$$

where the second term denotes non-linear operator, and the third term typify the linear operator, is converted to the dual-mode version as [4, 5]

$$\left(u_{tt} - \Theta^2 u_{xx}\right) + \left(\frac{\partial}{\partial t} - \lambda\Theta\frac{\partial}{\partial x}\right) A(u, u_x ...) + \left(\frac{\partial}{\partial t} - \kappa\Theta\frac{\partial}{\partial x}\right) B(u_{xx}, u_{xxx}, ...) = 0.$$

The authors of [6], have proposed the following dual-mode non-linear Schrödinger's equation (DMNLSEs) with Kerr law non-linearity such as:

$$u_{tt} - s^2 u_{xx} + 2i(\frac{\partial}{\partial t} - \alpha s\frac{\partial}{\partial x})u|u|^2 + i(\frac{\partial}{\partial t} - \beta s\frac{\partial}{\partial x})u_{xx} = 0 \tag{1}$$

where $u = u(x, t)$ is the envelope field, x and t are spatial and temporal variables, respectively and $i = \sqrt{-1}$.

In [6], Jaradat et al. presented the dark and singular soliton solutions, including the criteria for the existence of these solutions, with the help of the tanh-coth technique when $\alpha = \beta$.

In [5], Zayed and Shohib extended the work of [6] via solving Eq. (1) in the case of $\alpha \neq \beta$ by exploiting the tanh-coth method. Moreover, they considered Eq. (1) in the case of $\alpha \neq \beta$ applying the unified Riccati equation method, the modified simple equation method and the new extended auxiliary equation method.

The study is organized as follows: In the second and third parts of the study, we analysed governing equation and performed the ansatz method in question to the DMNLSE model. In the fourth section, the modulation instability analysis of the system is presented.The last section is devoted to conclusions and discussions.

2 Governing equation

In this section we seek solutions to the following NLEE that describe the propagations for dual-mode pulses in optical fiber [6]

$$u_{tt} - s^2 u_{xx} + 2i(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x})u|u|^2 + i(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x})u_{xx}, \text{ when } \alpha = \beta. \tag{2}$$

Using two wave transforms, $\zeta = x - ct$ and $\eta = \lambda(x + wt)$, we assume the solution of (2) as

$$u(x,t) = \exp(i\eta)v(\zeta). \tag{3}$$

Substituting (3) into (2) and separating complex part from real part, results in the following system

$$0 = \lambda^2(-s^2 + w(w-\lambda) + s\beta\lambda)v(\zeta) + 2(w - s\beta)v^3(\zeta) - (c^2 - s^2 + 2c\lambda - w\lambda + 3s\beta\lambda)v''(\zeta), \tag{4}$$

$$0 = \lambda(2s^2 + 2cw - c\lambda + 2w\lambda - 3s\beta\lambda)v(\zeta) + 2(c + s\beta)v(\zeta) + 2(c + s\beta)v^3(\zeta) + (c + s\beta)v''(\zeta)).$$

If we substitute the $c = -s\beta$ value obtained from the second equation in $\lambda(2s^2 + 2cw - c\lambda + 2w\lambda - 3s\beta\lambda) = 0$ and solve it according to w, we get

$$w = -\frac{s(\beta\lambda - s)}{\beta s - \lambda} \tag{5}$$

If we substitute the value of Eq.(5) into the (4) equation, the following reduced equation is obtained:

$$\begin{aligned} &-(\beta s - \lambda)((-2\beta s - 2c)\lambda^2 + (3\beta^2 s^2 + 2\beta c s - c^2)\lambda + \beta c^2 s - \beta s^3)v''(\zeta) \\ &-((2\beta s - 2\lambda)v(\zeta)^2 + s\lambda(-\beta\lambda + s))s^2\lambda(\beta+1)(\beta-1)v(\zeta) = 0 \end{aligned} \tag{6}$$

3 Solitary wave type solutions to DMNLSE

In this section, we will give the solitary wave solution forms, which will be presented in various forms, in ansatz form [7] and try to obtain them. These solutions are exact solution forms with various physical properties in the form of dark, bright and dark-bright.

Ansatz form I. The ansatz function of the DMNLSE can be taken in the form of bright solitary wave solution

$$v_1(x,t) = K \sec h(h\zeta), \tag{7}$$

where K,h, c are the amplitude, the pulse width and velocity of soliton in normalized unites.

Substituting Eq. (7) into Eq. (6), and then equating the coefficients of the $\sec h(h\zeta)$ terms to zero, we obtain the algebraic system in K,h,β,λ, and s. Solving this system by help of Maple, we get the coefficients in the form

$$K = \pm\sqrt{\frac{-(-\beta\lambda^2 s + \lambda s^2)}{(\beta s - \lambda)}},$$

$$h = \pm\sqrt{\frac{-(-\beta^3\lambda s + \beta^2 s^2 + \beta\lambda s - s^2)}{3\beta^3\lambda s^3 + \beta^2 c^2 s^2 + 2\beta^2 c\lambda s^2 - 5\beta^2\lambda^2 s^2 - \beta^2 s^4 - 2\beta c^2\lambda s - 4\beta c\lambda^2 s + 2\beta\lambda^3 s + \beta\lambda s^3 + c^2\lambda^2 + 2c\lambda^3)}}s\lambda$$

Then the solution of the DMNLSE as bright solitary wave solution is

$$u_1(x,t) = K \sec h(h\zeta)\exp(i\eta(x,t)).$$

The sufficient conditions for solitary wave solution's non-singularity is

$$\frac{-(-\beta^3\lambda s + \beta^2 s^2 + \beta\lambda s - s^2)}{3\beta^3\lambda s^3 + \beta^2 c^2 s^2 + 2\beta^2 c\lambda s^2 - 5\beta^2\lambda^2 s^2 - \beta^2 s^4 - 2\beta c^2\lambda s - 4\beta c\lambda^2 s + 2\beta\lambda^3 s + \beta\lambda s^3 + c^2\lambda^2 + 2c\lambda^3)} > 0$$

$$\frac{-(-\beta\lambda^2 s + \lambda s^2)}{(\beta s - \lambda)} > 0,$$

$$\beta s - \lambda \neq 0,$$

and

$$3\beta^3\lambda s^3 + \beta^2 c^2 s^2 + 2\beta^2 c\lambda s^2 - 5\beta^2\lambda^2 s^2 - \beta^2 s^4 - 2\beta c^2\lambda s - 4\beta c\lambda^2 s + 2\beta\lambda^3 s + \beta\lambda s^3 + c^2\lambda^2 + 2c\lambda^3 \neq 0.$$

Ansatz form II. Suppose that the following ansatz function of the DMNLSE in the form of dark solitary wave solution:

$$v_2(x,t) = K \tanh h(h\zeta), \tag{8}$$

where K,h, c are the amplitude, the pulse width and velocity of soliton in normalized unites.

Substituting Eq. (8) into Eq. (6), and then equating the coefficients of the $\tanh h(h\zeta)$ terms to zero, we obtain the algebraic system in K,h,β,λ, and s.Solving this system by help of Maple, we get the coefficients in the form

$$K = \pm\sqrt{\frac{-(-\beta\lambda^2 s + \lambda s^2)}{(2\beta s - 2\lambda)}},$$

$$h = \pm\sqrt{\frac{-(\beta^3\lambda s - \beta^2 s^2 - \beta\lambda s + s^2)}{6\beta^3\lambda s^3 + 2\beta^2 c^2 s^2 4\beta^2 c\lambda s^2 - 10\beta^2\lambda^2 s^2 - 2\beta^2 s^4 - 4\beta c^2\lambda s - 8\beta c\lambda^2 s + 4\beta\lambda^3 s + 2\beta\lambda s^3 + 2c^2\lambda^2 + 4c\lambda^3)}s\lambda}$$

Then the solution of the DMNLSE as dark solitary wave solution is

$$u_2(x,t) = K\tanh h(h\zeta)\exp(i\eta(x,t)).$$

The above dark solitary wave solution has the sufficient conditions for non-singularity as

$$\frac{-(\beta^3\lambda s - \beta^2 s^2 - \beta\lambda s + s^2)}{6\beta^3\lambda s^3 + 2\beta^2 c^2 s^2 + 4\beta^2 c\lambda s^2 - 10\beta^2\lambda^2 s^2 - 2\beta^2 s^4 - 4\beta c^2\lambda s - 8\beta c\lambda^2 s + 4\beta\lambda^3 s + 2\beta\lambda s^3 + 2c^2\lambda^2 + 4c\lambda^3} > 0,$$

$$\frac{-(-\beta\lambda^2 s + \lambda s^2)}{(2\beta s - 2\lambda)} > 0,$$

$$\beta s - \lambda \neq 0,$$

$$6\beta^3\lambda s^3 + 2\beta^2 c^2 s^2 + 4\beta^2 c\lambda s^2 - 10\beta^2\lambda^2 s^2 - 2\beta^2 s^4 - 4\beta c^2\lambda s - 8\beta c\lambda^2 s$$
$$+4\beta\lambda^3 s + 2\beta\lambda s^3 + 2c^2\lambda^2 + 4c\lambda^3 \neq 0.$$

Ansatz form III: The DMNLSE has the bright–dark solitary wave solution using the ansatz function in the form:

$$v_3(x,t) = K\sec h(h\zeta) + L\tanh(h\zeta), \tag{9}$$

Using the bright–dark solitary wave solution (9) in Eq. (6), we derive the coefficients such as

$$K = \pm\sqrt{-\frac{\beta\lambda^2 s - \lambda s^2}{2\beta s - 2\lambda}},$$

$$L = \pm\sqrt{-\frac{-\beta\lambda^2 s + \lambda s^2}{2\beta s - 2\lambda}},$$

$$h = \pm\sqrt{\frac{-(2\beta^3\lambda s - 2\beta^2 s^2 - 2\beta\lambda s + 2s^2)}{3\beta^3\lambda s^3 + \beta^2 c^2 s^2 + 2\beta^2 c\lambda s^2 - 5\beta^2\lambda^2 s^2 - \beta^2 s^4 - 2\beta c^2\lambda s - 4\beta c\lambda^2 s + 2\beta\lambda^3 s + \beta\lambda s^3 + -c^2\lambda^2 + 2c\lambda^3)}s\lambda}$$

Then we have the bright–dark solitary wave solution of the DMNLS equation as

$$u_3(x,t) = (K\sec h(h\zeta) + L\tanh(h\zeta))\exp(i\eta(x,t)).$$

The sufficient conditions for bright–dark solitary wave solutions non-singularity are

$$\beta s - 2\lambda \neq 0,$$

$$3\beta^3\lambda s^3 + \beta^2 c^2 s^2 + 2\beta^2 c\lambda s^2 - 5\beta^2\lambda^2 s^2 - \beta^2 s^4 - 2\beta c^2\lambda s - 4\beta c\lambda^2 s + 2\beta\lambda^3 s + \beta\lambda s^3 + -c^2\lambda^2 + 2c\lambda^3 \neq 0,$$

$$\frac{-(2\beta^3\lambda s - 2\beta^2 s^2 - 2\beta\lambda s + 2s^2)}{3\beta^3\lambda s^3 + \beta^2 c^2 s^2 + 2\beta^2 c\lambda s^2 - 5\beta^2\lambda^2 s^2 - \beta^2 s^4 - 2\beta c^2\lambda s - 4\beta c\lambda^2 s + 2\beta\lambda^3 s + \beta\lambda s^3 + -c^2\lambda^2 + 2c\lambda^3)} > 0,$$

$$-\frac{-\beta\lambda^2 s + \lambda s^2}{2\beta s - 2\lambda} > 0$$

4 Modulation instability

Numerous non-linear systems manifest an instability that cause to modulation of the steady state as a consequence of an interaction between the non-linear and dispersive effects. To derive the modulation instability of the Eq. (2) by performing the standard linear stability analysis [8]-[11] the DMNLSE has the perturbed steady-state solution

$$u(x,t) = (\sqrt{Q} + \theta(x,t))e^{i\sigma(t)},\ \sigma(t) = (Qq + \gamma\epsilon Q^2)t \tag{10}$$

where Q is the normalized optical power. We investigate the evolution of the perturbation $\theta(x,t)$ utilizing a linear stability schemes. By substituting Eq. (10) into Eq. (2) and linearizing in $\theta(x,t)$, we obtain the following form as:

$$-\theta_{xx}Q^2\gamma\epsilon + 2i\theta_t Qq - 18Q^3\theta\gamma\epsilon - 2\theta Q^3\gamma\epsilon q + \theta_{tt} - \theta Q^4\epsilon^2\gamma^2 - 18Q^2\theta q - \theta Q^2 q^2$$

$$+i\theta_{xxt} - \theta_{xx}s^2 - \theta_{xx}Qq + 6i\theta_t Q - is\beta\theta_{xxx} - 6i\theta_x s\beta Q + 2i\theta_t Q^2\gamma\epsilon = 0 \tag{11}$$

Let us take the solution of Eq. (11) in the form

$$Q(x,t) = \alpha_1 e^{i(kx-wt)} + \alpha_2 e^{-i(kx-wt)} \tag{12}$$

where w and k are the frequency of perturbation and normalized wave number. The dispersion relation $k = k(w)$ of a constant coefficient linear evolution equation forms how time oscillations $e^{ikx}$are connected to spatial oscillations e^{iwt} of a wave number. Substituting Eq. (12) into Eq. (11), we yield the following dispersion relation as:

$$\begin{aligned} w &= -Q^2\gamma\epsilon - Qq + \frac{1}{2}k^2 - 3Q \\ &\pm\frac{1}{2}\sqrt{-48Q^3\gamma\epsilon + 4\beta k^3 s - 24Q\beta ks + k^4 + 4k^2s^2 - 48Q^2q - 12Qk^2 + 36Q^2} \end{aligned} \tag{13}$$

The above dispersion relation in Eq. (13) also manifests the value of frequency w is real for values of

$$-48Q^3\gamma\epsilon + 4\beta k^3 s - 24Q\beta ks + k^4 + 4k^2s^2 - 48Q^2q - 12Qk^2 + 36Q^2 > 0 \tag{14}$$

which implies that steady solution is stable (see, Figure 3).

5 Conclusions and Discussions

In this study, we constructed dark, bright and dark-bright type solitonic solutions of DMNLSE which have some physical properties. In this context, we used the method of ansatz forms. We also performed graphical analyses of the obtained solutions based on the specific values of the parameters. In Figure 1, we presented the 3-dimensional and 2-dimensional density graphics of the dark soliton solution (The square of the module of (3)), and 3-dimensional and 2-dimensional contour graphics of the bright soliton solution (The square of the module of (3)) in Figure 2.

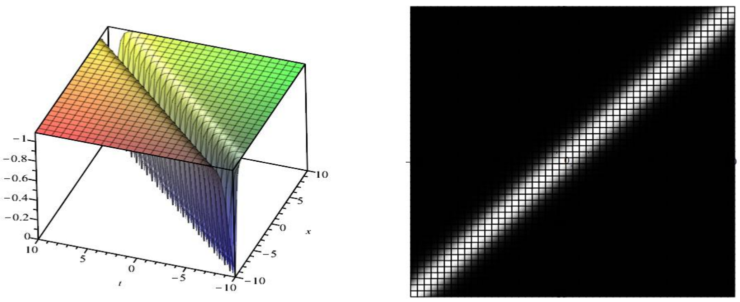

Fig. 1: 3D and countour plots of the dark soliton wave solution (3).

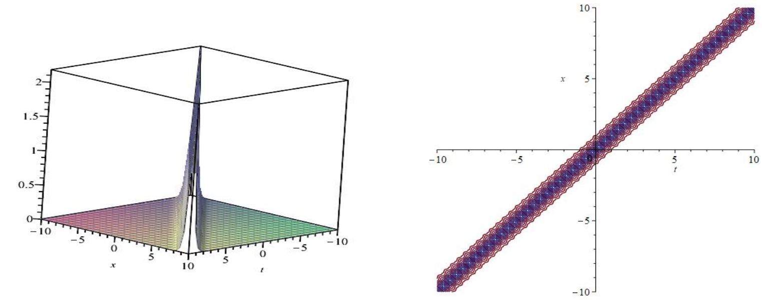

Fig. 2: 3D and density plots of the bright soliton wave solution (3).

In addition, we also performed the instability analysis of the model. The solutions obtained can be used in experimental studies related to dual mode waves modelled by the equation.

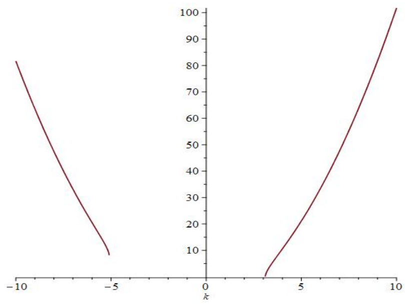

Fig. 3: Frequency and dispersion relation in Eq.(14) by taking all parameters unity. (13).

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$\mathcal{T}$-$\mathcal{Z}$-Semi-Symmetric Riemann Manifolds

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Abstract: $\mathcal{T}$-curvature tensorhas been defined as a generalization of many special curvature tensors which have been used in differential geometry for to understand geometric properties of manifolds and in theoretical physics with some applications to relativity theory. $\mathcal{Z}-$ tensor is a general notion of the Einstein gravitational tensor in General Relativity. In this study, we consider these two general tensor for symmetry conditions and we examine Riemannian manifolds under these conditions.

Keywords: Semi-symmetry, T-tensor, Z-tensor.

1 Introduction

Curvature is the most basic concept of Riemann geometry. The curvature of a Riemannian manifold gives the measure of its difference from Euclidean and is expressed by a (1,3)-type tensor. This tensor, called the Riemann curvature tensor, allows the calculation of the local and global curvatures of the manifold. The Riemann curvature tensor has very useful symmetry properties. Also, there are different (1,3)-type tensors with similar properties. These tensors have some geometric meanings as well as their algebraic properties. For example, the conformal curvature tensor remains invariant under conformal transformations, so it can also measure the resemblance of a manifold to Euclidean space under conformal transformations. Apart from the conformal curvature tensor, many different forms of curvature tensors have been defined. All these tensors are frequently used in the study of the differential geometry of Riemannian manifolds.

In [**?**], a general curvature tensor has been defined as follow;

$$\begin{aligned}\mathcal{T}(X_1,X_2)X_3 &= a_0R(X_1,X_2)X_3 + a_1Ric(X_2,X_3)X_1 + a_2Ric(X_1,X_3)X_2 \\ &+ a_3Ric(X_1,X_2)X_3 + a_4g(X_2,X_3)QX_1 + a_5g(X_1,X_3)QX_2 \\ &+ a_6g(X_1,X_2)QX_3 + a_7r\left(g(X_2,X_3)X_1 - g(X_1,X_3)X_2\right)\end{aligned} \tag{1}$$

where, a_0, ..., a_7 are some smooth functions on M, R, Ric, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator of type $(1,1)$ and the scalar curvature, respectively. This tensor is called by $\mathcal{T}$-curvature tensor. $\mathcal{T}$-tensor has been defined as a generalization of many special curvature tensors which have been used in differential geometry for to understand geometric properties of manifolds and in theoretical physics with some applications to relativity theory. $\mathcal{T}$-tensor has not exactly same symmetry properties of Riemannian curvature tensor. But, under special restriction on coefficients we obtain these symmetry properties. In [**?**], the authors presented these properties.

While examining the Riemannian geometry of manifolds, classification and characterizations are made by selecting special conditions on the curvature tensors. The fact that the curvature tensor is zero can give important information about the geometry of the manifold. If the $\mathcal{T}$-curvature tensoris identically zero, the manifold is called a $\mathcal{T}$-flat. On the other hand, the symmetry properties of manifolds can also be studied using the curvature tensor. If R is parallel, the manifold is said to be locally symmetrical. $R \cdot R$ product is defined as a generalization of this concept. If $\mathcal{T} \cdot R = 0$ then the manifold is said to be $\mathcal{T}$-semi-symmetric. Many classifications have been made on manifolds using special curvature tensors. Since the $\mathcal{T}$-tensor is a general expression, the results obtained using the $\mathcal{T}$-tensor can also include other results. In [**?**] K-contact and Sasakian manifolds classified the under some certain conditions on $\mathcal{T}$-curvature tensor. In [**?**], it was obtained some results on (κ,μ)-contact manifolds with $\mathcal{T}$-curvature tensor. φ-symmetric $\mathcal{T}$- curvature tensor in $N(\kappa)-$contact metric manifolds have been studied in [**?**]. Also, in [**?**], Lorentzian α-Sasakian manifolds have been classified with $\mathcal{T}$-curvature tensor . In 2019 Gupta studied on φ-$\mathcal{T}$-Symmetric ϵ-Para Sasakian manifolds [**?**]. The other special kind of contact manifolds is LP-Sasakian manifolds have been studied by using $\mathcal{T}$-curvature tensor in [**?**]. Presented author, Altin and Pandey classified $N(k)-$contact metric manifolds by using $\mathcal{T}$-curvature tensor in [**?**]. As can be seen from these studies, with $\mathcal{T}$-curvature tensor we can classify manifolds with special structures.

As it is known, the examination of special solutions of Einstein field equations is an important subject of general relativity. While examining these equations, a tensor called Einstein tensor, which has important geometrical properties, has been defined. A generalization of this tensor is is given by Mantica and Molinari as follow;

$$\mathcal{Z}(X_1,X_2) = Ric(X_1,X_2) + \psi g(X_1,X_2) \tag{2}$$

where, $X_1, X_2 \in \Gamma(TM)$ and ψ is an arbitrary function [**?**]. $\mathcal{Z}$- tensor has many important properties and applications. It is a general notion of the Einstein gravitational tensor in General Relativity.

In this study, we studied on Riemann manifolds under certain conditions which related to $\mathcal{T}$-curvature tensor and $\mathcal{Z}$-tensor.

2 Preliminaries

A Riemannian manifold M is called as locally symmetric if we have $\nabla R = 0$ for Levi-Civita connection ∇. Locally symmetric Riemannian manifolds are a generalization of manifolds of constant curvature. As a generalization of locally symmetric Riemannian manifolds, semi-symmetric Riemannian manifolds were defined by condition $R \cdot R = 0$. Here first R acts as a derivation on the second R and this operation is defined by

$$\begin{aligned}(R(X_1, X_2) \cdot R)(X_3, X_4)X_5 \quad &= \quad R(X_1, X_2)R(X_3, X_4)X_5 - R(R(X_1, X_2), X_3, X_4)X_5 \\ &\quad -R(X_3, R(X_1, X_2)X_4)X_5 - R(X_3, X_4)R(X_1, X_2)X_5\end{aligned}$$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$. It is known that locally symmetric manifolds are semi-symmetric manifolds, but the converse is not true. This notion could be generalized for other curvature tensor. A Riemannian manifold is said to be $\mathcal{P}-$semi-symmetric if $\mathcal{P} \cdot R = 0$ for any $(1, 3)-$type curvature tensor $\mathcal{P}$.

The other notion is Ricci semi-symmetry. A Riemannian manifold is said to be Ricci semi-symmetric if $R \cdot Ric = 0$, where $\cdot$ operation is given by

$$(R(X_1, X_2) \cdot Ric)(X_3, X_4) = -Ric(R(X_1, X_2)X_3, X_4) - Ric(X_3, R(X_1, X_2)X_4).$$

Simalarly, we recall a Riemannian manifold as $\mathcal{P}$-Ricci semi-symmetric if $\mathcal{P} \cdot Ric = 0$. By replace Ric with $\mathcal{Z}$, a Riemannian manifold said to be $\mathcal{P}$-$\mathcal{Z}$ semi-symmetric if $\mathcal{P} \cdot \mathcal{Z} = 0$. On the other hand, For a $(1, 1)-$tensor $\mathcal{K}$ we have

$$(R(X_1, X_2) \cdot \mathcal{K})X_3 = R(X_1, X_2)\mathcal{K}X_3 - \mathcal{K}R(X_1, X_2)X_3.$$

For the $\mathcal{T}$-curvature tensor is defined as in (**??**), we have

$$\begin{aligned}\mathcal{T}(X_1, X_2, X_3, X_4) &= a_0R(X_1, X_2, X_3, X_4) + a_1S(X_2, X_3)g(X_1, X_4) \\ &+ a_2S(X_1, X_3)g(X_2, X_4) + a_3S(X_1, X_2)g(X_3, X_4) \\ &+ a_4g(X_2, X_3)S(X_1, X_4) + a_5g(X_1, X_3)S(X_2, X_4) \\ &+ a_6S(X_3, X_4)g(X_1, X_2) \\ &+ a_7r(g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)).\end{aligned} \tag{3}$$

where $\mathcal{T}(X_1, X_2, X_3, X_4) = g(\mathcal{T}(X_1, X_2)X_3, X_4)$ and with the special values of coefficients $a_i,\ 1 \leq i \leq 7$ the $\mathcal{T}$-curvature tensor is reduced to quasi-conformal, conformal, conharmonic, concircular, pseudo-projective, projective, M-projective, W_i-curvature tensors $(i = 0, ..., 9)$ W_j-curvature tensors $(j = 0, 1)$quasi-conformal, conformal, conharmonic, concircular, pseudo-projective, projective, M-projective, W_i-curvature tensors $(i = 0, ..., 9)$ W_j-curvature tensors $(j = 0, 1)$ as in Table 1.

$\mathcal{T}$**-curvature tensor**	a_i coefficients
quasi-conformal curvature tensor C_*	$a_1 = -a_2 = a_4 = -a_5,\ a_3 = a_6 = 0,\ a_7 = -\frac{1}{2n+1}(\frac{a_0}{2n} + 2a_1)$
conformal curvature tensor C	$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1},\ a_3 = a_6 = 0, a_7 = \frac{1}{2n(2n-1)}$
conharmonic curvature tensor L	$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1},\ a_3 = a_6 = 0, \quad a_7 = 0$
concircular curvature tensor V	$a_0 = 1,\ a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0,\ a_7 = -\frac{1}{2n(2n+1)}$
pseudo-projective curvature tensor P_*	$a_1 = -a_2,\ a_3 = a_4 = a_5 = a_6 = 0, a_7 = -\frac{1}{2n+1}(\frac{a_0}{2n} + a_1)$
projective curvature tensor P	$a_0 = 1,\ a_1 = -a_2 = -\frac{1}{2n},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0$
$M-$*projective curvature tensor*	$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{4n},\ a_3 = a_6 = a_7 = 0$
W_0-*projective curvature tensor*	$a_0 = 1,\ a_1 = -a_5 = -\frac{1}{2n},\ a_2 = a_3 = a_4 = a_6 = a_7 = 0$
$W_0^\star-$*projective curvature tensor*	$a_0 = 1,\ a_1 = -a_5 = -\frac{1}{2n},\ a_2 = a_3 = a_4 = a_6 = a_7 = 0$
W_1-*projective curvature tensor*	$a_0 = 1,\ a_1 = -a_2 = \frac{1}{2n},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0$
$W_1^\star-$*projective curvature tensor*	$a_0 = 1,\ a_1 = -a_2 = \frac{-1}{2n},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0$
W_2-*projective curvature tensor*	$a_0 = 1,\ a_4 = -a_5 = \frac{-1}{2n},\ a_1 = a_2 = a_3 = a_6 = a_7 = 0$
W_3-*projective curvature tensor*	$a_0 = 1, \quad a_2 = -a_4 = \frac{-1}{2n}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0$
W_4-*projective curvature tensor*	$a_0 = 1, \quad a_5 = -a_6 = \frac{1}{2n}, \quad a_1 = a_2 = a_3 = a_4 = a_7 =$
W_5-*projective curvature tensor*	$a_0 = 1, \quad a_2 = -a_5 = \frac{-1}{2n}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0$
W_6-*projective curvature tensor*	$a_0 = 1, \quad a_1 = -a_6 = \frac{-1}{2n}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0$
W_7-*projective curvature tensor*	$a_0 = 1, \quad a_1 = -a_4 = \frac{-1}{2n}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0$
W_8-*projective curvature tensor*	$a_0 = 1, \quad a_1 = -a_3 = \frac{-1}{2n}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0$
W_9-*projective curvature tensor*	$a_0 = 1, \quad a_3 = -a_4 = \frac{-1}{2n}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0$

Table 1 Determination of $\mathcal{T}$-curvature tensor with special values of a_i

Einstein manifolds are special kind of Riemannian manifolds which are the special solutions of Einstein fields equations. An Einstein manifold is a Riemannian manifold with following Ricci tensor:

$$Ric(X_1, X_2) = \lambda g(X_1, X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$, where λ is a constant. It is point out that an Einstein manifold is Ricci semi-symmetric, but the converse is not true in generally. Under some special conditions such as Ricci semi-symmetry, we can classify Riemannian manifolds Einstein or quasi-Einstein.

3 Results

In this section, we give some conditions for $\mathcal{T}-$Ricci-semi-symmetric and $\mathcal{T}-\mathcal{Z}$-semi-symmetric Riemannian manifolds.

Theorem 1. *Let M be an $n-$dimensional Riemannian manifold. M is $\mathcal{T}-$Ricci-semi-symmetric if and only if $R \cdot Q = 0$ and we have $a_1 = a_2$, $a_4 = a_5 = a_7 = 0$, a_0, a_3, a_6 are constant.*

Proof: Let M be a $\mathcal{T}-$Ricci-semi-symmetric Riemannian manifold. From the definition of Ricci-semi-symmetry we get

$$\begin{aligned}
(\mathcal{T}(X_1, X_2) \cdot Ric)(X_3, X_4) &= \mathcal{T}(X_1, X_2, X_3, QX_4) - \mathcal{T}(X_1, X_2, X_4, QX_3) \\
&= a_0(R(X_1, X_2, X_3, QX_4) - R(X_1, X_2, X_4, QX_3)) \\
&\quad + a_1(Ric(X_2, X_3)g(X_1, QX_4) - Ric(X_2, X_4)g(X_1, QX_3)) \\
&\quad + a_2(Ric(X_1, X_3)g(X_2, QX_4) - Ric(X_1, X_4)g(X_2, QX_3)) \\
&\quad + a_3(Ric(X_1, X_2)g(X_3, QX_4) - Ric(X_1, X_2)g(X_4, QX_3)) \\
&\quad + a_4(g(X_2, X_3)Ric(X_1, QX_4) - g(X_2, X_4)Ric(X_1, QX_3)) \\
&\quad + a_5(g(X_1, X_3)Ric(X_2, QX_4) - g(X_1, X_4)Ric(X_2, QX_3)) \\
&\quad + a_6(Ric(X_2, QX_4)g(X_1, X_2) - Ric(X_4, QX_3)g(X_1, X_2)) \\
&\quad + a_7 r(g(X_2, X_3)g(X_1, QX_4) - g(X_1, X_3)g(X_2, QX_4) \\
&\quad - g(X_1, X_3)g(X_2, QX_4 + g(X_1, X_4)g(X_2, QX_3))).
\end{aligned}$$

Here the first part is equal to $g((R(X_1, X_2) \cdot Q)X_3, X_4)$. Thus, the proof is completed.

□

It is easy to verify that M is Ricci-semi-symmetric if and only if $R \cdot Q = 0$.

Theorem 2. *Let M be an $n-$dimensional Riemannian manifold. M is $\mathcal{T}-\mathcal{Z}$-semi-symmetric if and only if $\mathcal{T}-$Ricci-semi-symmetric and we have $a_1 = -a_5$, $a_2 = -a_4$, $a_3 = a_6 = 0$ and a_0, a_7 are constant.*

Proof: From the definition of $\mathcal{Z}-$tensor we have

$$\begin{aligned}
(\mathcal{T}(X_1, X_2) \cdot \mathcal{Z})(X_3, X_4) &= -\mathcal{Z}(\mathcal{T}(X_1, X_2)X_3, X_4) - \mathcal{Z}(X_3, \mathcal{T}(X_1, X_2)X_4) \\
&= (\mathcal{T}(X_1, X_2) \cdot Ric)(X_3, X_4) \\
&\quad - \psi(\mathcal{T}(X_1, X_2, X_3, X_4) + \mathcal{T}(X_1, X_2, X_4, X_3))
\end{aligned}$$

By using expression of $\mathcal{T}$-curvature tensor the folowing equlity is hold for $a_1 = -a_5$, $a_2 = -a_4$ and $a_3 = a_6 = 0$:

$$\mathcal{T}(X_1, X_2, X_3, X_4) + \mathcal{T}(X_1, X_2, X_4, X_3) = 0. \tag{4}$$

□

It is obvious that for Riemannian curvature tensor the equation **??** is hold. So, we can state; M is $\mathcal{Z}-$semi-symmetric if and only if it is Ricci-semi-symmetric. When we check the table we also see that quasi-conformal curvature tensor satisfies above theorem. So, we state; M is quasi-conformal-$\mathcal{Z}-$semi-symmetric if and only if quasi-conformal-Ricci-semi-symmetric.

A tensor $\mathcal{P}$ of a type $(1, 3)$ on a Riemannian manifold M is called a curvature tensor if it satisfies following properties;

- $\mathcal{P}(X_1, X_2) = -\mathcal{P}(X_2, X_1)$
- $g(\mathcal{P}(X_1, X_2)X_3, X_4) = -g(\mathcal{P}(X_1, X_2)X_4, X_3)$
- $\mathcal{P}(X_1, X_2)X_3 + \mathcal{P}(X_2, X_3)X_1 + \mathcal{P}(X_3, X_1)X_2 = 0$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$. The Riemannian curvature tensor is a well known example. Some tensors do not satisfy second identity. These type of tensors are called as curvature like tensors. Quasi- conformal ($\tilde{\mathcal{C}}$), conformal ($\mathcal{C}$), conharmonic ($\mathcal{K}$), concircular ($\mathcal{V}$) and M-projective ($\mathcal{W}$) curvature tensors satisfy second identity.

Therefore we state following result;

Corollary 1. *Let M be an $n-$dimensional Riemannian manifold. M is $\mathcal{T}-\mathcal{Z}$-semi-symmetric if and only if $\mathcal{T}-$Ricci-semi-symmetric and $\mathcal{T}$ is a curvature tensor.*

Suppose that M is an Einstein manifold. It is obvious that for an Einstein manifold we have

$$(\mathcal{T}(X_1, X_2) \cdot \mathcal{Z})(X_3, X_4) = (\lambda + \psi)(\mathcal{T}(X_1, X_2, X_3, X_4) + \mathcal{T}(X_1, X_2, X_4, X_3)).$$

By consider last equation we state;

Theorem 3. *An Einstein manifold is $\mathcal{T}-$ $\mathcal{Z}$-semi-symmetric if and only if $\lambda + \psi = 0$ or $\mathcal{T}$ is a curvature tensor.*

Thus we get following result;

Corollary 2. *Let M be an Einstein manifold. M is quasi-conformal-$\mathcal{Z}$-semi-symmetric, conformal-$\mathcal{Z}$-semi-symmetric, conharmonic-$\mathcal{Z}$-semi-symmetric,concircular-$\mathcal{Z}$-semi-symmetric and M-preojective-$\mathcal{Z}$-semi-symmetric.*

An Einstein manifold is not $\mathcal{T}$-Ricci-semi-symmetric in general. By consider the definition of $\mathcal{T}$-curvature tensor we have following result.

Theorem 4. *An Einstein manifold is $\mathcal{T}$-Ricci-semi-symmetric if and only if $a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6 = 0$.*

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On Eigenvalues of the Laplacian with the Dirichlet Condition at End-Vertices on Simple Quantum Trees

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Abstract: An eigenvalue problem of the edge-based laplacian on simple metric tree is considered. Of particular interest is the analysis of their spectral properties via adjacency calculus, and secular determinants. This investigation is carried out on an equilateral graph with Dirichlet boundary conditions at end-vertices. We show that the eigensystems obtained from the two quantitative analyzes are similar. Furthermore, the behavior of the eigenfunctions obtained from generic cases through several simulations are described. These results is the first step towards determining the solution of the edge-based wave equation on quantum trees.

Keywords: Dirichlet, Eigenvalue, Laplacian, Quantum tree.

1 Introduction

A metric graph is a combinatorial graph that is accompanied by mapping from an edge to a positive real number. Therefore, when the metric graph is given a differential operator that works at its edges and is accompanied by boundary conditions on each vertex, then it becomes a quantum graph. This quantum structure has appeared in scientific work in the first chemistry study in the 1930s. Furthermore, the structure is used to model the movement of electrons between atoms. In accordance with [1, 2], which contains a collection of quantum graph topics with an emphasis on spectral analysis, this paper presented spectral theory discussion on several types of quantum graphs. From a wide variety of graphs, this study focuses on quantum trees, graphs with no cycles. This selection is based on the quantum tree having several end-vertex that can be given the Dirichlet vertex boundary condition. Moreover, only two types of quantum trees are discussed, namely the simple star graph and the quantum tree $P_2 \rhd S_2$ with Dirichlet conditions at end-vertices. The differential operator assigned to each section is the negative Laplacian, which is a second derivative for the defined function in each section. This operator works on an edge, therefore it is called edge-based Laplacian as with the concept introduced by Friedmann in [3] about Calculus on graphs. The results obtained in this paper is a modification of quantitative spectral analysis via adjacency calculus to determine the eigenvalue of Laplacian on quantum graphs. This method was used by Friedmann, Wilson, and Aziz in [3]-[5], and introduced by von Below (1984). Also, this study modified the method to determine the eigenvalue of Laplacian on quantum trees that have Dirichlet conditions at the end vertices. To verify the results, the quantitative spectral analysis was used via secular determinant (see in [1, 6]), which is constructed by a scattering matrix from each vertex. In addition, a simulation of the eigenfunction is presented from the Laplacian's eigenvalue on the star graph and quantum tree $P_2 \rhd S_2$.

2 Theoretical review

This study refers to [1, 6, 7] which discussed two methods for calculating the spectrum of the Laplacian on a quantum graph, namely adjacency calculus and secular determinants. This assumes the finite graph notation as $G = (V, E)$, with $V, E \neq \emptyset$. Let $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, ..., e_N\}$ with $n, N \in \mathbb{N}$. Two vertices of $v_i, v_j \in V$ are said to be adjacent, denoted as $v_i \sim v_j$, when there is an edge $e \in E$ such that $e = \{v_i, v_j\}$. Each edge $e = \{v_i, v_j\}$ is marked with a certain direction, by classifying the two adjacent vertices v_i, v_j as the start and end vertex. When v_i is the initial vertex and v_j is the terminal vertex, it can be denoted as $e = (v_i, v_j)$. First we assume a combinatorial graph.

Assumption 1. *The combinatorial graph $G = (V, E)$ is assumed to be a non-empty, directed, connected, finite and simple graph.*

Suppose that $G=(V,E)$ is a combinatorial graph given the numbering of the vertices and the edges $I_V:=\{1,...,n\}$ and $I_E:=\{1,...,N\}$. The mapping index $p: I_V\times I_V\to I_E$ is defined as follows

$$p(i,j):=\begin{cases} l, \text{ if } e_l=\{v_i,v_j\}, \\ 1, \text{ others.}\end{cases}$$

Next is defined several matrices represent a graph, starting from the adjacency matrix.

Definition 2. *(Adjacency Matrix) Suppose that $G=(V,E)$ is a graph that satisfies the* Assumption 1 *with the set of vertices $V=\{v_1,v_2,...,v_n\}$. The adjacency matrix $A:=(a_{ij})\in\{0,1\}^{n\times n}$ is defined in the form*

$$a_{ij}=\begin{cases} 1, & \text{if } v_i\sim v_j; \\ 0, & \text{if } v_i\nsim v_j.\end{cases}$$

In this section a transition matrix is defined which describes the possible displacements from one vertex to another.

Definition 3. *(Transition Matrix) Suppose that $G=(V,E)$ is a graph that satisfies the* Assumption 1 *with the adjacency matrix A. Let $\overline{n}:=(1)_{n\times 1}$, then it can be defined the transition matrix as $\widetilde{A}\in[0,1]^{n\times n}$ with the form*

$$\widetilde{A}:=[Diag(A\overline{n})]^{-1}A.$$

The transition matrix entries can be written as follows,

$$\widetilde{a}_{ij}=\frac{a_{ij}}{\sum_j a_{ij}}.$$

Let $e=(v_i,v_j)$, is defined by $o(e):=v_i$ as the initial vertex and $t(e)=v_j$ as the terminal vertex of the directed edge $e:=(v_i,v_j)$.

Definition 4. *(Incidence matrix) Let $G=(V,E)$ is a graph that satisfies the* Assumption 1 *with the set of vertices $V=\{v_1,v_2,...,v_n\}$ and $E=\{e_1,e_2,...,e_N\}$. The incidence matrix $Q:=(q_{ij})\in\{-1,0,1\}^{n\times N}$ is defined with*

$$q_{ij}=\begin{cases} -1 & \text{if } v_i=o(e_j), \\ 1 & \text{if } v_i=t(e_j), \\ 0 & \text{others.}\end{cases}$$

This paper will discuss the concept of quantum graphs and point conditions that accompany differential operators.

Definition 5. *(Metric Graph) A metric graph $G_m:=(G,m)$ is an ordered pair containing a combinatorial graph $G=(V,E)$ and a mapping of $m:E\to(0,\infty]$ with $e\mapsto m(e):=m_e$. When $|V|<\infty$ and $m_e<\infty$ for each $e\in E$ then G_m is compact. Also, when $m_e=1$ for every e then the metric graph is equilateral.*

Definition 6. *Let $G_m:=(G,m)$ be a compact metric graph with G that satisfies the Assumptions (1) and each edge $e\in E$ identified by interval $[0,l_e]$. Suppose $1\le p<\infty$ and $k\in\mathbb{N}_0$ with $L^p(e):=L^p(0,l_e)$ and $H^k(e):=H^k(0,l_e)$. Then we define the graph function space $\left(\widetilde{L}^p(G),\|\cdot\|_{\widetilde{L}^p}\right)$ and $\left(\widetilde{H}^k(G),\|\cdot\|_{\widetilde{H}^k}\right)$ with the form*

$$\widetilde{L}^p(G):=\oplus_{e\in E}L^p(e) \text{ with } \|f\|^2_{\widetilde{L}^p(G)}:=\sum_{e\in E}\|f\|^2_{L^p(e)},\ f\in\widetilde{L}^p(G),$$
$$\widetilde{H}^k(G):=\oplus_{e\in E}H^k(e) \text{ with } \|f\|^2_{\widetilde{H}^k(G)}:=\sum_{e\in E}\|f\|^2_{H^k(e)},\ f\in\widetilde{H}^p(\Gamma).$$

A quantum graph is an ordered pair $(G_m,\mathcal{H})$ consisting of a graph with a metric G_m and a differential operator (Hamiltonian) (Hamiltonian) $\mathcal{H}$ accompanied by the vertex conditions. The following introduces some vertex conditions defined at $v\in V$ as follows, let $f\in\widetilde{H}^2(G)$, then some boundaries on the vertex include:

1. Neumann-Kirchoff condition, let $v\in V$
$$\begin{cases} f_e(v)=f_{\overline{e}}(v) \text{ for each } e,\overline{e}\in E_v, \\ \sum_{e\in E_v} d_{v,e}f'_e(v)=0.\end{cases}$$
2. Dirichlet condition, $f_e(v)=f_{\overline{e}}(v)=0$ for each $e,\overline{e}\in E_v$, with $E_v=\{e\in E\mid v\in e\}$.

Furthermore, the Hamilton operator $-\Delta_G$ with $f_e\mapsto -\frac{d^2}{dx_e^2}f_e$ for each $e\in E$ is called an edge-based Laplacian that works on edges in a quantum graph. The notation $-\Delta_G^K$ means the edge-based Laplacian with Neumann-Kirchoff vertex conditions at each vertex in the quantum graph. This concept was introduced by Friedmann and Tillich in [3].

2.1 Adjacency calculus for Neumann boundary conditions

In 2019, MA Klawonn provided a method for calculating the edge-based Laplacian spectrum on metric graphs G_m with Neumann-Kirchhoff boundary conditions. Furthermore, given a system of equations for the eigenvalues problem of the edge-based Laplacian with the following Neumann Kirchhoff conditions

$$\begin{array}{ll} -f_j'' = \lambda f_j & \text{for each } j \in \{1, ..., N\}, \\ f_i(v) = f_j(v) & \text{for each } v \in e_i \cap e_j, \\ \sum\limits_{j=1}^{N} d_{ij} f_j'(v_i) = 0 & \text{for each } i \in \{1, ..., n\}. \end{array} \tag{1}$$

In the above formula, the function f_j is defined on the edge e_j. Furthermore, the spectrum of the edge-based Laplacian is obtained from the transition matrix $\widetilde{A}(G)$ with the following relationship:

Proposition 7. *(Klawonn [7]) Let $\left(G_m, -\Delta_G^K\right)$ is a quantum graph of the Kirchoff condition with transition matrix $\widetilde{A}$ as in the problem of the system of eigen equations (1), then the spectrum of the operator is given by*

$$\sigma\left(-\Delta_G^K\right) = \{0\} \cup \left\{\lambda > 0 : \cos\sqrt{\lambda} \in \sigma\left(\widetilde{A}\right)\right\} \cup \left\{\lambda > 0 : \cos\sqrt{\lambda} = -1\right\}.$$

As a result, we get an explicit expression of the eigenvalues of the edge-based Laplacian on a metric graph G_m.

Corollary 1. *As illustrated in* Proposition 7*, if the eigenvalues of $\widetilde{A}$ are denoted by $1 = \mu_1 > \mu_2 > ... > \mu_n \geq -1$, then the eigenvalues $\lambda_{r,h}$ $(h \in \mathbb{N}_0)$ of the quantum graph $-\Delta_G^K$ are given by*

$$\lambda_{r,h} = \begin{cases} 0 & \text{if } r = 1, h = 0, \\ (2h)^2 \pi^2 & \text{if } r = 1, h \neq 0, \\ (2h\pi \pm \arccos(\mu_r))^2 & \text{if } 1 < r < n, \\ (2h+1)^2 \pi^2 & \text{if } r = n \text{ and } G \text{ is bipartite,} \\ (2h\pi \pm \arccos(\mu_r))^2 & \text{if } r = n \text{ and } G \text{ is not bipartite,} \\ (2h+1)^2 \pi^2 & \text{if } r = n+1 \text{ and } G \text{ is not bipartite.} \end{cases}$$

2.2 Secular determinant

In addition to using adjacency calculus analysis, the edge-based Laplacian spectrum on a quantum graph can be calculated through another approach. This method is used on compact metric graphs in general, namely for m_e which differs on each edge e. This approach is called Secular Determinant which can be a tool to verify the accuracy of the results to determine the spectrum of the Laplacian using a modified adjacency calculus. In this paper, the discussion of the method refers to [1, 6].

The eigenvalue problem $-\Delta f_j = \lambda f_j = k^2 f_j$ is given on each edge j. Furthermore, the solution to the eigenvalue problem is by using the plain wave, including,

$$f_j(x) = a_j e^{ikx} + a_{\bar{j}} e^{ik(L_j - x)}, \tag{2}$$

with L_j is the length of the edge j. Because the coefficient $a_j, a_{\bar{j}}$ will be searched for each j, then the relationship represented by the following matrix multiplication is obtained,

$$SD(k)\, a = a,$$

where $S := (s_{ij}) \in \mathbb{C}^{2|E| \times 2|E|}$ is the scattering matrix, the diagonal matrix $D(k) := Diag\left(e^{ikL_j}\right)$, and the vector $a := (a_j)_{2|E| \times 1}$. Meanwhile, the S matrix has the following entries:

$$s_{j',j} = \begin{cases} \frac{2}{d} - 1 & \text{if } j' = \bar{j}, \\ \frac{2}{d} & \text{if } j' \text{ follows } j \text{ and } j' \neq \bar{j}, \\ 0 & \text{others.} \end{cases}$$

With j' follows j when $t(j) = o(j')$ and d are denoted as degrees of the vertex $t(j)$.

Furthermore, the facts obtained for calculating the Laplacian spectrum are presented in the following theorem.

Theorem 8. *(Berkolaiko [1]) Suppose G is a compact quantum graph, then $k^2 \in \mathbb{C} \setminus \{0\}$ the eigenvalue of the operator $-\Delta$ if only if k satisfies the equation*

$$\det(I - SD(k)) = 0,$$

where $D(k) := Diag(z_j)$ with $z_j = e^{ikL_j}$ and L_j is the length of the edge corresponding to the coefficient a_j.

The function $\Sigma(k) := \det(I - SD(k))$ is called the Secular Determinant.

3 Results and discussion

3.1 Adjacency calculus modification

The adjacency calculus modification contains new concepts created by defining the Dirichlet adjacency and transition matrices. First in this section, a new adjacency matrix is defined. That is, the adjacency matrix for a given graph with Dirichlet conditions at its end-vertex.

Definition 9. *(Dirichlet adjacency matrix) Let $(G_m, \mathcal{H})$ is a quantum graph with the set of vertices $V = \{v_1, v_2, ..., v_n\}$. Suppose the Dirichlet boundary condition is given to $v \in V$, i.e $f(v) = 0$. The Dirichlet adjacency matrix $A_D := \left(a_{ij}^D\right) \in \{0,1\}^{n\times n}$ is defined by the form*

$$a_{ij}^D = \begin{cases} 1, & \text{if } v_i \sim v_j \text{ and } f(v_i) \neq 0; \\ 0, & \text{if } v_i \sim v_j \text{ and } f(v_i) = 0; \\ 0, & \text{if } v_i \nsim v_j. \end{cases}$$

In addition to the adjacency matrix, later in the modified adjacency calculus, we need a new definition of the transition matrix.

Definition 10. *(Dirichlet transition matrix) Let $(G_l, \mathcal{H})$ is a quantum graph with an adjacency matrix A. Suppose $\overline{n} := (1)_{n\times 1}$, define a Dirichlet transition matrix $\widetilde{A}_D \in [0,1]^{n\times n}$ with the form*

$$\widetilde{A}_D := [Diag\,(A\overline{n})]^{-1} A_D.$$

Lemma 11. *(Spectrum $\widetilde{A}_D$) Let $(G_m, \mathcal{H})$ is a quantum graph with adjacency matrix A, transition matrix $\widetilde{A}$ with eigenvalues $1 = \gamma_1 \geq ... \geq \gamma_n \geq -1$ and Dirichlet adjacency matrix A_D, then $\widetilde{A}_D$ is irreducible and has real eigenvalues $\mu_1 \geq ... \geq \mu_k \in \mathbb{R}$ (where k is the number of nonzero rows of matrix $\widetilde{A}_D$) which can be sorted and satisfies*

$$\gamma_{n-k+j} \geq \mu_j \geq \gamma_j \text{ for each } j \leq k.$$

Proof: This can be done in the same way as the proof of the transition matrix. Let $B := Diag\,(A\overline{n})$ and are given a symmetric matrix $X = B^{-1/2}AB^{-1/2}$. Then define the matrix $X_D := B^{-1/2}A_DB^{-1/2}$ and let $x \in \mathbb{C}^n$ is the eigenvector of $\widetilde{A}_D$ corresponding to the eigenvalue μ. Define the vector $\widetilde{x} := B^{1/2}x$, then

$$\widetilde{A}_Dx = \mu x \Leftrightarrow A_Dx = \mu Bx.$$

Next, consider that

$$A_DB^{-1/2}B^{1/2}x = \mu B^{1/2}B^{1/2}x.$$

Then obtained

$$B^{-1/2}A_DB^{-1/2}\widetilde{x} = \mu\widetilde{x} \Leftrightarrow X_D\widetilde{x} = \mu\widetilde{x}.$$

Therefore $\sigma\left(\widetilde{A}_D\right) = \sigma(X_D)$. X_D is the matrix obtained from the matrix X which converted a row into a zero row. Also, X_D is singular and has eigenvalue 0 as many rows of zeros. When a matrix X_D' is formed by removing the zero row and the corresponding column with the row index, then the main submatrix X is obtained. By using the Cauchy Interlacing Theorem, the aim was achieved. □

3.2 Investigating the Dirichlet boundary condition problem on a star graph and a quantum tree

As explained in the introduction, the type of quantum graph discussed is a quantum tree. Therefore, this study aims to determine the spectrum of the Laplacian on a star graph in two cases and one case on a tree $P_2 \rhd S_2$. These investigations were carried out to construct a modified Adjacency Calculus. In the first step, the three eigenvalue problem systems for each case were written. Furthermore, the calculation and analysis process was carried out towards the final result. This aims to obtain Adjacency Calculus for the eigenvalue problem of the Laplacian with Dirichlet conditions on end-vertices on simple quantum trees. Let an equilateral quantum star graph with N edges. Firstly, the system of eigenvalue problem of edge-based Laplacian with Dirichlet vertex conditions is given to all the end-vertices as follows,

$$\begin{array}{ll} -f_j'' = \lambda f_j & \text{for each } j \in \{1, ..., N\}, \\ f_i(v_{cen}) = f_j(v_{cen}) & \text{for } v_{cen} \in e_i \cap e_j, \\ \displaystyle\sum_{j=1}^{N} d_{ij} f_j'(v_i) = 0 & \text{for } v_i = v_{cen}, \\ f_j(v_i) = 0 & \text{for each } i = j \in \{1, ..., N\}. \end{array} \tag{3}$$

Secondly, the quantum star graph with the Dirichlet condition at one end-vertex. The system of eigenvalue problem of edge-based Laplacian includes:

$$\begin{array}{ll} -f_j'' = \lambda f_j & \text{for each } j \in \{1, ..., N\}, \\ f_i(v_{cen}) = f_j(v_{cen}) & \text{for } v_{cen} \in e_i \cap e_j, \\ \displaystyle\sum_{j=1}^{N} d_{ij} f_j'(v_i) = 0 & \text{for each } i \in \{2, ..., n\}, \\ f_j(v_i) = 0 & \text{for each } i = j = 1. \end{array} \tag{4}$$

Finally, the quantum tree $P_2 \rhd S_2$ has a Dirichlet condition at its two end-vertices. The system of eigenvalue problem of edge-based Laplacian is given as follows

$$\begin{array}{ll} -f_j'' = \lambda f_j & \text{for each } j \in \{1, ..., 5\}, \\ f_i(v_{int}) = f_j(v_{int}) & \text{for } v_{int} \in e_i \cap e_j, \\ \displaystyle\sum_{j=1}^{N} d_{ij} f_j'(v_i) = 0 & \text{for each } i \in \{3, ..., 6\}, \\ f_j(v_i) = 0 & \text{for each } i = j \in \{1, 2\}. \end{array} \tag{5}$$

In the three formulas above, f_j is defined on the edge e_j. Also, the vertex labeling for the three cases above starts from the end-vertices of each graph which is illustrated in the following figure:

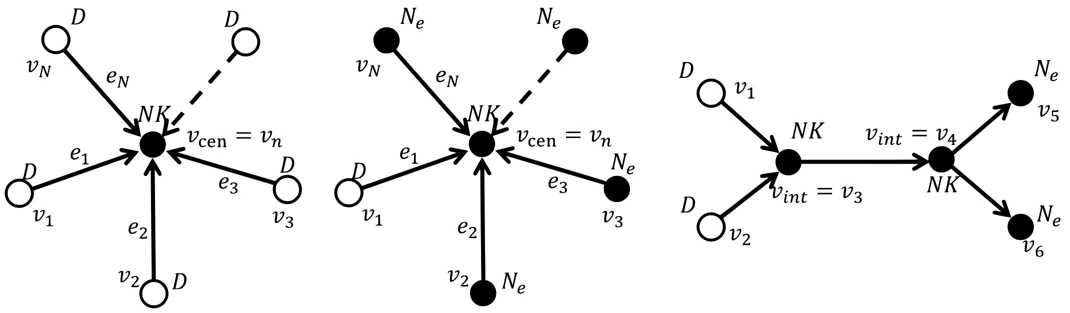

Fig. 1: D for the Dirichlet, N_e for Neumann, and NK for the Neumann-Khirchoff conditions.

Furthermore, to simplify calculations and analysis, this study follows the process and references [7] by using several new concepts and slightly different steps. These steps are performed on an equilateral quantum star graph with N edges which can generally represent a quantum tree that has Dirichlet condition at end-vertices.

Definition 12. *In the system of equations (3) and (4), the matrix valued-function $W := \left(w_{ij}\right)$ is defined using the entries from the adjacency matrix $A = \left(a_{ij}\right)$ and the sign of the incidence matrix $Q = \left(q_{ij}\right)$ as follows,*

$$\begin{aligned} &w_{ij} : [0,1] \to \mathbb{C}^{n\times n} \\ &\textit{where } x \longmapsto w_{ij}(x) := a_{ij} f_{p(i,j)} \left(\frac{1 + q_{i,p(i,j)}}{2} - x q_{i,p(i,j)} \right). \end{aligned}$$

Lemma 13. *In the situation given in the* Definition 12*, it can be concluded that*

$$\begin{aligned} W^*(x) &= W(1-x), \\ W'(x)^* &= -W'(1-x). \end{aligned}$$

In the following discussion, we will calculate the general matrix W of the systems (3) and (4) in a star graph. Given an Adjacency and Incident Matrix as follows:

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$\begin{aligned} &w_{ij}(x) = 0 \text{ if } a_{ij} = 0, \\ &w_{14}(x) = 1.f_1\left(\frac{1+q_{11}}{2} - x q_{11}\right) = f_1(x), \text{etc.} \end{aligned}$$

$$W(x) = \begin{bmatrix} 0 & \cdots & 0 & f_1(x) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & f_N(x) \\ f_1(1-x) & \cdots & f_N(1-x) & 0 \end{bmatrix}$$

Therefore:

$$\begin{aligned} -W'' &= \begin{bmatrix} 0 & \cdots & 0 & -f_1''(x) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -f_N''(x) \\ -f_1''(1-x) & \cdots & -f_N''(1-x) & 0 \end{bmatrix} \\ &= \lambda \begin{bmatrix} 0 & \cdots & 0 & f_1(x) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & f_N(x) \\ f_1(1-x) & \cdots & f_N(1-x) & 0 \end{bmatrix} = \lambda W. \end{aligned}$$

So that

$$-W'' = \lambda W. \tag{6}$$

Let

$$u := (f(v_i))_{n\times 1} = \begin{bmatrix} f(v_i) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix} \text{ and } \overline{n} = (1)_{n\times 1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

where $f(v_i) = 0$ for i (Dirichlet condition at ith vertex). Hence, the relationship is obtained

$$\begin{aligned} W(0) &= \begin{bmatrix} 0 & \cdots & 0 & f_1(0) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & f_N(0) \\ f_1(1) & \cdots & f_N(1) & 0 \end{bmatrix} \\ &= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(v_n) \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & 1 \end{pmatrix} \cdot A. \end{aligned}$$

Therefore

$$U(0) = u\overline{n}^* \cdot A, \tag{7}$$

where $\cdot$ is Hadamard's multiplication on the matrix.

Let

$$W'(0) = \begin{bmatrix} 0 & \cdots & 0 & f_1'(0) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & f_N'(0) \\ -f_1'(1) & \cdots & -f_N'(1) & 0 \end{bmatrix}, \ \overline{n} := \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}_{n\times 1},$$

Hence, the relationship is obtained

$$\left[W'(0) \cdot A_D\right] \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

namely

$$\left[U'(0) \cdot A_D\right] \overline{n} = 0. \tag{8}$$

From (6), (7), and (8), we get the system of eigenvalue problem of the matrix valued-function as follow

$$\begin{array}{ll} -W'' = \lambda W, & \text{Eigenvalue problem } a) \\ W(0) = u\overline{n}^* \cdot A, & \text{continuity and Dirichlet condition } b) \\ \left[W'(0) \cdot A_D\right] \overline{n} = 0. & \text{Neumann-Kirchoff condition } c) \end{array} \tag{9}$$

Where $u := (f(v_i))_{n\times 1} \in \mathbb{C}^n$ and $\overline{n} := (1)_{n\times 1}$ and $w_{ij}(0) = f_{s(i,j)}(v_i) = f(v_i)$.

In the context of the matrix valued-function and its eigensystems, we define

$$M_1 := W(0) \text{ and } M_2 := W'(0).$$

The general solution of system (6) is shown in the following lemma,

Lemma 14. *(General Solution) Let W is the nontrivial solution of the system of eigenvalue problem of the matrix valued-function related to the eigenvalues λ, then $\lambda \in [0, \infty)$ and W is of the form*

$$W(x) = \begin{cases} M_1 + x(M_1^* - M) \text{ if } \lambda = 0, \\ \cos\left(\sqrt{\lambda}x\right) M_1 + \dfrac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}} M_2 \text{ if } \lambda > 0 \end{cases}$$

Proof: Assume that $\lambda = 0$, thus

$$W'' = 0,$$

therefore, the form is as follows

$$W(x) = M_1 + xM, \text{ for } M \in \mathcal{M}$$

where $\mathcal{M} := \left\{M = (m_{ij}) \in \mathbb{C}^{n\times n} : a_{ij} = 0 \Longrightarrow m_{ij} = 0\right\}$. Furthermore, using $W^*(x) = W(1-x)$, and substitute $x = 1$ in the above equation to get

$$W(1) = M_1^* = M_1 + M, \text{ therefore } M = M_1^* - M_1,$$

which gives the form for $\lambda = 0$.

All solutions $W'' = -\lambda W$ and their derivatives are given as

$$W(x) = \sin\left(\sqrt{\lambda}x\right) A + \cos\left(\sqrt{\lambda}x\right) B,\ A, B \in \mathcal{M} \tag{10}$$
$$W'(x) = \sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) A - \sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) B,\ A, B \in \mathcal{M}.$$

Substituting $x = 0$ into (10) to get $A = \dfrac{M_2}{\sqrt{\lambda}}$ and $B = M_1$, which gives an expression for $\lambda > 0$. □

It is necessary to know the fact in the Hadamard multiplication of the matrix that,

Lemma 15. *If $C = \left(c_{ij}\right)_{m\times n}$, $B = \left(b_{ij}\right)_{m\times n} \in \mathbb{R}^{m\times n}$ and $x = (x_i)_{n\times 1} \in \mathbb{R}^n$, then*

$$((C \cdot B)\, x)_i = \left(C Diag(x)\, B^T\right)_{ii}. \tag{11}$$

3.3 The spectrum of the Laplacian with the Dirichlet condition at end-vertices on simple quantum trees

From the previous discussion, we can obtain the spectrum of the edge-based Laplacian on the equilateral quantum tree G_m with the Dirichlet boundary conditions at the end-vertices from the transition matrix $\widetilde{A}_D$ with the following relationship,

Proposition 16. *(Dirichlet Spectrum end-vertices) Suppose $(G_m, -\Delta_G)$ is a quantum tree of the Dirichlet condition at end-vertices with a transition matrix $\widetilde{A}_D$ obtained from matrix A_D and $u = \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix}_{n\times 1}$ with $f(v_i) = 0$ for i (Dirichlet condition at the ith vertex). Then the operator spectrum is given as follows*

$$\sigma(-\Delta_G) = \left\{\lambda > 0 : \cos\sqrt{\lambda} \in \sigma\left(\widetilde{A}_D\right)\right\},$$

with multiplicity $(m(\lambda))$ is given by

$$m(\lambda) = \#\left\{u = \alpha\eta \mid \eta \textit{ eigenvector of } \cos\sqrt{\lambda}\right\},\ \textit{if } \lambda > 0 \textit{ and } \sin\sqrt{\lambda} \neq 0.$$

With $\#$ is denoted as the number of elements of a set (cardinality).

Proof: For $\lambda > 0$, $\sin\sqrt{\lambda} \neq 0$, the following solutions and derivatives were given,

$$W(x) = \cos\left(\sqrt{\lambda}x\right) M_1 + \frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}} M_2 \tag{12}$$
$$W'(x) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) M_1 + \cos\left(\sqrt{\lambda}x\right) M_2.$$

Substitute $x = 1$ into the equation of the first form, using $W(1) = W^*(0) = M_1^*$ and as a result of the continuity of the system of eigenvalue problem of the matrix valued-function, the results are as follow

$$W(1) = M_1^* = \cos\left(\sqrt{\lambda}\right) M_1 + \frac{\sin\left(\sqrt{\lambda}\right)}{\sqrt{\lambda}} M_2$$
$$M_2 = \frac{\sqrt{\lambda}}{\sin\left(\sqrt{\lambda}\right)}\left(M_1^* - \cos\left(\sqrt{\lambda}\right) M_1\right).$$

Because $W(0) = u\overline{n}^* \cdot A = M_1$, then

$$M_2 = \frac{\sqrt{\lambda}}{\sin\left(\sqrt{\lambda}\right)}\left(M_1^* - \cos\left(\sqrt{\lambda}\right) M_1\right) \tag{13}$$
$$= \frac{\sqrt{\lambda}}{\sin\left(\sqrt{\lambda}\right)}\left(\left[u\overline{n}^* \cdot A\right]^* - \cos\left(\sqrt{\lambda}\right) u\overline{n}^* \cdot A\right).$$

Form (13) is a new expression for M_2.

Substitute the second form into the Neumann-Kirchoff rule and according to equation (11), it can be calculated that

$$\left[W'(0)\cdot A_D\right]\overline{n}=0 \Longleftrightarrow \left[\left(\overline{n}u^*\cdot A-\cos\left(\sqrt{\lambda}\right)u\overline{n}^*\cdot A\right)\cdot A_D\right]\overline{n}=0,$$

thus

$$\left(\overline{n}u^*\cdot A_D\right)\overline{n}=\left(\cos\left(\sqrt{\lambda}\right)u\overline{n}^*\cdot A_D\right)\overline{n}.$$

By using the Hadamard multiplication property, then

$$\left(A_D u\overline{n}^*\right)_{ii}=\cos\left(\sqrt{\lambda}\right)\left(A_D\overline{n}u^*\right)_{ii}.$$

It means

$$\sum_{j=1}^{n}a_{ij}^{D}u_j=\cos\left(\sqrt{\lambda}\right)u_i\sum_{j=1}^{n}a_{ij}^{D}\Longleftrightarrow\sum_{j=1}^{n}a_{ij}^{D}u_j=\cos\left(\sqrt{\lambda}\right)u_i\sum_{j=1}^{n}a_{ij}.$$

Then

$$A_D u=\cos\left(\sqrt{\lambda}\right)Diag\left(A\overline{n}\right)u$$

which means that $\widetilde{A}_D u=\cos\left(\sqrt{\lambda}\right)u$.

To ascertain the multiplicity of λ , the eigenvectors corresponding to $\cos\left(\sqrt{\lambda}\right)$ need to be determined. □

Due to Proposition 16, the eigenfunction form of each side was obtained and presented in the following facts:

Corollary 2. *As illustrated in* Proposition 16*, denoted that $1>\mu_1>...>\mu_n>-1$ are the eigenvalues of $\widetilde{A}_D$, then the eigenvalue $\lambda_{r,h}$ $(h\in\mathbb{N}_0)$ of the quantum tree $-\Delta_G$ is given by*

$$\lambda_{r,h}=\left(2h\pi\pm\arccos\left(\mu_r\right)\right)^2.$$

If $\eta_r=\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{bmatrix}_{n\times 1}$ *with $\eta_i=0$ for i is an eigenvector of $\widetilde{A}_D$ corresponding to μ_r. Then the eigenfunction $W_{r,h}(x)$ is given as follows*

$$W_{r,h}(x)=\alpha\left(\cos\left(\sqrt{\lambda_{r,h}}x\right)\eta\overline{n}^*+\frac{\sin\left(\sqrt{\lambda_{r,h}}x\right)}{\sin\left(\sqrt{\lambda_{r,h}}\right)}\left(\overline{n}\eta^*-\cos\left(\sqrt{\lambda_{r,k}}\right)\eta\overline{n}^*\right)\right)\cdot A.$$

3.4 Comparison of modified adjacency calculus with secular determinants on a star graph

The following presents a verification of the accuracy of Adjacency Calculus modified by quantitative analysis of secular determinants. It determines the edge-based Laplacian spectrum with Dirichlet conditions at the edge vertices of a simple quantum star graph.

Let $z_1=e^{ik}$ and $k=\sqrt{\lambda}$, then spectrum of the star graph from the modified adjacency calculus and the secular determinants are presented in the form of the secular factor equation below.

For all end-vertices Dirichlet condition

edge	Secular Determinant
N	$\left(z_1^2+1\right)\left(z_1-1\right)^{N-1}\left(z_1+1\right)^{N-1}$

edge	Modified Adjacency Calculus
N	$\left(z_1^2+1\right)$

For one end-vertex Dirichlet condition

n	Secular Determinant
4	$\frac{1}{3}\left(z_1^2+1\right)\left(3z_1^4-2z_1^2+3\right)$
5	$\left(z_1^2+1\right)^2\left(z_1^4-z_1^2+1\right)$
6	$\frac{1}{5}\left(z_1^2+1\right)^3\left(5z_1^4-6z_1^2+5\right)$

n	Modified Adjacency Calculus
4	$\left(3z_1^4-2z_1^2+3\right)$
5	$\left(z_1^4-z_1^2+1\right)$
6	$\left(5z_1^4-6z_1^2+5\right)$

3.5 Eigen function simulation

In this section, one of the results is given in the form of an eigenfunctions simulation. This simulation uses Python which is presented in Figures 2, 3, and 4. Furthermore, the eigenfunctions used were obtained from the effect calculation Corollary 2 using the eigenvalues of the edge-based Laplacian in the three cases.

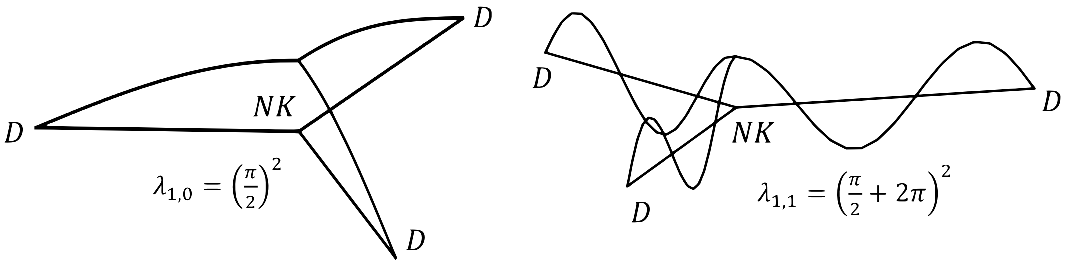

Fig. 2: Eigenfunction, $\lambda_{r,h}$ for $r = 1, h = 0, 1$ on a quantum star graph C_3 with Dirichlet conditions on the three end-vertices.

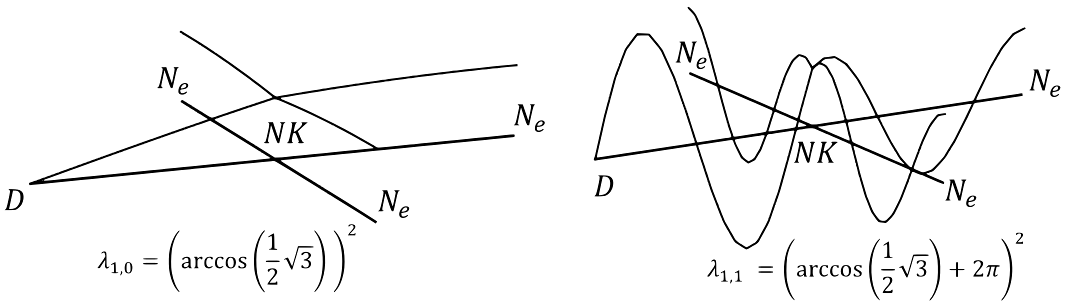

Fig. 3: Eigenfunction, $\lambda_{r,h}$ for $r = 1, h = 0, 1$ on a quantum star graph C_4 with Dirichlet conditions at one end-vertices.

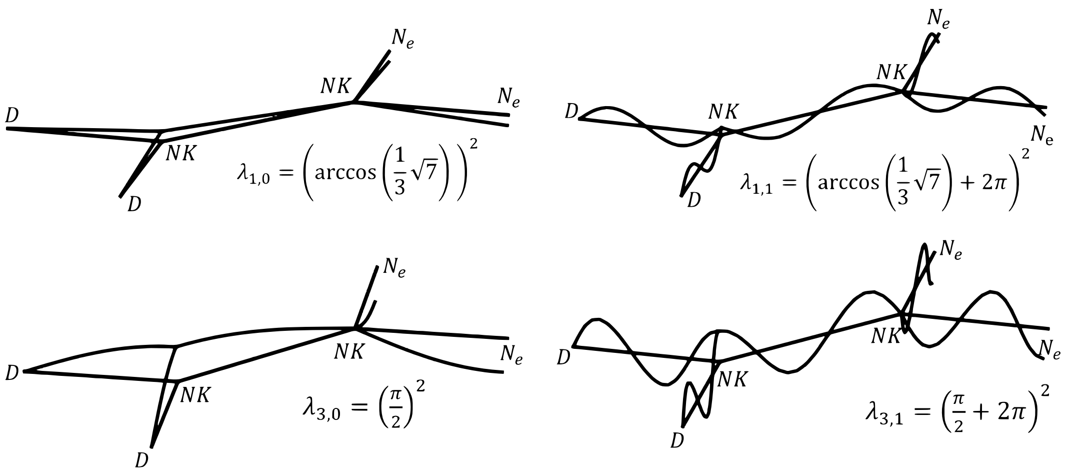

Fig. 4: Eigenfunction, $\lambda_{r,h}$ for $r = 1, h = 0, 1$ and $r = 3, h = 0, 1$ on a quantum tree $P_2 \triangleright S_2$ with Dirichlet conditions at two end-vertices.

From this simulation, it can be seen that the end-vertex with Dirichlet condition makes the eigenfunction value on the vertex is zero.

4 Conclusion

This study modifies adjacency calculus to solve the eigenvalues problem of the Laplacian with Dirichlet conditions at the end-vertices of a quantum tree. The results are not significantly different from the quantitative analysis calculation of the secular determinants. Also, the Laplacian spectrum with Dirichlet conditions at the ende-vertices using modifies adjacency calculus is a subset of the secular Determinant analysis results. This modification resulted in new concepts regarding the representation of a quantum tree, namely the Dirichlet adjacency and transition matrices. Therefore, these results can be used in future studies to solve wave equations in quantum trees with Dirichlet condition at end-vertices.

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Evaluation of Termination Proposal for Pregnant Women Exposed to Radiation with Comparison of Machine Learning Methods

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Abstract: Objectives: The objective of the study is to establish a model that can directly predict the termination recommendation of physicians to whom women refer to our center for teratological counseling making the most accurate model with the least possible variable.
Study design: Machine learning methods were performed to comprehensively analyze predict the termination recommendation of physicians to whom women refer to our center for teratological counseling.
Methods: We determined the week of exposure to radiation, exposure dose, day of admission to hospital, age, and referring institution as a dependent variables. We compared the model we created using these variables with Naïve Bayes, kNN, SVM, ANN, and Logistic regression methods.
Results: We estimated the physicians' termination recommendation with 80% accuracy in ANN method.
Conclusion: By using this model, pregnant women exposed to radiation can be detected before termination and non-indicative general losses can be prevented with appropriate teratological counseling.

Keywords: Machine learning, Pregnancy, Radiation, Termination.

1 Introduction

Most people expose ionizing radiation willingly or unwillingly in their life. This exposure mostly occurs via diagnostic radiological procedures. It is estimated that more than 3.5 billion diagnostic radiological procedures were implemented, worldwide. With the advancement of modern technology, this ratio will be more increase [1].

Radiation is a well-described physical teratogen and showed its teratogenic effect via deterministic way and have a threshold value [2], 50 mGy accepted by NRCP [3]. In general, none of the radiological procedures for diagnostic purposes exceed this threshold value. Furthermore, this teratogenic effect is also depends on radiation dose and duration of exposure. Besides,the first 28 days of pregnancy, which is also described as an all-or-none phenomenon, does not result in fetal anomalies [4]. Considering that reproductive age women make up about one quarter of the society [5],it is highly essential to provide a counselling to pregnant women appropriately by physicians [6]-[8].

Understanding, evaluating, and interpreting radiation exposure for diagnostic purposes in pregnancy and providing appropriate counseling still have a great problem for physicians [9] due to their overrated risk perception of radiation [6, 7, 9, 10]. This perception of radiation may result in the recommendation of termination in pregnancies by physicians without any scientific evidence[11]. Although many factors put forward on radiation perception of physicians [7, 9, 10, 12, 13], there is limited information on behind the possible factors on the recommendation of termination by first-contact physicians to pregnant women exposed to ionizing radiation for diagnostic purposes. However, based on the previous studies, perception of radiation does not always indicative for evaluating recommendation of termination by a physician, examination of the factors which may directly affect the decision of physicians on the recommendation of termination may more beneficial [14]. Therefore, in this article, we aimed to carry out a generalized model which explains based on influential factors on the recommendation of termination of pregnancy by first-contact physicians and compared the model with the other machine learning methods.

2 Methods

This study was implemented in Istanbul University-Cerrahpasa, Cerrahpasa Faculty of Medicine, Genetic Diseases and Diagnosis Center (GETAM) 2009-2018. Among the 10,784 women who met the eligibility criteria which are 461 pregnant women exposed to ionizing radiation for the diagnostic purpose were included in a study for teratological counseling. The study was carried out in accordance with the Declaration of Helsinki, with the approval of the Istanbul University-Cerrahpasa Faculty of Medicine ethics committee.

The database with all 5 independent variables and one dependent dichotomous variable was obtained from 461 participants. The 5 independent variables used in creating the database are the age of the pregnant, application day to GETAM, fetal dose, week, health institution that referred to GETAM for teratological consultation. The reason for the pregnant to come to GETAM was selected as the dependent(outcome) variable. The reasons for the pregnancy to come to GETAM.

Machine Learning Analysis: Machine learning (ML) is the process of using mathematical models to help a computer learn without direct instructions. This is considered a subset of artificial intelligence (AI). Machine learning uses algorithms to identify patterns in data. These patterns are also used to create a predictive data model. Just as people improve as they practice more, as the amount of data and experience increases machine learning results are, more accurate.

In this study, optimizable K-Nearest Neighbor, logistic regression, kernel Naive Bayes, quadratic support vector machine, and artificial neural network methods were used. These are classification methods in supervised learning. There are several criteria that can be used to compare the performance of machine learning models, and their choice depends on the structure of the data and the nature of the work to be done [15, 16]. With the help of the confusion matrix, we can calculate different parameters for the model, such as accuracy and precision. These values indicate how effective the methods used are. To calculate these criteria, "True positive (TP)", "True negative (TN)", "False positive (FP)" and "False negative (FN)" values in the confusion matrix are used. Used in this study, "Accuracy (ACC)", "Error Rate (ERR)", "Precision (PREC)", "Sensitivity (SENS)", "Specificity (SPEC)", "F-Measure (FM)", "Youden's Index (YI)","Kappa (κ) statistics", "True Negative Rate (TNR)","False Positive Rate (FPR)" and "Receiver Operating Characteristic Area (ROC)"criteria are given in Table 1.

Criteria	*Formula*
Accuracy	$ACC = (TP + TN)/(P + N)$
Error Rate	$EER = (FP + FN)/(TP + TN + FP + FN)$
Precision	$PREC = TP/(TP + FP)$
Sensitivity	$SENS = TP/(TP + FN)$
Specificity	$SPEC = TN/(FP + TN)$
F-Measure	$F - Measure = 2 \times TP/(2TP + FP + FN)$
Youdens Index	$YI = TPR + TNR - 1$
Kappa	$Kappa = 2 \times (TP \times TN\breve{}FNFP)/(TP \times FN + TPFP+$ $2 \times TP \times TN + FN2 + FN \times TN + FP2 + FP \times TN)$
False Positive Rate	$FPR = FP/(FP + TN)$
False Negative Rate	$FNR = FN/(FN + TP)$

Table 1 Evaluation criteria formulas

3 Results

The classification performances of the algorithms according to the classification criteria stated previously are given in Table 2 and Table 3. The confusion matrix results are shown in Table 2. The results of the comparison criteria according to machine learning algorithms are shown in Table 3. The best classification results according to the criteria were obtained with ANN.

	Logistic Regression		k-Nearest Neighbors		Naive Bayes		Support Vector Machine		Artificial Neural Network	
	1	2	1	2	1	2	1	2	1	2
1	119	137	180	76	231	25	188	68	191	72
2	117	301	88	330	180	238	89	239	65	346

Table 2 Classification performances

	Logistic Regression	k-Nearest Neighbors	Naive Bayes	Support Vector Machine	Artificial Neural Network
ACC	0.62	0.76	0.70	0.77	0.80
ROC	0.72	0.83	0.77	0.82	0.85
Precision	0.46	0.70	0.90	0.73	0.72
Sensitivity	0.50	0.67	0.56	0.68	0.75
Specificity	0.69	0.81	0.90	0.83	0.83
F1	0.485	0.69	0.69	0.71	0.74
Kappa	0.19	0.49	0.42	0.51	0.57
YI	0.19	0.49	0.42	0.51	0.57
ERR	0.38	0.24	0.30	0.23	0.20
FPR	0.31	0.18	0.09	0.17	0.17
FNR	0.49	0.32	0.43	0.32	0.25

Table 3 Overall comparisons for methods

When the results were examined in general, it was seen that the ANN method produced more successful results than other methods. It was observed that the results of the SVM and kNN methods were close to each other. It was seen that the worst classification process was obtained by the Logistic regression method. The polykernel is used for the SVM method. In the KNN method, the most suitable solution as the closest neighbor is obtained for three values.

When Table 3 is examined, it is seen that the ANN method has 80% ACC. The results of the kNN and SVM methods are also close to the results of the ANN method. The same results are valid for the weighted ROC value (Figure 1). As a result of the estimation, the sensitivity, which is the estimate of the true positives, got its highest value with the ANN method. The precision values that positively predicted the positive ones in the data set were obtained in the highest Naïve Bayes algorithm. The highest specificity value expressing confidence in the results was obtained in the Naïve Bayes algorithm. In other methods except for the logistic regression method, specificity value was obtained above 80% [17]. However, when the sensitivity and specificity rates are evaluated together, it is seen that the most optimal result is in the ANN method. When results are evaluated as the F1 measure showing a good classification performance, the Kappa measure showing that the classification is not entirely due to chance, the YI index showing the misclassification potential of classification, it is seen that the best result is in the ANN algorithm. Other algorithms are listed as SVM, NB, kNN, and LR, respectively. The approach of measurement values to 1 for these three measurement criteria indicates that the relevant model has made a good classification. Since kappa values for NB, kNN, and LR methods in Table 3 are below 0.50, it is understood that the results obtained with these methods are random. ANN value was obtained as higher than 0.50.

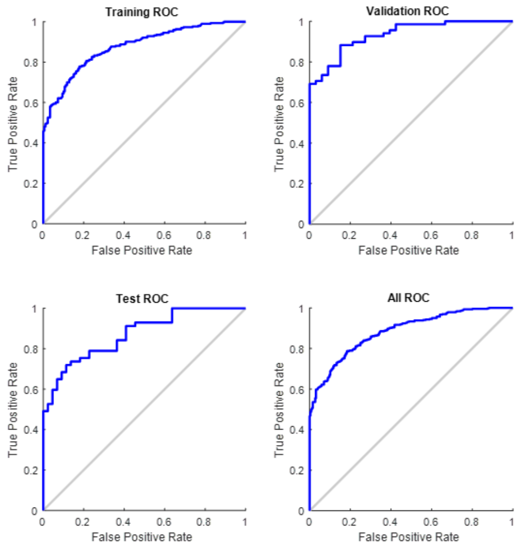

Fig. 1: Model

4 Discussion

In this study, ANN, kNN, SVM, naive Bayes, and logistic regression, which are the most frequently used machine learning methods, were used in the evaluation of termination suggestions for pregnant women exposed to radiation. The results of each machine learning method were obtained and compared with each other. Among these methods, ANN was obtained as the best performance machine learning method in predicting and explaining teratological counseling.

When machine learning algorithms are performed on a particular data set, the most important parameter that matters is the accuracy. Therefore, the outcome indicates that ANN the highest accuracy of 80% for the data set. The method proposes to use the variables age of the pregnant, application day to GETAM, fetal dose, week, referred health institute to GETAM for teratological consultation as input to the ANN, which can predict the correct teratological counseling to avoid unnecessary termination of pregnancy by future radiation-exposed women after being properly trained. The ANN classifier gave a performed as expected in terms of all parameters. Hence, the output was obtained as per the expectations.

This study did not reveal a direct relationship between the perception levels of physicians and the recommendation for termination [14]. In addition, the perception level of physicians is a subjective concept and is evaluated as a result of questionnaires. New methods with more objective variables are needed to evaluate increasing termination rates. In a study conducted in Israel, it was shown that physicians recommended termination to pregnant women who were exposed to low dose radiation of 40-70%. However, possible factors behind the recommendation of termination were not evaluated with a generalized model[18].

5 Conclusion

In this article, we tried to create a model that can accurately predict the termination recommendations of physicians. While creating this model, we wanted to make the most accurate estimation with the least possible variables. The level of awareness in the choice of variables, about radiation instead of questioning subjective conditions such as knowledge level, we questioned variables such as radiation exposure week, exposure dose, day of application, age, and referring institution. In this model, we predicted the physicians' recommendation with an accuracy of 80%. This model can be used in risk modeling of pregnant women who apply to physicians for counseling. Therefore, pregnant women exposed to radiation can be detected before termination and non-indicative general losses can be prevented with appropriate teratological counseling. Considering together that reproductive age women make up about one quarter of the society and the number of implemented radiodiagnostic methods worldwide , it is obviusly clear that we have facing a global health problem that may waited for a solve. To the best of our knowledge, there is no risk modeling and machine learning algorithm that previously predicts the termination recommendation of physicians.

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A Neural Network-Based Comparative Analysis for the Diagnosis of Emerging Different Diseases Based on COVID-19

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Abstract: Dermatological diseases are frequently encountered in children and adults for various reasons. Many factors cause the onset of these diseases and different symptoms are generally seen in each age group. Artificial neural networks can provide expert-level accuracy in the diagnosis of dermatological findings of patients with COVID-19 disease. Therefore, the use of neural network classification methods can give the best estimation method in dermatology. In this study, the prediction of cutaneous diseases caused by COVID-19 was analyzed by Scaled Conjugate Gradient, Levenberg Marquardt, Bayesian Regularization neural networks. In this investigation, the prediction capabilities of artificial neural networks were compared. At some points, Bayesian Regularization and Levenberg Marquardt were almost equally effective, but Bayesian Regularization performed better than Levenberg Marquard and called Conjugate Gradient in performance. It is seen that neural network model predictions achieve the highest accuracy. For this reason, artificial neural networks can classify these diseases as accurately as human experts in an experimental setting.

Keywords: Bayesian regularization neural network, COVID-19, Dermatological findings, Levenberg-–Marquardt neural network, Scaled conjugate gradient neural network.

1 Introduction

Artificial neural networks are one of the artificial intelligence technologies that simulate the working structure of the human brain, analyze new data, and create new information with different learning algorithms. Artificial neural networks are inspired by biological neurons (nerve cells) and have emerged as a result of artificial simulation studies of the brain's work system. The artificial neural network has started to be used frequently in medical diagnostics recently and it will be seen more in all biomedical systems shortly.

Artificial neural networks have two learning types as unsupervised and supervised. The neural network is trained according to patterns related to input and output, in supervised learning. During this process, the neural networks calculate the weights differently until appropriate results are produced and stop the process when it achieves the most appropriate output. In unsupervised learning, there are only data and no information is given about them. Generally, feed-forward neural networks are trained to employ Back Propagation.This operation is the process of learning the relationships between input and output values. The workflow in the Back Propagation training algorithm is as follows [1, 2]: (a) Input values propagate towards a hidden layer, (b) Propagate sensitivities back to minimize the error, (c) It updates the weights at the end of the process.

As a new and highly accurate tool that can assist physicians in analyzing, modeling, and facilitating the understanding of complex clinical data in various medical diagnosis practices, artificial neural networks have been used frequently. Artificial neural networks are useful in simplifying the prognosis, diagnosis, and treatment of many diseases [3]-[17].

In this study, Cutaneous manifestations of diagnosed patients as COVID-19 were investigated with Scaled Conjugate Gradient, Levenberg–Marquardt, Bayesian Regularization neural networks. In this investigation, the prediction capabilities of artificial neural networks were compared.

2 Methods

In this work, dermatological findings of the inpatient and intensive care patients in the hospital of Cerrahpaşa Medical Faculty in the COVID-19 pandemic were investigated. The database used in the study was created in Cerrahpaşa Medical Faculty. Database records were composed with the knowledge of dermatological findings of inpatient and intensive care patients. The database consists of eight independent variables and one dependent variable(dermatological findings). Information of 210 patients was taken from the database. The independent variables are in-patient and intensive care patients, gender, age, comorbidity, SARS/COVID/PCR(positive, negative), computed tomography(CT)(positive, negative), time of dermatological findings(before COVID infection, after COVID infection). Variables and their codes are defined in Table 1.

Variables	*Coding*
in hospital	inpatient (1) intensive care (2)
age	18–24 (1) 25–34 (2) 35–44 (3) 45–54 (4)
	55–64 (5) 65–74 (6) 75–84 85+ (8)
comorbidity	none (0) one (1) two+more (3)
SARS, COV2, PCR	negative (1) positive (2)
computed tomography	negative (1) positive (2)
before / after COVID infection	none (1) before (2) after (3)
gender	male (1) female (2)

Table 1 Definitions and codes of variables

In this presented work, neural network algorithms were used: Scaled Conjugate Gradient, Levenberg-–Marquardt, Bayesian Regularization neural networks.

Artificial neural networks propose a different calculation method than known calculation methods. It is possible to see successful applications of this calculation method that adapts to the environment they are in, are adaptive, can work with incomplete information, make decisions under uncertainties, and is tolerant of errors, in almost every area of life. In particular, artificial neural networks can be considered among the most powerful techniques in classification, pattern recognition, signal processing, data compression, and optimization studies. It is possible to come across successful examples of artificial neural networks in many areas such as data mining, optical character handling, optimum route determination, fingerprint recognition, material analysis, job scheduling, and quality control, medical analysis.

Levenberg Marquardt-Back Propagation algorithm: Levenberg Marquardt-Back Propagation algorithm is conceived to application the second-order training method and does not need to use the Hessian matrix. The performance function can be expressed as the sum of squares. Thus the Hessian matrix comes near and the gradient can be calculated:

$$H = M^T M$$
$$g = M^T h.$$

M is a Jacobian matrix in which entries of this matrix are first-order derivatives of network errors according to weights and biases. In this case, the h becomes a vector of network errors. With the standard Back Propagation method, the Jacobian matrix is more easily computable than the Hessian matrix. The Levenberg Marquardt-Back Propagation algorithm employs the Hessian matrix approach in the

$$x_{k+1} = x_k - [M^T M + \rho I]^{-1} M^T h. \tag{1}$$

Newton-like equation. x in this equation indicates connection weights. If ρ is selected as zero, Equation 1 is Newton's method, which uses the approximation Hessian matrix. ρ large can also be chosen. In this case, the equation will be gradient descent with a small step size. To obtain the minimum error, Newton's method is faster and more correct. Therefore, the objective is to move on to Newton's method. After successful steps, ρ is minimized. When ρ increases, it will increase the tentative step performance function. Therefore, every time, the performance function will be decreased with every iteration. Considering the gradient descent technique, it can be said that the Levenberg-Marquardt technique is more powerful. The use of the Levenberg-Marquardt Training Algorithm in disease diagnosis is very common in the literature [18]-[20].

Scaled Conjugate Gradient-Back Propagation algorithm: The basic backpropagation algorithm adjusts the weights in the steepest descent direction (negative of the gradient). This is the direction in which the performance function is decreasing most rapidly. It turns out that, although the function decreases most rapidly along with the negative of the gradient, this does not necessarily produce the fastest convergence. In the conjugate gradient algorithms, a search is performed along with conjugate directions, which produces generally faster convergence than steepest descent directions. In this case, a search is usually implemented along such a direction producing most rapid convergence than the steepest descent direction.

Conjugate direction is the naming given for this direction. Regulating the step size at every iteration is existing in almost every Conjugate Gradient algorithm. Determining the step size is a process performed along the Conjugate Gradient direction and the performance function will be minimized during this process. Starting by searching in the steepest descent direction on the first iteration is the main characteristic of Conjugate Gradient algorithms (Equation 2). It is very common to use Conjugate Gradient algorithms inline search (Equations 3, 4).

$$p_0 = -g_0 \tag{2}$$
$$x_{k+1} = x_k + \alpha_k g_k \tag{3}$$
$$p_k = -g_k + \beta_k p_{k-1} \tag{4}$$

How to factor β_k is calculated allows one to distinguish Conjugate Gradient algorithms.

There are also different approaches to estimating step size except for the line search method. By integrating the model trust region way known from the Levenberg-–Marquardt algorithm with Conjugate Gradient, a new technique can be created. The new technique is called Scaled Conjugate Gradient. For Scaled Conjugate Gradient, factor β_k computation and direction of new search are as follows:

$$\beta_k = \frac{(|g_{k+1}|^2 - g_{k+1}^T g_k)}{g_k^T g_k}$$
$$p_{k+1} = -g_{k+1} + \beta_k p_k.$$

For the success of the algorithm, it is very important to update the design parameters independently at each iteration user. This is a great advantage considering the operation of line search-based algorithms. The Scaled Conjugate Gradient-Back Propagation Algorithm can be examined as examples of use in disease diagnosis in the studies of [21, 22]:

Bayesian Regularization-Back Propagation algorithm: Bayesian Regularization is an example of a training algorithm in which weight and bias values are updated. Bayesian Regularization minimizes the composition of weights and squared errors and after that identifies an accurate composition to conceive a network that can be best generalized. Bayesian Regularization presents network weights into the second-order training objective function which is shown as $F(\omega)$ in Equation 5:

$$F(\omega) = \alpha_\omega + \beta E_D \tag{5}$$

In this equation, the sum of network errors, the sum of the squared network weights, and objective function parameters are denoted E_D, E_ω, α and β, respectively. In Bayesian Regularization, the weights of the network are random variables. In this case, the second-order training set and the distribution of net weights are taken as the Gaussian distribution.

Let's use Bayes' theorem while defining α and β factors. For the variables U and V:

$$P(U|V) = \frac{P(V|U)P(U)}{P(V)}$$

In 6, the posterior probability of U conditional on V, the prior of V conditional on U and the non-zero prior probability of event V are denoted $P(U|V)$, $P(V|U)$ and $P(V)$, respectively. Where $P(U|V)$ is the posterior probability of U conditional on V, $P(V|U)$ the prior of V conditional on U, and $P(V)$ the non-zero prior probability of event V, which functions as a normalizing constant. Minimizing the objective function (Equation 5) is necessary to find the optimum weight. This means that the posterior probability function

$$P(\alpha, \beta U|W, M) = \frac{P(W|\alpha, \beta, M)P(\alpha, \beta|M)}{P(W|M)} \tag{6}$$

is maximized. In 6, the factors needed be to optimized, the weight distribution, the particular neural network architecture, the normalization factor, uniform prior density for the regularization parameters and the likelihood function of W for given α, β, M are denoted α and β, W, M, $P(W|M)$, $P(\alpha, \beta|M)$, and $P(W|\alpha, \beta, M)$, respectively. If we maximize the posterior function $P(\alpha, \beta|W, M)$, it means that the likelihood function $P(W|\alpha, \beta, M)$ is maximized. After doing this operation, optimal values for α and β to a certain weight space are obtained. Subsequently, the algorithm moves into Levenberg-–Marquardt stage where Hessian matrix computations occur and update the weight space in order to minimize the objective function. Thereafter, if the convergence is not met, the algorithm anticipates new values for α and β and the whole transaction recaps itself until convergence is achieved. Studies [23]-[25] can be reviewed for examples of using Bayesian Regularization-Back Propagation algorithm.

Analyses: The network model was created using MATLAB(2015a) neural network toolbox. The training algorithms targeted in this work and the MATLAB functions used for these algorithms are $trainlm$, $trainbr$, $trainscg$. A neural network consisting of layers (1 input, 1 hidden, 1 output) is presented. The number of neurons in the hidden layer of each selected Training Algorithm is collimated to achieve the best performance. The training and validation phases intend to compose an optimal weight space to build the mapping of the taken-out noise constituents from input and target datasets. A neural network is created and then pre-normalized datasets are distributed across the network. The dataset is disunited into the subgroups as Training, Validation, Testing.

In these models established, the number of hidden neurons was taken as 10 for each neural network (Figure 1). The data are divided as follows: training 70%, validation 15%, testing 15%.

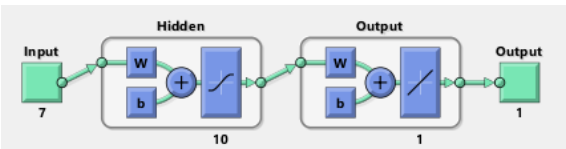

Fig. 1: Model

3 Results and discussion

The status of the variables in the data set is given in Table 2. The age range and frequencies in dataset are shown in Table 3:

Gender	male	123
	female	87
in-patient/intensive care	in-patient	163
	intensive care	47
comorbidity	none	88
	one	48
	two + more	74
SARS/COV2/PCR	negative	88
	positive	122
computed tomography	negative	8
	positive	202
After/Before COVID infectious	after	52

Table 2 Status of the Variables

Range of age	Frequencies
18–24	5
25–34	18
35–44	26
45–54	34
55–64	50
65–74	43
75–84	27
85+	7
total	210

Table 3 Frequency of Age

A trial and error method was made on the network and it was decided how many neurons would be in the hidden layer with this method. Optimal results were achieved in 10 neurons in all three algorithms. R-squared values were found for Scaled Conjugate Gradient, Levenberg-–Marquardt, and Bayesian Regularization algorithms as 0.99596, 1, and 0.99905, respectively. It is understood from the obtained R-squared values that there is a very good correlation between observed and predicted values. That is, it explains that the artificial neural network algorithms applied to the data are a very suitable model for predicting optimal conditions for dermatological findings based on COVID-19.

According to the results obtained when the artificial neural network methods were applied, they were revealed that the network performed very well in learning patterns related to the symptoms of diseases. The network was simulated in situations that it has not encountered before, namely in the test data. The results are great. MSE is the average square difference between outputs and targets. A zero value means that there are no errors and a low value means the best result.

The outputs were compared and it was seen that Bayesian Regularization was better than Levenberg-–Marquardt and Scaled Conjugate Gradient from the results obtained. Thus, $BayesianRegularization > Levenberg - \breve{}Marquardt > ScaledConjugateGradient$. The fact that $R^2 = 1$ means that an artificial neural network gives better results. The fact that the Bayesian Regularization has performed well on the dataset explains that by applying regularization, the over-fitting problem in Bayesian Regularization is eliminated. The reason for this is due to the probabilistic nature of Bayesian Regularization.

4 Conclusion

This work, it is aimed to create models that can predict cutaneous diseases caused by COVID-19 with different Back Propagation Training Algorithms and to select the most suitable of these models. Therefore, the performance of an artificial neural network in obtaining dermatological findings in COVID-19 patients was evaluated. That is, this work is to demonstrate the predictive abilities of Levenberg-–Marquardt, Scaled Conjugate Gradient, Bayesian Regularization neural network training algorithms. The ability to predict reflective thinking in the data was employed with three different back propagation-based algorithms. The best model was obtained according to the highest correlation coefficient between predicted and real data sets. The best model was scrutinized by the Bayesian Regularization neural network not only by examining the highest correlation but also by examining the least effective number of parameters.

The results obtained in this study show that artificial neural network approaches can more accurately identify COVID-19 patients with dermatological diseases. That is, these outcomes demonstrate that the proposed method can procreate efficient rules with the highest accuracy for the detection of dermatological findings of COVID-19 patients. Artificial neural networks can classify dermatological diseases at a level similar to dermatologists. This proposed method predicts that physicians can help differentiate dermatological findings in diseases such as COVID.

In conclusion, it was found that a hidden layer acquired the best classification performance provided that it was placed with neurons on the hidden layers. In summary, the obtained results proved that the artificial neural network model is a good choice for performing classification.

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Notes on BLUPs and OLSPs under SUR Models

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Abstract: A seemingly unrelated regressions (SUR) system consists of a set of SUR models which appear unrelated however they can be linked to their correlated error terms. In this study, we consider two SUR models and combine them by making use of block matrices. We investigate connections between the ordinary least squares predictors (OLSPs) and the best linear unbiased predictors (BLUPs) under the SUR system and its individual SUR models by using various rank formulas of block matrices and elementary matrix operations. One of our main aims is to determine the equalities between the predictors and the other is to establish additive decomposition equalities for predictors.

Keywords: BLUP, Equality of predictor, OLSP, Rank, SUR model.

1 Introduction

A seemingly unrelated regressions (SUR) model, proposed by [22], is a generalization of a linear regression model that consists of several regression equations. A SUR system comprising a group of SUR models allows correlated error terms between the models. Such kind of models can be encountered in some statistical problems. For example, an experiment can be considered that data collected from the same units that are investigated before and after the experiment can be simultaneously correlated. It can be expected for this experiment that the error terms of these models can be simultaneously correlated. The unknown vectors in individual SUR models can be predicted by ordinary least squares methods but, in general, efficient prediction can require joint prediction from the entire system. A method for joint estimation of unknown parameters by combining the SUR models was given by [22]-[24].

In this study, we consider two SUR models in which the error terms of the models are correlated across each other and their SUR system obtained by combining the models by making use of block matrices. There are connections between inference results obtained from the SUR system and individual SUR models. Therefore, it is natural to consider the problems of establishing certain links among predictors under the system and its individual models. One of the approaches to determining the connections among predictors is to establish the equalities between them and another approach is to establish additive decomposition equalities for predictors. Such equalities provide us to determine the role of partial unknown vectors under the SUR system. Our main purpose is to establish the equalities of predictors under a system of linear regression models and to consider the problem of additive decompositions of the predictors. We consider the best linear unbiased predictors (BLUPs) and the ordinary least squares predictors (OLSPs) as predictors of unknown vectors under the considered models. The derivation process of the results in this study consists of heavy matrix operations including the Moore-Penrose generalized inverses of matrices. Therefore, we use some rank formulas of block matrices, given in the following lemma; see [12], to simplify various matrix expressions.

Lemma 1. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, *and* $\mathbf{D} \in \mathbb{R}^{l\times k}$. *Then,*

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A}, & \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{E_A B}) = \boldsymbol{r}(\mathbf{B}) + \boldsymbol{r}(\mathbf{E_B A}), \tag{1}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{C E_{A'}}) = \boldsymbol{r}(\mathbf{C}) + \boldsymbol{r}(\mathbf{A E_{C'}}), \tag{2}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{D} - \mathbf{C}\mathbf{A}^{+}\mathbf{B}) \text{ if } \mathcal{C}(\mathbf{B}) \subseteq \mathcal{C}(\mathbf{A}) \text{ and } \mathcal{C}(\mathbf{C}') \subseteq \mathcal{C}(\mathbf{A}'). \tag{3}$$

In particular,

(a) $\boldsymbol{r}\begin{bmatrix}\mathbf{A}, & \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) \Leftrightarrow \mathbf{E_A B} = \mathbf{0} \Leftrightarrow \mathbf{A}\mathbf{A}^{+}\mathbf{B} = \mathbf{B} \Leftrightarrow \mathcal{C}(\mathbf{B}) \subseteq \mathcal{C}(\mathbf{A})$.
(b) $\boldsymbol{r}\begin{bmatrix}\mathbf{A}, & \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{B}) \Leftrightarrow \mathcal{C}(\mathbf{A}) \cap \mathcal{C}(\mathbf{B}) = \{\mathbf{0}\}$.
(c) $\boldsymbol{r}\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) \Leftrightarrow \mathbf{C E_{A'}} = \mathbf{0} \Leftrightarrow \mathbf{C}\mathbf{A}^{+}\mathbf{A} = \mathbf{C} \Leftrightarrow \mathcal{C}(\mathbf{C}') \subseteq \mathcal{C}(\mathbf{A}')$.

SUR models have attracted considerable interest in recent years; see e.g., [8]-[11] and [17]. For other studies on SUR models, we may refer to [2]-[6], [16, 19, 21]. We may also refer to the studies [13, 15, 18] for a system of linear regression models.

The following terminology and notation are used throughout the paper. Let $\mathbb{R}^{m\times n}$ be the set of all real matrices of dimension $m\times n$. $\mathbf{I}_m$ denotes the $m\times m$ identity matrix. For $\mathbf{A}\in\mathbb{R}^{m\times n}$, the notations $\mathbf{A}'$, $\mathbf{A}^+$, $r(\mathbf{A})$, and $\mathcal{C}(\mathbf{A})$, denote the transpose, the Moore–Penrose generalized inverse, the rank, and the column space, respectively. $\mathbf{E}_\mathbf{A}=\mathbf{A}^\perp=\mathbf{I}_m-\mathbf{A}\mathbf{A}^+$ stands for the orthogonal projectors. $\mathbf{A}\otimes\mathbf{B}=(a_{ij}\mathbf{B})$ denotes the Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$.

2 Predictors under SUR models

Consider the following two SUR models

$$\mathcal{S}_1:\ \mathbf{y}_1=\mathbf{X}_1\boldsymbol{\beta}_1+\boldsymbol{\varepsilon}_1\quad\text{and}\quad\mathcal{S}_2:\ \mathbf{y}_2=\mathbf{X}_2\boldsymbol{\beta}_2+\boldsymbol{\varepsilon}_2 \tag{4}$$

and their combined model

$$\mathcal{S}:\ \mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon} \tag{5}$$

by considering the following block matrices

$$\mathbf{y}=\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix},\ \mathbf{X}=\begin{bmatrix}\mathbf{X}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_2\end{bmatrix},\ \boldsymbol{\beta}=\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix},\ \boldsymbol{\varepsilon}=\begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix},$$

where $\mathbf{y}_i\in\mathbb{R}^{n\times 1}$ is an observable random vector and $\mathbf{y}\in\mathbb{R}^{2n\times 1}$, $\mathbf{X}_i\in\mathbb{R}^{n\times p_i}$ is a known matrix of arbitrary rank and $\mathbf{X}\in\mathbb{R}^{2n\times p}$, $\boldsymbol{\beta}_i\in\mathbb{R}^{p_i\times 1}$ is an unknown parameter vector and $\boldsymbol{\beta}\in\mathbb{R}^{p\times 1}$ with $p_1+p_2=p$, $\boldsymbol{\varepsilon}_i\in\mathbb{R}^{n\times 1}$ is an error vector and $\boldsymbol{\varepsilon}\in\mathbb{R}^{2n\times 1}$, $i=1,2$. The combined model $\mathcal{S}$ can be called as SUR system since this system comprises a pair of SUR models $\mathcal{S}_1$ and $\mathcal{S}_2$ by making use of block matrices. The assumptions on expectations and dispersion matrices under the two SUR models $\mathcal{S}_1$ and $\mathcal{S}_2$ in (4) are

$$\mathrm{E}(\boldsymbol{\varepsilon}_i)=\mathbf{0}\quad\text{and}\quad\mathrm{cov}(\boldsymbol{\varepsilon}_i,\boldsymbol{\varepsilon}_i)=\mathrm{D}(\boldsymbol{\varepsilon}_i)=\sigma_{ii}\mathbf{I}_n, \tag{6}$$

where σ_{ii} is a positive unknown parameter, $i=1,2$. Although $\mathcal{S}_1$ and $\mathcal{S}_2$ seem unrelated, they can have correlated error terms among each other, i.e., the covariance matrix between the error terms of these models can be written as $\mathrm{cov}(\boldsymbol{\varepsilon}_i,\boldsymbol{\varepsilon}_j)=\sigma_{ij}\mathbf{I}_n$, $i,j=1,2, i\neq j$. In this case, according to (6)

$$\mathrm{E}(\boldsymbol{\varepsilon})=\mathbf{0}\quad\text{and}\quad\mathrm{D}(\boldsymbol{\varepsilon})=\boldsymbol{\Sigma}\otimes\mathbf{I}_n \tag{7}$$

is obtained, where $\boldsymbol{\Sigma}=\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}\in\mathbb{R}^{2\times 2}$ is a known or unknown nonnegative definite matrix of arbitrary rank. In what follows, it is assumed that $\mathcal{S}$ is consistent, i.e., $\mathbf{y}\in\mathcal{C}\begin{bmatrix}\mathbf{X}, & \boldsymbol{\Sigma}\otimes\mathbf{I}_n\end{bmatrix}$ holds with probability 1; see [14]. If $\mathcal{S}$ is consistent, then $\mathcal{S}_i$ is consistent; see [20].

To establish some general results on simultaneous predictions of all unknown vectors under models $\mathcal{S}$ and $\mathcal{S}_i$, we can consider the following vector

$$\boldsymbol{\phi}_i=\mathbf{K}_i\boldsymbol{\beta}_i+\mathbf{H}_i\boldsymbol{\varepsilon}_i \tag{8}$$

for given matrices $\mathbf{K}_i\in\mathbb{R}^{k\times p_i}$ and $\mathbf{H}_i\in\mathbb{R}^{k\times n}$. The vector $\boldsymbol{\phi}_i$ in (8) is equivalently written as $\boldsymbol{\phi}_i=\widehat{\mathbf{K}}_i\boldsymbol{\beta}+\widehat{\mathbf{H}}_i\boldsymbol{\varepsilon}$ with $\widehat{\mathbf{K}}_1=\begin{bmatrix}\mathbf{K}_1, & \mathbf{0}\end{bmatrix}$, $\widehat{\mathbf{K}}_2=\begin{bmatrix}\mathbf{0}, & \mathbf{K}_2\end{bmatrix}$, $\widehat{\mathbf{H}}_1=\begin{bmatrix}\mathbf{H}_1, & \mathbf{0}\end{bmatrix}$, and $\widehat{\mathbf{H}}_2=\begin{bmatrix}\mathbf{0}, & \mathbf{H}_2\end{bmatrix}$. Let $\mathbf{K}=\begin{bmatrix}\mathbf{K}_1, & \mathbf{K}_2\end{bmatrix}\in\mathbb{R}^{k\times p}$ and $\mathbf{H}=\begin{bmatrix}\mathbf{H}_1, & \mathbf{H}_2\end{bmatrix}\in\mathbb{R}^{k\times 2n}$, then we can write

$$\boldsymbol{\phi}=\mathbf{K}\boldsymbol{\beta}+\mathbf{H}\boldsymbol{\varepsilon}=\mathbf{K}_1\boldsymbol{\beta}_1+\mathbf{H}_1\boldsymbol{\varepsilon}_1+\mathbf{K}_2\boldsymbol{\beta}_2+\mathbf{H}_2\boldsymbol{\varepsilon}_2=\boldsymbol{\phi}_1+\boldsymbol{\phi}_2. \tag{9}$$

According to the assumptions in (6) and (7),

$$\mathrm{E}(\boldsymbol{\phi})=\mathbf{K}\boldsymbol{\beta},\ \ \mathrm{E}(\boldsymbol{\phi}_i)=\mathbf{K}_i\boldsymbol{\beta}_i=\widehat{\mathbf{K}}_i\boldsymbol{\beta},$$

$$\mathrm{D}(\boldsymbol{\phi})=\mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{H}',\ \ \mathrm{D}(\boldsymbol{\phi}_i)=\sigma_{ii}\mathbf{H}_i\mathbf{H}_i'=\widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\widehat{\mathbf{H}}_i',$$

$$\mathrm{cov}(\boldsymbol{\phi},\mathbf{y})=\mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n),\ \ \mathrm{cov}(\boldsymbol{\phi}_i,\mathbf{y})=\widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n),\ \ \mathrm{cov}(\boldsymbol{\phi}_i,\mathbf{y}_i)=\sigma_{ii}\mathbf{H}_i=\widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{T}_i',$$

where $\mathbf{T}_1=\begin{bmatrix}\mathbf{I}_n, & \mathbf{0}\end{bmatrix}$ and $\mathbf{T}_2=\begin{bmatrix}\mathbf{0}, & \mathbf{I}_n\end{bmatrix}$.

The predictability conditions of vectors $\boldsymbol{\phi}_i$ in (8) and $\boldsymbol{\phi}$ in (9) under the models are given as follows, see, e.g., [1].

(a) $\boldsymbol{\phi}_i$ is predictable under $\mathcal{S}_i\iff\mathcal{C}(\mathbf{K}_i')\subseteq\mathcal{C}(\mathbf{X}_i')$.
(b) $\boldsymbol{\phi}_i$ is predictable under $\mathcal{S}\iff\mathcal{C}(\widehat{\mathbf{K}}_i')\subseteq\mathcal{C}(\mathbf{X}')$.
(c) $\boldsymbol{\phi}$ is predictable under $\mathcal{S}\iff\mathcal{C}(\mathbf{K}')\subseteq\mathcal{C}(\mathbf{X}')$.

We also note that $\mathcal{C}(\mathbf{K}_i')\subseteq\mathcal{C}(\mathbf{X}_i')\iff\mathcal{C}(\widehat{\mathbf{K}}_i')\subseteq\mathcal{C}(\mathbf{X}')\iff\mathcal{C}(\mathbf{K}')\subseteq\mathcal{C}(\mathbf{X}')$. Further, if there exists a matrix $\mathbf{L}$ such that

$$\mathrm{D}(\mathbf{L}\mathbf{y}-\boldsymbol{\phi})=\ \min \text{ s.t. } \mathrm{E}(\mathbf{L}\mathbf{y}-\boldsymbol{\phi})=\mathbf{0} \tag{10}$$

holds in the Löwner partial ordering, then the linear statistic $\mathbf{L}\mathbf{y}$ is defined to be the BLUP of $\boldsymbol{\phi}$, a term introduced by [7], and is denoted by $\mathbf{L}\mathbf{y}=\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\phi})=\mathrm{BLUP}_{\mathcal{S}}(\mathbf{K}\boldsymbol{\beta}+\mathbf{H}\boldsymbol{\varepsilon})$.

Let the vector $\boldsymbol{\phi}$ be predictable under $\mathcal{S}$. The fundamental BLUP equation of the vector $\boldsymbol{\phi}$ under $\mathcal{S}$ is given as follows, for details, see, [20].

$$\mathrm{D}(\mathbf{L}\mathbf{y}-\boldsymbol{\phi}) = \text{ min s.t. } \mathrm{E}(\mathbf{L}\mathbf{y}-\boldsymbol{\phi}) = \mathbf{0} \Longleftrightarrow \mathbf{L}\left[\mathbf{X},\ (\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right] = \left[\mathbf{K},\ \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right]. \tag{11}$$

The general solution of $\mathbf{L}$ and the corresponding $\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\phi})$ can be written as

$$\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\phi}) = \mathbf{L}\mathbf{y} = \left(\left[\mathbf{K},\quad \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right]\mathbf{W}^{+} + \mathbf{U}\mathbf{W}^{\perp}\right)\mathbf{y}, \tag{12}$$

where $\mathbf{U}\in\mathbb{R}^{k\times 2n}$ is an arbitrary matrix and $\mathbf{W} = \left[\mathbf{X},\ (\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right]$. Let the vector $\boldsymbol{\phi}_i$ be predictable under $\mathcal{S}_i$ (also predictable under $\mathcal{S}$). According to (11) and (12), we can write the following equality, see also [9].

$$\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\phi}_i) = \mathbf{L}_i\mathbf{y} = ([\widehat{\mathbf{K}}_i,\ \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}]\mathbf{W}^{+} + \mathbf{U}_i\mathbf{W}^{\perp})\mathbf{y}, \tag{13}$$

where $\mathbf{U}_i\in\mathbb{R}^{k\times n}$ is an arbitrary matrix, $i=1,2$. The OLSP of $\boldsymbol{\phi}_i$ under the models is given as follows, see also [9].

$$\mathrm{OLSP}_{\mathcal{S}}(\boldsymbol{\phi}_i) = (\widehat{\mathbf{K}}_i\mathbf{X}^{+} + \widehat{\mathbf{H}}_i\mathbf{X}^{\perp})\mathbf{y}, \tag{14}$$

$$\mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\phi}_i) = (\mathbf{K}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})\mathbf{y}_i. \tag{15}$$

We note that the BLUP and the OLSP of $\boldsymbol{\phi}_i$ under model $\mathcal{S}_i$ coincide since $\mathrm{D}(\mathbf{y}_i) = \sigma_{ii}\mathbf{I}_n$, that is, (15) can also be written as

$$\mathrm{BLUP}_{\mathcal{S}_i}(\boldsymbol{\phi}_i) = \mathbf{G}_i\mathbf{y}_i = \left(\left[\mathbf{K}_i,\ \sigma_{ii}\mathbf{H}_i\mathbf{X}_i^{\perp}\right]\mathbf{W}_i^{+} + \mathbf{U}_i\mathbf{W}_i^{\perp}\right)\mathbf{y}_i, \tag{16}$$

where $\mathbf{U}_i\in\mathbb{R}^{k\times n}$ is an arbitrary matrix and $\mathbf{W}_i = \left[\mathbf{X}_i,\ \sigma_{ii}\mathbf{X}_i^{\perp}\right]$, $i=1,2$.

3 Main results

In this section, we give the main results on equalities between BLUPs and OLSPs under SUR models and the combined model.

Theorem 1. *Let $\mathcal{S}_i$ and $\mathcal{S}$ be as given in* (4) *and* (5)*, respectively. Assume that $\boldsymbol{\phi}_i$ is predictable under $\mathcal{S}_i$ and $\mathcal{S}$, $i=1,2$. Let $\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\phi}_i)$, $\mathrm{OLSP}_{\mathcal{S}}(\boldsymbol{\phi}_i)$, and $\mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\phi}_i)$ be as given in* (13)*,* (14)*, and* (15)*, respectively. Then the following statements are equivalent.*

(a) $\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\phi}_i) = \mathrm{OLSP}_{\mathcal{S}}(\boldsymbol{\phi}_i) = \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\phi}_i)$,
(b) $(\widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X})\mathbf{X}^{+}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([(\widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X})\mathbf{X}^{+}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)]'\right)\subseteq\mathcal{C}(\mathbf{X}')$,
(c) $\sigma_{ij}(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\mathbf{X}_j^{\perp} = \mathbf{0}$, *i.e.,* $\mathcal{C}\left([\sigma_{ij}(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}]'\right)\subseteq\mathcal{C}(\mathbf{X}_j')$ *for* $j=1,2$, $i\neq j$,
(d) $\boldsymbol{r}\begin{bmatrix}\mathbf{X}_i'\mathbf{X}_i & \sigma_{ij}\mathbf{X}_i'\mathbf{X}_j^{\perp}\\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0}\end{bmatrix} = \boldsymbol{r}(\mathbf{X}_i)$ *for* $j=1,2$, $i\neq j$,
(e) $\boldsymbol{r}\begin{bmatrix}\mathbf{X}_i'\mathbf{X}_i & \sigma_{ij}\mathbf{X}_i'\\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_j'\end{bmatrix} = \boldsymbol{r}(\mathbf{X}_1) + \boldsymbol{r}(\mathbf{X}_2)$ *for* $j=1,2$, $i\neq j$,
(f) $\mathcal{C}\left([\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i,\ \mathbf{0}]'\right)\subseteq\mathcal{C}\left(\left[\mathbf{X}_i'\mathbf{X}_i,\ \sigma_{ij}\mathbf{X}_i'\mathbf{X}_j^{\perp}\right]'\right)$ *for* $j=1,2$, $i\neq j$.

Proof: The expressions $\widehat{\mathbf{K}}_i\mathbf{X}^{+}\mathbf{y}$ and $\widehat{\mathbf{H}}_i\mathbf{X}^{\perp}\mathbf{y}$ are equivalently written as $\mathbf{K}_i\mathbf{X}_i^{+}\mathbf{y}_i$ and $\mathbf{H}_i\mathbf{X}_i^{\perp}\mathbf{y}_i$, respectively. Then $\mathrm{OLSP}_{\mathcal{S}}(\boldsymbol{\phi}_i) = \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\phi}_i)$ in (a) is obtained. The equivalence between (a) and (b) is written from [20, Theorem 6.1] by using our notations and considerations. The equivalency of expressions in (b) is obvious. The equality in (b) is equivalent to the equality in (c) for $i,j=1,2$, $i\neq j$. The equivalency of expressions in (c) is obvious. The equality in (c) can be written as $\sigma_{ij}(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)(\mathbf{X}_i'\mathbf{X}_i)^{+}\mathbf{X}_i'\mathbf{X}_j^{\perp} = \mathbf{0}$. Applying (3) to this equality, we obtain

$$\boldsymbol{r}\left(\sigma_{ij}(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\mathbf{X}_j^{\perp}\right) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_i'\mathbf{X}_i & \sigma_{ij}\mathbf{X}_i'\mathbf{X}_j^{\perp}\\ \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i & \mathbf{0}\end{bmatrix} - \boldsymbol{r}(\mathbf{X}_i'\mathbf{X}_i). \tag{17}$$

The equality in (d) is seen from (17) since $\boldsymbol{r}\left(\sigma_{ij}(\mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i)\mathbf{X}_i^{+}\mathbf{X}_j^{\perp}\right) = \mathbf{0}$. The equivalency between (d) and (e) is seen from (2). The equivalency between (d) and (f) follows from Lemma 1 (a) and (b). □

Theorem 2. *Let $\mathcal{S}_i$ and $\mathcal{S}$ be as given in* (4) *and* (5)*, $i=1,2$. Assume that $\boldsymbol{\phi}_i$ in* (8) *is predictable under $\mathcal{S}_i$ and $\boldsymbol{\phi}$ in* (9) *is predictable under $\mathcal{S}$. Then the following statements are equivalent.*

(a) $\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\phi}) = \mathrm{BLUP}_{\mathcal{S}_1}(\boldsymbol{\phi}_1) + \mathrm{BLUP}_{\mathcal{S}_2}(\boldsymbol{\phi}_2)$,
(b) $\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \begin{bmatrix}\sigma_{11}\mathbf{I}_n & \mathbf{0}\\ \mathbf{0} & \sigma_{22}\mathbf{I}_n\end{bmatrix}\mathbf{X}^{\perp} & (\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\\ \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} & \mathbf{0}\end{bmatrix} = 2n$,
(c) $\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \begin{bmatrix}\sigma_{11} & \mathbf{0}\\ \mathbf{0} & \sigma_{22}\end{bmatrix}\otimes\mathbf{I}_n & \boldsymbol{\Sigma}\otimes\mathbf{I}_n\\ \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}' & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{bmatrix} = 2\boldsymbol{r}(\mathbf{X}) + 2n$.

Proof: The equality in (a) holds if and only if the coefficient matrices in (16) satisfy the equation in (11):

$$(\mathbf{G}_1\mathbf{T}_1 + \mathbf{G}_2\mathbf{T}_2)\left[\mathbf{X},\ (\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right] = \left[\mathbf{K},\ \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right], \text{ i.e.,}$$

$$\left[\mathbf{K}_1,\ \sigma_{11}\mathbf{H}_1\mathbf{X}_1^{\perp}\right]\mathbf{W}_1^{+}\left[\widehat{\mathbf{X}}_1 \quad \sigma_{11}\mathbf{X}_1^{\perp} \quad \sigma_{12}\mathbf{X}_2^{\perp}\right] + \left[\mathbf{K}_2,\ \sigma_{22}\mathbf{H}_2\mathbf{X}_2^{\perp}\right]\mathbf{W}_2^{+}\left[\widehat{\mathbf{X}}_2, \quad \sigma_{21}\mathbf{X}_1^{\perp} \quad \sigma_{22}\mathbf{X}_2^{\perp}\right] = \left[\mathbf{K},\ \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right] \tag{18}$$

holds, where $\widehat{\mathbf{X}}_1 = \left[\mathbf{X}_1, \quad \mathbf{0}\right]$ and $\widehat{\mathbf{X}}_2 = \left[\mathbf{0}, \quad \mathbf{X}_2\right]$. (18) is equivalently written as

$$\begin{aligned}&\left[\left[\mathbf{K}_1, \quad \sigma_{11}\mathbf{H}_1\mathbf{X}_1^{\perp}\right], \quad \left[\mathbf{K}_2, \quad \sigma_{22}\mathbf{H}_2\mathbf{X}_2^{\perp}\right]\right]\begin{bmatrix}\mathbf{W}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{W}_2\end{bmatrix}^{+}\begin{bmatrix}\widehat{\mathbf{X}}_1 & \sigma_{11}\mathbf{X}_1^{\perp} & \sigma_{12}\mathbf{X}_2^{\perp}\\ \widehat{\mathbf{X}}_2 & \sigma_{21}\mathbf{X}_1^{\perp} & \sigma_{22}\mathbf{X}_2^{\perp}\end{bmatrix}\\ &-\left[\mathbf{K},\ \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\right] = \mathbf{0}.\end{aligned} \tag{19}$$

Applying (3) to (19), setting $\mathbf{W}_i$ and using Lemma 1, the rank of (19) is equivalently written as

$$\begin{aligned}
&\boldsymbol{r}\begin{bmatrix}\mathbf{X}_1 & \sigma_{11}\mathbf{X}_1^{\perp} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1 & [\sigma_{11}\mathbf{I}_n,\ \sigma_{12}\mathbf{I}_n]\mathbf{X}^{\perp}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}_2 & \sigma_{22}\mathbf{X}_2^{\perp} & \widehat{\mathbf{X}}_2 & [\sigma_{21}\mathbf{I}_n,\ \sigma_{22}\mathbf{I}_n]\mathbf{X}^{\perp}\\ \mathbf{K}_1 & \sigma_{11}\mathbf{H}_1\mathbf{X}_1^{\perp} & \mathbf{K}_2 & \sigma_{22}\mathbf{H}_2\mathbf{X}_2^{\perp} & \mathbf{K} & \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\mathbf{X}_1 & \sigma_{11}\mathbf{X}_1^{\perp} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}_2 & \sigma_{22}\mathbf{X}_2^{\perp}\end{bmatrix}\\
&= \boldsymbol{r}\begin{bmatrix}\widehat{\mathbf{X}}_1 & \sigma_{11}\mathbf{X}_1^{\perp} & \mathbf{0} & [\sigma_{11}\mathbf{I}_n,\ \sigma_{12}\mathbf{I}_n]\mathbf{X}^{\perp}\\ \widehat{\mathbf{X}}_2 & \mathbf{0} & \sigma_{22}\mathbf{X}_2^{\perp} & [\sigma_{21}\mathbf{I}_n,\ \sigma_{22}\mathbf{I}_n]\mathbf{X}^{\perp}\\ \mathbf{K} & \sigma_{11}\mathbf{H}_1\mathbf{X}_1^{\perp} & \sigma_{22}\mathbf{H}_2\mathbf{X}_2^{\perp} & \mathbf{H}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\end{bmatrix} - \boldsymbol{r}\left[\mathbf{X}_1,\ \sigma_{11}\mathbf{X}_1^{\perp}\right] - \boldsymbol{r}\left[\mathbf{X}_2,\ \sigma_{22}\mathbf{X}_2^{\perp}\right]\\
&= \boldsymbol{r}\begin{bmatrix}\widehat{\mathbf{X}}_1 & \sigma_{11}\mathbf{X}_1^{\perp} & \mathbf{0} & [\sigma_{11}\mathbf{I}_n,\ \sigma_{12}\mathbf{I}_n]\mathbf{X}^{\perp}\\ \widehat{\mathbf{X}}_2 & \mathbf{0} & \sigma_{22}\mathbf{X}_2^{\perp} & [\sigma_{21}\mathbf{I}_n,\ \sigma_{22}\mathbf{I}_n]\mathbf{X}^{\perp}\\ \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} - 2n\\
&= \boldsymbol{r}\begin{bmatrix}\mathbf{X} & \begin{bmatrix}\sigma_{11}\mathbf{I}_n & \mathbf{0}\\ \mathbf{0} & \sigma_{22}\mathbf{I}_n\end{bmatrix}\mathbf{X}^{\perp} & (\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}\\ \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} & \mathbf{0}\end{bmatrix} - 2n\\
&= \boldsymbol{r}\begin{bmatrix}\mathbf{X} & \begin{bmatrix}\sigma_{11} & \mathbf{0}\\ \mathbf{0} & \sigma_{22}\end{bmatrix}\otimes\mathbf{I}_n & \boldsymbol{\Sigma}\otimes\mathbf{I}_n\\ \mathbf{K}-\mathbf{H}\mathbf{X} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}' & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{bmatrix} - 2\boldsymbol{r}(\mathbf{X}) - 2n.
\end{aligned}$$

Setting both sides of the last two expressions to zero leads to (b) and (c), respectively. □

Many consequences can be derived from the previous two theorems for different choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}_i$, $i = 1, 2$. If $\mathbf{H} = \mathbf{0}$ in $\boldsymbol{\phi}$, then the linear statistic $\mathbf{Ly}$ in (10) corresponds to the best linear unbiased estimator (BLUE) of $\mathbf{K}\boldsymbol{\beta}$ under $\mathcal{S}$, if $\mathbf{H}_i = \mathbf{0}$ in $\boldsymbol{\phi}_i$, then the OLSPs in (14) and (15) reduce the ordinary least squares estimators (OLSEs) of $\mathbf{K}_i\boldsymbol{\beta}_i$ under $\mathcal{S}$ and $\mathcal{S}_i$, respectively, and if $\mathbf{K}_i = \mathbf{0}$ and $\mathbf{H}_i = \mathbf{I}_n$ in $\boldsymbol{\phi}_i$, then the OLSPs in (14) and (15) correspond to the OLSPs of $\boldsymbol{\varepsilon}_i$ under $\mathcal{S}$ and $\mathcal{S}_i$, respectively. In this case, by setting $\mathbf{H}_i = \mathbf{0}$ in Theorems 1 and 2, the results are obtained for the BLUEs and the OLSEs of $\mathbf{K}_i\boldsymbol{\beta}_i$ under $\mathcal{S}$ and $\mathcal{S}_i$. By setting $\mathbf{K}_i = \mathbf{X}_i$ and $\mathbf{H}_i = \mathbf{0}$ in Theorems 1 and 2, the results are obtained for the BLUEs and the OLSEs of $\mathbf{X}_i\boldsymbol{\beta}_i$ under $\mathcal{S}$ and $\mathcal{S}_i$. Furthermore, we obtain the results for the BLUPs and the OLSPs of $\boldsymbol{\varepsilon}_i$ when setting $\mathbf{K}_i = \mathbf{0}$ and $\mathbf{H}_i = \mathbf{I}_n$ in Theorems 1 and 2.

4 Conclusion

In this study, we establish the results on the equality of the BLUPs and the OLSPs, and additive decomposition of the BLUPs of unknown vectors within the concept of the SUR models. We consider two SUR models and the SUR system. We accept that these two SUR models have correlated error terms among each other. We use some known rank formulas for simplification of complicated operations of matrices. We have obtained a variety of results on the equivalence of predictors in SUR models and the SUR system from a theoretical point of view. Our approach provides a general framework on related subjects for describing the BLUPs/OLSPs' properties and connections under SUR models.

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A New Contribution to the Discontinuity Problem on S-metric Spaces

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Abstract: Recently, new solutions to Rhoades' Open Problem have been given for self-mappings on metric and generalized metric spaces. In this paper, we give a new solution to the Rhoades' Open Problem on the existence of a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point on S-metric spaces. We present an illustrative example supporting our theoretical results.

Keywords: Activation function, Discontinuity, Fixed point, Metric space, S-metric space.

1 Introduction and preliminaries

One of the interesting problems of fixed point theory is the Rhoades' problem on discontinuity at fixed point. In [30], Rhoades mentioned the question whether there exists a contractive condition that is strong enough to generate a fixed point but that does not force the map to be continuous at the fixed point. After the first solution given by R. P. Pant in [25], several solutions of this open problem have been presented via different approaches. For example, in [24], some solutions were given by means of the notion of an S-metric and a new technique inspired by the notion of a Zamfirescu mapping. In [4], a new setting was provided to answer the Rhoades' question by means of the class of convex contractions of order $m \in \mathbb{N}$. For more details, we indicate the interested readers to study [1, 2, 8, 15, 19, 22], [24]-[28],[31] and the references therein.

On the other hand, since Stefan Banach, various generalizations have been presented using many perspectives on fixed point theory. In many of these generalizations, the contraction condition is generalized, and in some, metric spaces are generalized. One of the generalizations of metric spaces is the notion of an S-metric space given in [33].

Definition 1. *[33] Let X be a nonempty set and $\mathcal{S} : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:*
$(S1)$ $\mathcal{S}(x, y, z) = 0$ *if and only if* $x = y = z$,
$(S2)$ $\mathcal{S}(x, y, z) \leq \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$.
Then $\mathcal{S}$ is called an S-metric on X and the pair $(X, \mathcal{S})$ is called an S-metric space.

Example 1. *[34] Let $X = \mathbb{R}$ (or $\mathbb{C}$) and the function $\mathcal{S} : X \times X \times X \to [0, \infty)$ be defined as*

$$\mathcal{S}(x, y, z) = |x - z| + |y - z|,$$

for all $x, y, z \in \mathbb{R}$ (or $\mathbb{C}$). Then the function $\mathcal{S}$ is an S-metric on $\mathbb{R}$ (or $\mathbb{C}$). This S-metric is called the usual S-metric on $\mathbb{R}$ (or $\mathbb{C}$).

Lemma 1. *[33] Let $(X, \mathcal{S})$ be an S-metric space and $x, y \in X$. Then we have*

$$\mathcal{S}\,(x, x, y) = \mathcal{S}\,(y, y, x)\,.$$

The relationships between a metric and an S-metric were given in different studies (see [11], [12] and [23] for more details). In [12], one of these results was given as follows:

Lemma 2. *[12] Let (X, d) be a metric space. Then the following properties are satisfied*:

1. $\mathcal{S}_d(x, y, z) = d(x, z) + d(y, z)$ *for all $x, y, z \in X$ is an S-metric on X.*
2. $x_n \to x$ *in* (X, d) *if and only if* $x_n \to x$ *in* $(X, \mathcal{S}_d)$.
3. $\{x_n\}$ *is Cauchy in* (X, d) *if and only if* $\{x_n\}$ *is Cauchy in* $(X, \mathcal{S}_d)$.
4. (X, d) *is complete if and only if* $(X, \mathcal{S}_d)$ *is complete.*

The S-metric $\mathcal{S}_d$ was called the S-metric generated by the metric d [23]. Lemma 2 (1) shows that every metric d generates an S-metric. But, the converse does not hold in general, that is, each S-metric cannot be generated by a metric. We note that there exists an S-metric which is not generated by any metric d as seen in the following example.

Example 2. *[10] Let $X = \mathbb{R}$ (or $\mathbb{C}$) and the function $\mathcal{S} : X \times X \times X \to [0, \infty)$ be defined as*

$$\mathcal{S}(x, y, z) = \max\{|x-y|, |y-z|, |x-z|\},$$

for all $x, y, z \in \mathbb{R}$ (or $\mathbb{C}$). Then the function $\mathcal{S}$ is an S-metric on $\mathbb{R}$ (or $\mathbb{C}$) and this S-metric can not be generated by any metric d.

In [11], it was asserted that the function $d_S : X \times X \longrightarrow [0, \infty)$ defined by $d_S(x, y) = \mathcal{S}(x, x, y) + \mathcal{S}(y, y, x)$ for all $x, y \in X$, where $(X, \mathcal{S})$ is any S-metric space, defines a metric on X. But, in [23], it was illustrated by an example that the function d_S does not always define a metric on X (see Example 2 on page 9). In the context of this difference, it is important to study fixed point results on S-metric spaces. Our aim in this paper is to present a new solution to the Rhoades' open question on S-metric spaces.

2 Main results

Let $(X, \mathcal{S})$ be an S-metric space. In this section, we use the following number

$$M_s(x, x, y) = \max\left\{ \mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x,x,Tx)\mathcal{S}(y,y,Ty)}{1+\mathcal{S}(x,x,y)}, \frac{\mathcal{S}(x,x,Tx)\mathcal{S}(y,y,Ty)}{1+\mathcal{S}(Tx,Tx,Ty)} \right\},$$

defined for all $x, y \in X$. By means of this number, in the following theorem, we give a new solution to the Rhoades' question on S-metric spaces.

Theorem 1. *Let $(X, \mathcal{S})$ be a complete S-metric space. If T is a self-mapping on X satisfying the following conditions*

1. $\mathcal{S}(Tx, Tx, Ty) \le \psi(M_s(x, x, y))$*, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a self-mapping such that $\psi(t) < t$ for each $t > 0$,*
2. *For a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M_s(x, x, y) < \varepsilon + \delta$ implies $\mathcal{S}(Tx, Tx, Ty) \le \varepsilon$,*
then T has a unique fixed point $z \in X$ and $T^n x \to z$ for each $x \in X$. Additionally, T is continuous at z if and only if $\lim\limits_{x \to z} M_s(x, x, z) = 0$.

Proof: Let $x_0 \in X$ be a point such that $x_0 \neq Tx_0$ and consider the sequence $\{x_n\}$ in X given by the rule $T^n x_0 = Tx_n = x_{n+1}$ for $n = 0, 1, 2, 3, ...$ Using the condition (1) and definition of the number $M_s(x, x, y)$ we get

$$\begin{aligned}
\mathcal{S}(x_n, x_n, x_{n+1}) &= \mathcal{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) \qquad (1)\\
&\le \psi(M_s(x_{n-1}, x_{n-1}, x_n))\\
&< M_s(x_{n-1}, x_{n-1}, x_n)\\
&= \max\left\{ \begin{array}{c} \mathcal{S}(x_{n-1}, x_{n-1}, x_n), \mathcal{S}(x_{n-1}, x_{n-1}, Tx_{n-1}), \mathcal{S}(x_n, x_n, Tx_n), \\ \frac{\mathcal{S}(x_{n-1},x_{n-1},Tx_{n-1})\mathcal{S}(x_n,x_n,Tx_n)}{1+\mathcal{S}(x_{n-1},x_{n-1},x_n)}, \frac{\mathcal{S}(x_{n-1},x_{n-1},Tx_{n-1})\mathcal{S}(x_n,x_n,Tx_n)}{1+\mathcal{S}(Tx_{n-1},Tx_{n-1},Tx_n)} \end{array} \right\}\\
&= \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}.
\end{aligned}$$

Assume that $\mathcal{S}(x_{n-1}, x_{n-1}, x_n) \le \mathcal{S}(x_n, x_n, x_{n+1})$. Then by the inequality (1) we obtain

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_n, x_n, x_{n+1}),$$

which is a contradiction. So we find $\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n)$ and

$$M_s(x_{n-1}, x_{n-1}, x_n) = \mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

If we put $\mathcal{S}(x_n, x_n, x_{n+1}) = s_n$ then by the inequality (1) we get

$$s_n < s_{n-1}, \qquad (2)$$

and so, s_n is a strictly decreasing sequence of positive real numbers. The sequence s_n tends to a limit $s \ge 0$. Suppose that $s > 0$. Then there exists a positive integer k providing the following inequality for $n \ge k$:

$$s < s_n < s + \delta(s). \qquad (3)$$

Combining the condition (2) and the inequality (2), we have

$$\mathcal{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) = \mathcal{S}(x_n, x_n, x_{n+1}) = s_n < s, \qquad (4)$$

for $n \ge k$. But, the inequality (4) contradicts to the inequality (3) and hence we obtain $s = 0$.

Now we shall show that $\{x_n\}$ is a Cauchy sequence. To do this, let us fix an $\varepsilon > 0$. Without loss of generality, we assume that $\delta = \delta(\varepsilon) < \varepsilon$. Since $s_n \to 0$, there exists $k \in \mathbb{N}$ satisfying the following inequality for $n \geq k$:

$$\mathcal{S}(x_n, x_n, x_{n+1}) = s_n < \frac{\delta}{2} \ (0 < \delta < 1).$$

Following Jachymski's technique (see [13, 14]), we use the mathematical induction to show that, for any $n \in \mathbb{N}$,

$$\mathcal{S}(x_k, x_k, x_{k+n}) < \varepsilon + \delta. \tag{5}$$

Inequality (5) holds for $n = 1$ because of

$$\mathcal{S}(x_k, x_k, x_{k+1}) = s_k < \frac{\delta}{2} < \varepsilon + \delta.$$

Suppose that the inequality (5) is true for some n. Using the triangle inequality, we get

$$\mathcal{S}(x_k, x_k, x_{k+n+1}) \leq \mathcal{S}(x_k, x_k, x_{k+1}) + \mathcal{S}(x_k, x_k, x_{k+1}) + \mathcal{S}(x_{k+1}, x_{k+1}, x_{k+n+1}).$$

If we show that

$$\mathcal{S}(x_{k+1}, x_{k+1}, x_{k+n+1}) \leq \varepsilon,$$

then we deduce that the inequality (5) holds for $n + 1$. Now we show that

$$M_s(x_k, x_k, x_{k+n}) < \varepsilon + \delta,$$

where

$$M_s(x_k, x_k, x_{k+n}) = \max\left\{ \begin{array}{c} \mathcal{S}(x_k, x_k, x_{k+n}), \mathcal{S}(x_k, x_k, Tx_k), \mathcal{S}(x_{k+n}, x_{k+n}, Tx_{k+n}), \\ \frac{\mathcal{S}(x_k, x_k, Tx_k)\mathcal{S}(x_{k+n}, x_{k+n}, Tx_{k+n})}{1+\mathcal{S}(x_k, x_k, x_{k+n})}, \frac{\mathcal{S}(x_k, x_k, Tx_k)\mathcal{S}(x_{k+n}, x_{k+n}, Tx_{k+n})}{1+\mathcal{S}(Tx_k, Tx_k, Tx_{k+n})} \end{array} \right\}. \tag{6}$$

Then by the mathematical induction hypothesis, we have

$$\begin{aligned} \mathcal{S}(x_k, x_k, x_{k+n}) &< \varepsilon + \delta, \qquad (7) \\ \mathcal{S}(x_k, x_k, x_{k+1}) &< \frac{\delta}{2}, \\ \mathcal{S}(x_{k+n}, x_{k+n}, x_{k+n+1}) &< \frac{\delta}{2}, \\ \frac{\mathcal{S}(x_k, x_k, Tx_k)\mathcal{S}(x_{k+n}, x_{k+n}, Tx_{k+n})}{1 + \mathcal{S}(x_k, x_k, x_{k+n})} &< \frac{\delta^2}{4(1 + \mathcal{S}(x_k, x_k, x_{k+n}))} < \delta, \\ \frac{\mathcal{S}(x_k, x_k, Tx_k)\mathcal{S}(x_{k+n}, x_{k+n}, Tx_{k+n})}{1 + \mathcal{S}(Tx_k, Tx_k, Tx_{k+n})} &< \frac{\delta^2}{4(1 + \mathcal{S}(Tx_k, Tx_k, Tx_{k+n}))} < \delta. \end{aligned}$$

Using (6) and (7), we get $M_s(x_k, x_k, x_{k+n}) < \varepsilon + \delta$ and considering the condition (2), we obtain

$$\mathcal{S}(Tx_k, Tx_k, Tx_{k+n}) = \mathcal{S}(x_{k+1}, x_{k+1}, x_{k+n+1}) \leq \varepsilon.$$

Consequently, the inequality (5) indicate that $\{x_n\}$ is a Cauchy sequence. Then there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$ since $(X, \mathcal{S})$ is a complete S-metric space. Furthermore, we have $Tx_n \to z$.

Now we prove that $Tz = z$. On the contrary, assume $Tz \neq z$, that is, z is not a fixed point of T. Using the condition (1) we have

$$\begin{aligned} \mathcal{S}(Tz, Tz, Tx_n) &\leq \psi(M_s(z, z, x_n)) < M_s(z, z, x_n) \\ &= \max\left\{ \begin{array}{c} \mathcal{S}(z, z, x_n), \mathcal{S}(z, z, Tz), \mathcal{S}(x_n, x_n, Tx_n), \\ \frac{\mathcal{S}(z,z,Tz)\mathcal{S}(x_n,x_n,Tx_n)}{1+\mathcal{S}(z,z,x_n)}, \frac{\mathcal{S}(z,z,Tz)\mathcal{S}(x_n,x_n,Tx_n)}{1+\mathcal{S}(Tz,Tz,Tx_n)} \end{array} \right\} \end{aligned}$$

and taking limit for $n \to \infty$ we obtain

$$\mathcal{S}(Tz, Tz, z) < \mathcal{S}(z, z, Tz),$$

a contradiction. Consequently, z is a fixed point of T. Now we show that the fixed point is unique. Let w be another fixed point of T such that $z \neq w$. Using the condition (1), we find

$$\begin{aligned} \mathcal{S}(Tz, Tz, Tw) &= \mathcal{S}(z, z, w) \leq \psi(M_s(z, z, w)) < M_s(z, z, w) \\ &= \max\left\{ \begin{array}{c} \mathcal{S}(z, z, w), \mathcal{S}(z, z, Tz), \mathcal{S}(w, w, Tw), \\ \frac{\mathcal{S}(z,z,Tz)\mathcal{S}(w,w,Tw)}{1+\mathcal{S}(z,z,w)}, \frac{\mathcal{S}(z,z,Tz)\mathcal{S}(w,w,Tw)}{1+\mathcal{S}(Tz,Tz,Tw)} \end{array} \right\} \\ &= \mathcal{S}(z, z, w), \end{aligned}$$

which is a contradiction. This shows that z is the unique fixed point of T.

Finally, we show that T is discontinuous at z if and only if $\lim_{x\to z} M_s(x,x,z) \neq 0$. Equivalently, we prove that T is continuous at z if and only if $\lim_{x\to z} M_s(x,x,z) = 0$. Let T be continuous at the fixed point z and $x_n \to z$. Then $Tx_n \to Tz = z$ and

$$\mathcal{S}(x_n, x_n, Tx_n) \leq 2\mathcal{S}(x_n, x_n, z) + \mathcal{S}(Tx_n, Tx_n, z) \to 0.$$

Hence, we get $\lim_{x_n\to z} M_s(x_n, x_n, z) = 0$. On the other hand, if $\lim_{x_n\to z} M_s(x_n, x_n, z) = 0$ then $\mathcal{S}(x_n, x_n, Tx_n) \to 0$ as $x_n \to z$. This implies $Tx_n \to z = Tz$, that is, T is continuous at z. □

Example 3. *Let $X = [0, 6]$ be the S-metric space with the S-metric defined in Example 2. Let us define a self-mapping $T : X \to X$ by*

$$Tx = \begin{cases} 3 & ; \quad x \leq 3 \\ 0 & ; \quad x > 3 \end{cases}.$$

Then T satisfies the conditions of Theorem 1 and has a unique fixed point $x = 3$ at which T is discontinuous. The mapping T satisfies the contractive condition (1) with

$$\psi(t) = \begin{cases} 3 & ; \quad t > 3 \\ \frac{t}{3} & ; \quad t \leq 3 \end{cases}.$$

Also, T satisfies condition (2) with

$$\delta(\varepsilon) = \begin{cases} 35 & ; \quad \varepsilon \geq 3 \\ 10 - \varepsilon & ; \quad \varepsilon < 3 \end{cases}.$$

It is easy to check that $\lim_{x\to 3} M_s(x,x,3) \neq 0$ and T is discontinuous at the fixed point $x = 3$.

To show that the condition (1) of Theorem 1 is satisfied, we consider the following 4 different cases:

1. *If $x \leq 3$ and $y \leq 3$ then we have $Tx = 3$ and $Ty = 3$ and so $\mathcal{S}(Tx,Tx,Ty) = 0$. Thus the inequality $\mathcal{S}(Tx,Tx,Ty) \leq \psi(M_s(x,x,y))$ is satisfied.*
2. *If $x > 3$ and $y > 3$ then we have $Tx = 0$ and $Ty = 0$ and so $\mathcal{S}(Tx,Tx,Ty) = 0$. Thus the inequality $\mathcal{S}(Tx,Tx,Ty) \leq \psi(M_s(x,x,y))$ is satisfied.*
3. *If $x \leq 3$ and $y > 3$ then we have $Tx = 3$ and $Ty = 0$ and so $\mathcal{S}(Tx,Tx,Ty) = 3$. Since*

$$M_s(x,x,y) = \max\left\{|x-y|, |x-3|, |y|, \frac{|x-3||y|}{1+|x-y|}, \frac{|x-3||y|}{4}\right\}$$

and $|y| > 3$ then we have $M_s(x,x,y) > 3$. Thus, the inequality $\mathcal{S}(Tx,Tx,Ty) \leq \psi(M_s(x,x,y))$ is satisfied.

4. *If $x > 3$ and $y \leq 3$ then we have $Tx = 0$ and $Ty = 3$ and so $\mathcal{S}(Tx,Tx,Ty) = 3$. Since*

$$M_s(x,x,y) = \max\left\{|x-y|, |x|, |y-3|, \frac{|x||y-3|}{1+|x-y|}, \frac{|x||y-3|}{4}\right\}$$

and $|x| > 3$ then we have $M_s(x,x,y) > 3$. Thus the inequality $\mathcal{S}(Tx,Tx,Ty) \leq \psi(M_s(x,x,y))$ is satisfied.

To show that the condition (2) of Theorem 1 is satisfied, we consider the following 4 different cases:

1. *If $x \leq 3$ and $y \leq 3$ then we have $Tx = 3$ and $Ty = 3$ and so $\mathcal{S}(Tx,Tx,Ty) = 0$. Since*

$$M_s(x,x,y) = \max\left\{|x-y|, |x-3|, |y-3|, \frac{|x-3||y-3|}{1+|x-y|}, \frac{|x-3||y-3|}{1}\right\}$$

and

$$\begin{aligned} 0 &\leq |x-y| \leq 3, 0 \leq |x-3| \leq 3, \\ 0 &\leq |y-3| \leq 3, 0 \leq \frac{|x-3||y-3|}{1+|x-y|} \leq 9, \\ 0 &\leq \frac{|x-3||y-3|}{1} \leq 9, 0 \leq M_s(x,x,y) \leq 9, \end{aligned}$$

then we get

$$\varepsilon < M_s(x,x,y) < \varepsilon + \delta \Longrightarrow \mathcal{S}(Tx,Tx,Ty) = 0 \leq \varepsilon.$$

2. *If $x > 3$ and $y > 3$ then we have $Tx = 0$ and $Ty = 0$ and so $\mathcal{S}(Tx, Tx, Ty) = 0$. Since*

$$M_s(x, x, y) = \max\left\{|x-y|, |x|, |y|, \frac{|x||y|}{1+|x-y|}, \frac{|x||y|}{1}\right\}$$

and

$$\begin{aligned} 0 &\leq |x-y| < 3, 3 < |x| \leq 6, \\ 3 &< |y| \leq 6, \frac{9}{2} < \frac{|x||y|}{1+|x-y|} \leq 36, \\ 9 &< \frac{|x||y|}{1} \leq 36, 9 < M_s(x, x, y) \leq 36, \end{aligned}$$

then we get

$$\varepsilon < M_s(x, x, y) < \varepsilon + \delta \Longrightarrow \mathcal{S}(Tx, Tx, Ty) = 0 \leq \varepsilon.$$

3. *If $x \leq 3$ and $y > 3$ then we have $Tx = 3$ and $Ty = 0$ and so $\mathcal{S}(Tx, Tx, Ty) = 3$. Since*

$$M_s(x, x, y) = \max\left\{|x-y|, |x-3|, |y|, \frac{|x-3||y|}{1+|x-y|}, \frac{|x-3||y|}{4}\right\}$$

and

$$\begin{aligned} 0 &< |x-y| \leq 6, 0 \leq |x-3| \leq 3, \\ 3 &< |y| \leq 6, 0 \leq |x-3||y| \leq 18, \\ 1 &< 1+|x-y| \leq 7, 0 \leq \frac{|x-3||y|}{1+|x-y|} < 3, \\ 0 &\leq \frac{|x-3||y|}{4} \leq \frac{9}{2}, 3 < M_s(x, x, y) \leq 6, \end{aligned}$$

then we get

$$\varepsilon < M_s(x, x, y) < \varepsilon + \delta \Longrightarrow \mathcal{S}(Tx, Tx, Ty) = 3 \leq \varepsilon.$$

4. *If $x > 3$ and $y \leq 3$ then we have $Tx = 0$ and $Ty = 3$ and so $\mathcal{S}(Tx, Tx, Ty) = 3$. Since*

$$M_s(x, x, y) = \max\left\{|x-y|, |x|, |y-3|, \frac{|x||y-3|}{1+|x-y|}, \frac{|x||y-3|}{4}\right\}$$

and

$$\begin{aligned} 0 &< |x-y| \leq 6, 3 < |x| \leq 6, \\ 0 &\leq |y-3| \leq 3, 0 \leq |x||y-3| \leq 18, \\ 1 &< 1+|x-y| \leq 7, 0 \leq \frac{|x||y-3|}{1+|x-y|} < 3, \\ 0 &\leq \frac{|x||y-3|}{4} \leq \frac{9}{2}, 3 < M_s(x, x, y) \leq 6, \end{aligned}$$

then we get

$$\varepsilon < M_s(x, x, y) < \varepsilon + \delta \Longrightarrow \mathcal{S}(Tx, Tx, Ty) = 3 \leq \varepsilon.$$

Now we present the following consequences of Theorem 1.

Corollary 1. *Let $(X, \mathcal{S})$ be a complete S-metric space. If T is a self-mapping on X satisfying the following conditions*

1. $\mathcal{S}(Tx, Tx, Ty) < M_s(x, x, y)$*, for any $x, y \in X$ and $M_s(x, x, y) > 0$,*
2. *For a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M_s(x, x, y) < \varepsilon + \delta$ implies $\mathcal{S}(Tx, Tx, Ty) \leq \varepsilon$,*

then T has a unique fixed point $z \in X$ and $T^n x \to z$ for each $x \in X$. Additionally, T is discontinuous at z if and only if $\lim\limits_{x \to z} M_s(x, x, z) \neq 0$.

Corollary 2. *Let $(X, \mathcal{S})$ be a complete S-metric space and T be a self-mapping on X satisfying the following conditions*:

1. $\mathcal{S}(Tx, Tx, Ty) \leq \psi(\mathcal{S}(x, x, y))$*, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a self-mapping such that $\psi(\mathcal{S}(x, x, y)) < \mathcal{S}(x, x, y)$,*
2. *For a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < t < \varepsilon + \delta$ implies $\psi(t) \leq \varepsilon$ for any $t > 0$.*

Then T has a unique fixed point $z \in X$ and $T^n x \to z$ for each $x \in X$.

Remark 1. *Notice that T^m has also a fixed point under the hypothesis of Theorem 1. This fixed point can be unique in some cases. For example, if we consider the self-mapping T defined in Example 3, we obtain $T^m x = 3$ $(m \geq 2)$ for all $x \in X$. Consequently, T^m has a unique fixed point $x = 3$.*

In the following theorem we show that the power contraction allows also the possibility of discontinuity at the fixed point.

Theorem 2. *Let $(X, \mathcal{S})$ be a complete S-metric space and T be a self-mapping on X satisfying the following conditions*:

1. $\mathcal{S}(T^m x, T^m x, T^m y) \leq \psi(M_S^*(x,x,y))$ *, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a self-mapping such that $\psi(t) < t$ for each $t > 0$ and*

$$M_s^*(x,x,y) = \max\left\{ \begin{array}{c} \mathcal{S}(x,x,y), \mathcal{S}(x,x,T^m x), \mathcal{S}(y,y,T^m y), \\ \frac{\mathcal{S}(x,x,T^m x)\mathcal{S}(y,y,T^m y)}{1+\mathcal{S}(x,x,y)}, \frac{\mathcal{S}(x,x,T^m x)\mathcal{S}(y,y,T^m y)}{1+\mathcal{S}(T^m x, T^m x, T^m y)} \end{array} \right\}.$$

2. *For a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M_s^*(x,x,y) < \varepsilon + \delta$ implies $\mathcal{S}(T^m x, T^m x, T^m y) \leq \varepsilon$.*

Then T has a unique fixed point. Additionally, T is discontinuous at z if and only if $\lim\limits_{x \to z} M_s^*(x,x,z) \neq 0$.

Proof: From Theorem 1, it is clear that the function T^m has a unique fixed point z, that is, $T^m z = z$. Then, we have

$$Tz = TT^m z = T^m Tz$$

and hence Tz is a fixed point of T^m. From the uniqueness of the fixed point, we obtain $Tz = z$. Consequently, T has a unique fixed point. □

3 Fixed points of discontinuous activation functions

Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. Consider the fixed point set $Fix(T) = \{x \in X : Tx = x\}$ and a circle $C_{x_0,r}^S = \{x \in X : \mathcal{S}(x,x,x_0) = r\}$. One of the generalizations of fixed point theory is the problem of finding some contraction conditions that make the set $Fix(T)$ contains a circle. This problem known as the fixed-circle problem (see [8–10, 16, 17, 19–22, 27, 28] and the references therein). In this section, contrary to the previous section, if the number of elements of the set $Fix(T)$ is more than one, we study on the geometric properties of the set $Fix(T)$. We note that the number $M_s(x,x,y)$ can also be used to determine discontinuity (or continuity) of a self-mapping T on its fixed points without any hypothesis on the S-metric space and the self-mapping. The following proposition can be proved easily.

Proposition 1. *Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. Then T is continuous at $z \in Fix(T)$ if and only if* $\lim_{x \to z} M_s(x,x,z) = 0$.

Corollary 3. *Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. Then T is discontinuous at $z \in Fix(T)$ if and only if* $\lim_{x \to z} M_s(x,x,z) \neq 0$ *when the limit* $\lim_{x \to z} M_s(x,x,z)$ *exists. If the limit* $\lim_{x \to z} M_s(x,x,z)$ *does not exist then T is discontinuous at z.*

In recent years, discontinuous functions and fixed points of self-mappings have been studied in neural networks (see, for example, [18]). Some current solutions to the Rhoades' Open Problem are used to provide some applications for neural networks with discontinuous activation functions (see, for instance, [3, 4, 6, 7, 19, 22, 27, 28, 31, 32]).

In [18], the stability problem of multiple equilibria for delayed neural networks with discontinuous activation functions was discussed. A general class of discontinuous activation functions were introduced and defined as follows:

$$f_i(x) = \begin{cases} u_i & , \quad -\infty < x < p_i \\ l_{i,1}x + c_{i,1} & , \quad p_i \leq x \leq r_i \\ l_{i,2}x + c_{i,2} & , \quad r_i < x \leq q_i \\ v_i & , \quad q_i < x < +\infty \end{cases},$$

where $p_i, r_i, q_i, u_i, v_i, l_{i,1}, l_{i,2}, c_{i,1}, c_{i,2}$ are constants with $-\infty < p_i < r_i < q_i < +\infty$, $l_{i,1} > 0$, $l_{i,2} < 0$, $u_i = f_i(p_i) = f_i(q_i)$, $f_i(r_i) = l_{i,2}r_i + c_{i,2}$ and $v_i > f_i(r_i)$, $i = 1, 2, ..., n$.

Now we consider the discontinuous activation function $f(x)$ defined by

$$f(x) = \begin{cases} -2 & ; \quad -\infty < x < -4 \\ 2x+6 & ; \quad -4 \leq x \leq -3 \\ -2x-6 & ; \quad -3 < x \leq -2 \\ 6 & ; \quad -2 < x < +\infty \end{cases},$$

which is obtained by choosing $u_i = -2, l_{i,1} = 2, l_{i,2} = -2, c_{i,1} = 6, c_{i,2} = -6, p_i = -4, r_i = -3, q_i = -2, v_i = 6$. Notice that the fixed point set of f is not a singleton, especially we have $Fix(f) = \{-2, 6\}$. If we calculate the circle $C_{2,4}^S$ according to the S-metric given in Example 2 over the set $\mathbb{R}$ then we get $Fix(f) = C_{2,4}^S$. We can easily determine the continuity of f at its fixed points using the number $M_s(x,x,y)$. Since $\lim_{x \to -2} M_s(x,x,-2)$ does not exit, f is discontinuous at $x = -2$. We have $\lim_{x \to 6} M_s(x,x,6) = 0$ and hence f is continuous at $x = 6$.

4 Conclusion

In this paper, we have presented new solutions of Rhoades' Open Problem on S-metric spaces. We have used Jachymski's technique to obtain new fixed point results. As a future work, new solutions to Rhoades' Open Problem can be investigated by a new constructed technique.

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Boundedness of Modified Hilbert–Type Operators and Their Dual in Herz Spaces

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Abstract: In 1968, Herz introduced a class of function spaces to identify the convergence of Fourier Transform in the norm Lipschitz class function. Later on, they were known as Herz spaces. They contain more functions than Lebesgue spaces with power weight $|x|^{\alpha p}$, so in a sense, they are larger than the Lebesgue spaces with power weight $|x|^{\alpha p}$. Two kinds of Herz spaces based on the type of dyadic annuli decomposition of the domain $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^n$, they are termed homogeneous and non–homogeneous. In this article, we establish boundedness for modified Hilbert-type operators and their duals on homogeneous Herz and non–homogeneous Herz space. Our results are the generalization parameter of the previous Hilbert operators.

Keywords: Homogeneous Herz space, Modified Hilbert-type operators, Non-homogeneous Herz space.

1 Preliminaries and main results

Herz introduced function spaces where the Fourier transforms of a function in these spaces convergence in Lipschitz class functions spaces [1, Proposition 3.1']. Later these spaces were known as Herz spaces. Two type Herz spaces from Lu and Yang [6] based on how we decompose the spatial domain. First, $\mathbb{R}^n \setminus \{0\}$ is decomposed over annulus $B_j := \{x \in \mathbb{R}^n : 2^{j-1} < |x| \leq 2^j\}$ where $j \in \mathbb{Z}$. In the second decomposition, $\mathbb{R}^n$ is divided into a unit ball $\widetilde{B}_0$ and annulus $\widetilde{B}_j := B_j$ where $j \in \mathbb{N}$. The spaces with the first decomposition are named a homogeneous Herz space, while the second is called a non-homogeneous Herz space.

The Herz spaces are defined using 3 parameters, namely α, p, q with $0 < p, q < \infty$ and $\alpha \in \mathbb{R}$. The quasi–norm of homogeneous Herz space is constructed through a quasi-norm sequence in $\ell^q(\mathbb{Z})$ with the power weight of each term is $2^{j\alpha q}$. The terms of this sequence are $\|f\|_{L^p(B_j)}$ for every $j \in \mathbb{Z}$. Similarly, quasi-norm of non-homogeneous Herz space is constructed via quasi-norm sequence in $\ell^q(\mathbb{N}_0)$ with the power weight of each term being $2^{j\alpha q}$. The terms of this sequence are $\|f\|_{L^p(\widetilde{B}_j)}$ for every $j \in \mathbb{N}_0$.

The descriptions of the quasi-norms of the Herz spaces above can be written in a simple form through the definition below. Let $\chi_j := \chi_{B_j}$ and $\widetilde{\chi}_j := \chi_{\widetilde{B}_j}$, where χ_E is a characteristic function of E. Suppose $0 < p, q < \infty$ and $\alpha \in \mathbb{R}$.

1. The homogeneous Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ is defined as follows

$$\dot{K}_p^{\alpha,q}(\mathbb{R}^n) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} < \infty \right\}$$

with

$$\|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} := \left\| \left\{ 2^{\alpha j} \|f\chi_j\|_{L^p(\mathbb{R}^n)} \right\} \right\|_{\ell^q(\mathbb{Z})} = \left(\sum_{j=-\infty}^{\infty} 2^{\alpha j q} \|f\chi_j\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}},$$

and $L^p_{\text{loc}}(\Omega) := \{f \in L^p(K),\ \text{for every } K \subseteq \Omega, \text{compact K}\}$.

2. The non-homogeneous Herz space $K_p^{\alpha,q}(\mathbb{R}^n)$ is defined as follows

$$K_p^{\alpha,q}(\mathbb{R}^n) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K_p^{\alpha,q}(\mathbb{R}^n)} < \infty \right\}$$

with

$$\|f\|_{K_p^{\alpha,q}(\mathbb{R}^n)} := \left\|\left\{2^{\alpha j}\|f\widetilde{\chi}_j\|_{L^p(\mathbb{R}^n)}\right\}\right\|_{\ell^q(\mathbb{N}_0)} = \left(\sum_{j=0}^{\infty} 2^{\alpha j q}\|f\widetilde{\chi}_j\|^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}}.$$

Note that $\dot{K}_p^{\alpha,p}(\mathbb{R}^n) = L_p^{\alpha}(\mathbb{R}^n)$ where $L_p^{\alpha}(\mathbb{R}^n)$ is Lebesgue space with power weight $|x|^{\alpha p}$ with

$$\|f\|_{L_p^{\alpha}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^{\alpha p}\right)^{\frac{1}{p}}.$$

The Herz spaces contain more function in Lebesgue spaces with power weight $|x|^{\alpha p}$ for $\alpha \le 0$. So in a sense, they are larger than the Lebesgue spaces with power weight $|x|^{\alpha p}$. In other side, we can give many examples of functions that are included in Herz spaces but not in Lebesgue spaces power weight $|x|^{\alpha p}$. The examples as follows: suppose $0 < p < \infty$ and $\alpha < 0$. If $f(x) = \frac{1}{|x|^{\alpha+\frac{n}{p}}}\chi_{\widetilde{B}_0}(x)$ then $f \in K_p^{\alpha,p}(\mathbb{R}^n) \setminus L_p^{\alpha}(\mathbb{R}^n)$.

Suppose f be measurable function in $\mathbb{R}^n$. Suppose also that $\beta \in \mathbb{R}$ and $0 \le \gamma < n$. *Hilbert-type operator* in Samko [4] as follows

$$\mathcal{H}^{\beta} f(x) = |x|^{\beta} \int_{\mathbb{R}^n} \frac{f(y)}{|x|^n + |y|^n} \frac{dy}{|y|^{\beta}}.$$

If $\beta = 0$ then the operator is Hilbert-type operator $\mathcal{H}^0 = \mathcal{H}$ in [3]. From this definition, easy to see that dual of Hilbert operator is

$$\widetilde{\mathcal{H}}^{\beta} f(x) = \frac{1}{|x|^{\beta}} \int_{\mathbb{R}^n} \frac{f(y)}{|x|^n + |y|^n} |y|^{\beta}\, dy.$$

That a sense $\langle \mathcal{H}f, g\rangle = \langle f, \widetilde{\mathcal{H}}g\rangle$, where $\langle h, g\rangle := \int_{\mathbb{R}^n} f(x)g(x)\,dx$. Obviously, $\widetilde{\mathcal{H}}^{\beta} = \mathcal{H}^{-\beta}$.

Let $0 < \alpha < \infty$ and $1 \le p < \infty$, a function a_j with $j \in \mathbb{N}_0$ on $\mathbb{R}^n$ is said dyadic central (α, p)-block of restrict type if

$$\operatorname{supp}(a_j) \subseteq C_j \quad \text{and} \quad \|a_j\|_{L^p} \lesssim 2^{-\alpha j},$$

where $C_j := \{x \in \mathbb{R}^n : |x| \le 2^j\}$.

Proposition 1. *[6, Proposition 1.3.2] Let $0 < \alpha < \infty$, $1 \le p < \infty$, $0 < q < \infty$. The following two statements are equivalent:*

i. $f \in K_p^{\alpha,p}(\mathbb{R}^n)$
ii. f *can be representated by*

$$f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x),$$

where a_k is dyadic central (α, q)-block of restrict type with support contain in C_k and $\sum_{k\ge 0} |\lambda_k|^q < \infty$.

Morever, $\|f\|_{K_p^{\alpha,q}} \sim \inf\left(\sum_{k=0}^{\infty} |\lambda_k|^q\right)^{1/q}.$

The following Theorems are main result of our paper

Theorem 1. *Let $1 < p < \infty$, $0 < q < \infty$.*

1. *If $0 < \frac{n}{p} + \alpha + \beta < n$, then there exists $C > 0$ such that for every $f \in \dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ satisfy*

$$\|\mathcal{H}^{\beta} f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)}.$$

2. *If $0 < \frac{n}{p} + \alpha - \beta < n$, then there exists $C > 0$ such that for every $f \in \dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ satisfy*

$$\|\widetilde{\mathcal{H}}^{\beta} f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)}.$$

Theorem 2. *Let $1 < p < \infty$, $0 < q < \infty$, $0 < \alpha < \infty$.*

1. *If $0 < \frac{n}{p} + \alpha + \beta < n$, then there exists $C > 0$ such that for every $f \in K_p^{\alpha,q}(\mathbb{R}^n)$ satisfy*

$$\|\mathcal{H}^{\beta} f\|_{K_p^{\alpha,q}(\mathbb{R}^n)} \le C\|f\|_{K_p^{\alpha,q}(\mathbb{R}^n)}.$$

2. *If $0 < \frac{n}{p} + \alpha - \beta < n$, then there exists $C > 0$ such that for every $f \in K_p^{\alpha,q}(\mathbb{R}^n)$ satisfy*

$$\|\widetilde{\mathcal{H}}^{\beta} f\|_{K_p^{\alpha,q}(\mathbb{R}^n)} \leq C\|f\|_{K_p^{\alpha,q}(\mathbb{R}^n)}.$$

The following inequality will be used in this article: let $0 < r \leq 1$. For any positive sequence $\{u_j\}_{j\in\mathbb{Z}}$ then

$$\sum_{j=-\infty}^{\infty} |u_j| \leq \left(\sum_{j=-\infty}^{\infty} |u_j|^r\right)^{\frac{1}{r}}. \tag{1}$$

The following notations are used in this article. Suppose $A, B > 0$

- $A \lesssim B$ means that there is $C_1 > 0$ such that $A \leq C_1 B$.
- $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

This article is arranged in the following order. In Section 1, it is introduced the object in this article like modification of Hilbert-type operators, definitions of homogeneous Herz space, and non-homogeneous Herz space. In Section 2, it is shown the boundedness of Modified Hilbert–type operator on homogeneous Herz space. In Section 3, it is shown the boundedness of Modified Hilbert–type operator on non–homogeneous Herz space.

2 Proof of Theorem 1

1. Let fixed x. Split $\mathbb{R}^n$ into $|y| \leq |x|$ and $|y| > |x|$.

$$\begin{aligned}
|\mathcal{H}^{\beta} f(x)| &\leq |x|^{\beta} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x|^n + |y|^n} \frac{dy}{|y|^{\beta}} \\
&\leq \int_{|y|\leq|x|} \frac{|f(y)|}{|x|^{n-\beta}} \frac{dy}{|y|^{\beta}} + \int_{|y|>|x|} \frac{|f(y)|}{|x|^{-\beta}} \frac{dy}{|y|^{n-\beta}} \\
&:= I_1 f(x) + I_2 f(x).
\end{aligned}$$

Let we estimate I_1. Note that

$$I_1 f(x) = \int_{|y|\leq|x|} \frac{1}{|x|^{\frac{n}{p'}} |x|^{-\frac{\beta}{p'}} |y|^{\frac{n}{pp'} + \frac{\alpha+\beta}{p'}}} \frac{|y|^{\frac{n}{pp'} + \frac{\alpha}{p'} - \frac{\beta}{p}}}{|x|^{\frac{n}{p}} |x|^{-\frac{\beta}{p}}} |f(y)|\, dy.$$

By Hölder Inequality

$$\begin{aligned}
I_1 f(x) &\leq \left(\int_{|y|\leq|x|} \frac{1}{|x|^{n-\beta}|y|^{\frac{n}{p}+\alpha+\beta}} dy\right)^{\frac{1}{p'}} \left(\int_{|y|\leq|x|} \frac{|y|^{\frac{n}{p'} + \frac{\alpha p}{p'} - \beta}}{|x|^{n-\beta}} |f(y)|^p\, dy\right)^{\frac{1}{p}} \\
&\lesssim |x|^{-\frac{n}{pp'} - \frac{\alpha}{p'}} \left(\int_{|y|\leq|x|} \frac{|y|^{\frac{n}{p'} + \frac{\alpha p}{p'} - \beta}}{|x|^{n-\beta}} |f(y)|^p\, dy\right)^{\frac{1}{p}}.
\end{aligned}$$

By Hölder Inequality and since $|x| \sim 2^j$ on annulus B_j we have

$$2^{j\alpha} \|(I_1 f)\chi_j\|_{L^p} \lesssim 2^{j\alpha} \left(\int_{B_j} |x|^{-\frac{n}{p'} - \frac{\alpha p}{p'}} \int_{|y|\leq|x|} \frac{|y|^{\frac{n}{p'} + \frac{\alpha p}{p'} - \beta}}{|x|^{n-\beta}} |f(y)|^p\, dy\, dx\right)^{\frac{1}{p}} := 2^{j\alpha} V_j^{\frac{1}{p}}.$$

Case 1. Let $q \leq p$. By change of order integration and for any $y \in B_k$ we have $|y| \sim 2^k$ then

$$\begin{aligned}
V_j &\leq \int_{|y|\leq 2^j} |y|^{\frac{n}{p'} + \frac{\alpha p}{p'} - \beta} |f(y)|^p \int_{2^{j-1}<|x|\leq 2^j} |x|^{-\frac{n}{p'} - \frac{\alpha p}{p'} - n + \beta}\, dx\, dy \\
&\lesssim \int_{|y|\leq 2^j} |y|^{\frac{n}{p'} + \frac{\alpha p}{p'} - \beta} |f(y)|^p (2^j)^{-\frac{n}{p'} - \frac{\alpha p}{p'} + \beta}\, dy \\
&\lesssim \sum_{k=-\infty}^{j} \left(2^{k-j}\right)^{\frac{n}{p'} - \alpha - \beta} 2^{k\alpha p} 2^{-j\alpha p} \|f\chi_k\|_{L^p}^p.
\end{aligned}$$

Then from Inequality (1) and $\frac{n}{p'} - \alpha - \beta > 0$

$$\|I_1 f\|_{\dot{K}_p^{\alpha,q}} \lesssim \left[\sum_{j=-\infty}^{\infty} 2^{j\alpha q} \sum_{k=-\infty}^{j} \left(2^{k-j}\right)^{\left(\frac{n}{p'}-\alpha-\beta\right)\frac{q}{p}} 2^{k\alpha q} 2^{-j\alpha q} \|f\chi_k\|_{L^p}^q \right]^{\frac{1}{q}}$$

$$= \left[\sum_{k=-\infty}^{\infty} \sum_{j=k}^{\infty} \left(2^{k-j}\right)^{\left(\frac{n}{p'}-\alpha-\beta\right)\frac{q}{p}} 2^{k\alpha q} \|f\chi_k\|_{L^p}^q \right]^{\frac{1}{q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}.$$

Case 2. Let $p \leq q$. By Hölder's Inequality and $\frac{n}{p'} - \alpha - \beta > 0$

$$V_j \lesssim \sum_{k=-\infty}^{j} \left(2^{k-j}\right)^{\frac{1}{2}\left(\frac{n}{p'}-\alpha-\beta\right)} \left(2^{k-j}\right)^{\frac{1}{2}\left(\frac{n}{p'}-\alpha-\beta\right)} 2^{k\alpha p} 2^{-j\alpha p} \|f\chi_k\|_{L^p}^p$$

$$\lesssim \left[\sum_{k=-\infty}^{j} \left(2^{k-j}\right)^{\left(\frac{n}{p'}-\alpha-\beta\right)\frac{q}{2p}} 2^{k\alpha q} 2^{-j\alpha q} \|f\chi_k\|_{L^p}^q \right]^{\frac{p}{q}}.$$

$$\|I_1 f\|_{\dot{K}_p^{\alpha,q}} \lesssim \left(\sum_{j=-\infty}^{\infty} 2^{\alpha j q} V_j^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

$$= \left[\sum_{j=-\infty}^{\infty} 2^{j\alpha q} \sum_{k=-\infty}^{j} \left(2^{k-j}\right)^{\left(\frac{n}{p'}-\alpha-\beta\right)\frac{q}{2p}} 2^{k\alpha q} 2^{-j\alpha q} \|f\chi_k\|_{L^p}^q \right]^{\frac{1}{q}}$$

$$= \left[\sum_{k=-\infty}^{\infty} \sum_{j=k}^{\infty} \left(2^{k-j}\right)^{\left(\frac{n}{p'}-\alpha-\beta\right)\frac{q}{2p}} 2^{k\alpha q} \|f\chi_k\|_{L^p}^q \right]^{\frac{1}{q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}.$$

Let we estimate I_2. Note that

$$I_2 f(x) = \int_{|y|>|x|} \frac{|x|^{\frac{\beta}{p'}}}{|y|^{\frac{n+\beta}{p'}} |y|^{\frac{n}{pp'}+\frac{\alpha}{p'}}} \frac{|x|^{\frac{\beta}{p}} |y|^{\frac{n}{pp'}+\frac{\alpha}{p'}} |f(y)|}{|y|^{\frac{n+\beta}{p}}} \, dy.$$

By Hölder Inequality

$$I_2 f(x) = \left(\int_{|y|>|x|} \frac{|x|^{\beta}}{|y|^n |y|^{\frac{n}{p}+\alpha+\beta}} \, dy \right)^{\frac{1}{p'}} \left(\int_{|y|>|x|} \frac{|x|^{\beta} |y|^{\frac{n}{p'}+\frac{\alpha p}{p'}} |f(y)|^p}{|y|^{n+\beta}} \, dy \right)^{\frac{1}{p}}$$

$$\lesssim |x|^{-\frac{n}{pp'}-\frac{\alpha}{p'}} \left(\int_{|y|>|x|} \frac{|x|^{\beta} |y|^{\frac{n}{p'}+\frac{\alpha p}{p'}} |f(y)|^p}{|y|^{n+\beta}} \, dy \right)^{\frac{1}{p}}.$$

It implies in annulus B_j

$$\|(I_2 f)\chi_j\|_{L^p} \lesssim \left(\int_{B_j} |x|^{-\frac{n}{p'}-\frac{\alpha p}{p'}+\beta} \int_{|y|>|x|} \frac{|y|^{\frac{n}{p'}+\frac{\alpha p}{p'}} |f(y)|^p}{|y|^{n+\beta}} \, dy \, dx \right)^{\frac{1}{p}} := W_j^{\frac{1}{p}}.$$

By change of order integration and for any $y \in B_k$ we have $|y| \sim 2^k$ and

$$W_j \leq \int_{|y|>2^{j-1}} |y|^{\frac{n}{p'}+\frac{\alpha p}{p'}-n-\beta} |f(y)|^p \int_{2^{j-1}<|x|\leq 2^j} |x|^{-\frac{n}{p'}-\frac{\alpha p}{p'}+\beta} \, dx \, dy$$

$$\lesssim \int_{|y|>2^{j-1}} |y|^{\frac{n}{p'}+\frac{\alpha p}{p'}-n-\beta} |f(y)|^p (2^j)^{-\frac{n}{p'}-\frac{\alpha p}{p'}+n+\beta} \, dy$$

$$\lesssim \sum_{k=j-1}^{\infty} \left(2^{k-j}\right)^{\frac{n}{p'}-\alpha-\beta-n} 2^{k\alpha p} 2^{-j\alpha} \|f\chi_k\|_{L^p}^p$$

$$= \sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{n}{p}+\alpha+\beta} 2^{k\alpha p} 2^{-j\alpha p} \|f\chi_k\|_{L^p}^p. \tag{2}$$

So that, by (2) we have

$$2^{\alpha_2 j}\|(I_2 f)\chi_j\|_{L^p} \lesssim \sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{n}{p}+\alpha+\beta} 2^{k\alpha p}2^{-j\alpha p}\|f\chi_k\|_{L^p}^p$$
$$= \sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{n}{p}+\alpha+\beta} 2^{k\alpha p}\|f\chi_k\|_{L^p}^p. \tag{3}$$

Case 1. Let $q \le p$, from (3), (1), and $\frac{n}{p}+\alpha+\beta > 0$

$$\|I_2 f\|_{\dot{K}_p^{\alpha,q}} \lesssim \sum_{j=-\infty}^{\infty}\left[\sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{n}{p}+\alpha+\beta} 2^{k\alpha p}\|f\chi_k\|_{L^p}^p\right]^{\frac{q}{p}}$$
$$= \left[\sum_{j=-\infty}^{\infty}\sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{n}{p}+\alpha+\beta} 2^{k\alpha p}\|f\chi_k\|_{L^p}^p\right]^{\frac{1}{q}}$$
$$= \left[\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k+1} \left(2^{j-k}\right)^{\frac{n}{p}+\alpha+\beta} 2^{k\alpha p}\|f\chi_k\|_{L^p}^p\right]^{\frac{1}{q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}.$$

Case 2. Let $p \le q$. By Hölder Inequality and $\frac{n}{p'}-\alpha-\beta > 0$

$$W_j \lesssim \sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{1}{2}\left(\frac{n}{p}+\alpha+\beta\right)} \left(2^{j-k}\right)^{\frac{1}{2}\left(\frac{n}{p}+\alpha+\beta\right)} 2^{k\alpha p}2^{-j\alpha p}\|f\chi_k\|_{L^p}^p$$
$$\lesssim \left[\sum_{k=-\infty}^{j} \left(2^{j-k}\right)^{\left(\frac{n}{p'}+\alpha+\beta\right)\frac{q}{2p}} 2^{k\alpha q}2^{-j\alpha q}\|f\chi_k\|_{L^p}^q\right]^{\frac{p}{q}}.$$

By evaluating the Herz norm, we obtain

$$\|I_2 f\|_{\dot{K}_p^{\alpha,q}} \lesssim \sum_{j=-\infty}^{\infty}\left[\sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{1}{2}\left(\frac{n}{p}+\alpha+\beta\right)} 2^{k\alpha p}\|f\chi_k\|_{L^p}^p\right]^{\frac{q}{p}}$$
$$= \left[\sum_{j=-\infty}^{\infty}\sum_{k=j-1}^{\infty} \left(2^{j-k}\right)^{\frac{1}{2}\left(\frac{n}{p}+\alpha+\beta\right)} 2^{k\alpha p}\|f\chi_k\|_{L^p}^p\right]^{\frac{1}{q}}$$
$$= \left[\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k+1} \left(2^{j-k}\right)^{\frac{1}{2}\left(\frac{n}{p}+\alpha+\beta\right)} 2^{k\alpha p}\|f\chi_k\|_{L^p}^p\right]^{\frac{1}{q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}.$$

So, we have

$$\|\mathcal{H}^\beta f\|_{\dot{K}_p^{\alpha,q}} \le \|I_1 f\|_{\dot{K}_p^{\alpha,q}} + \|I_2 f\|_{\dot{K}_p^{\alpha,q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}.$$

2. From $\widetilde{\mathcal{H}}^\beta = \mathcal{H}^{-\beta}$, Theorem 1 Part (i) and $0 < \frac{n}{p}+\alpha-\beta < n$ then

$$\|\widetilde{\mathcal{H}}^\beta f\|_{\dot{K}_p^{\alpha,q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}.$$

The following are the corollaries from Theorem 1,

Corollary 1. *Let* $1 < p < \infty$.

1. *If* $0 < \frac{n}{p}+\alpha+\beta < n$ *then there exists* $C > 0$ *such that for every* $f \in L_p^\alpha(\mathbb{R}^n)$ *satisfy*

$$\|\mathcal{H}^\beta f\|_{L_p^\alpha(\mathbb{R}^n)} \le C\|f\|_{L_p^\alpha(\mathbb{R}^n)}.$$

2. *If* $0 < \frac{n}{p}+\alpha-\beta < n$ *then there exists* $C > 0$ *such that for every* $f \in L_p^\alpha(\mathbb{R}^n)$ *satisfy*

$$\|\widetilde{\mathcal{H}}^\beta f\|_{L_p^\alpha(\mathbb{R}^n)} \le C\|f\|_{L_p^\alpha(\mathbb{R}^n)}.$$

Corollary 2. *Let* $1 < p < \infty$.

1. *If* $0 < \frac{n}{p} + \beta < n$ *then there exists* $C > 0$ *such that for every* $f \in L^p(\mathbb{R}^n)$ *satisfy*
$$\|\mathcal{H}^{\beta} f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$
2. *If* $0 < \frac{n}{p} - \beta < n$ *then there exists* $C > 0$ *such that for every* $f \in L^p(\mathbb{R}^n)$ *satisfy*
$$\|\widetilde{\mathcal{H}}^{\beta} f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

3 Proof of Theorem 2

The method in Theorem 1 is not work since $|y| \not\sim 1$ for every $k \in \widetilde{B}_0$. The proof of this Theorem by dyadic decomposition in Proposition 1.

i. Let $f \in K_p^{\alpha,p}(\mathbb{R}^n)$. By Proposition 1, write $f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x)$, where $\operatorname{supp}(a_k) \subseteq C_k$, the norms $\|a_k\|_{L^p} \lesssim 2^{-k\alpha}$ and
$$\left(\sum_{k=0}^{\infty} |\lambda_k|^q\right)^{1/q} \sim \|f\|_{K_p^{\alpha,q}}.$$

Note that
$$\begin{aligned}
\|\mathcal{H}^{\beta} f\|_{K_p^{\alpha,q}}^q &= \sum_{j=0}^{\infty} 2^{j\alpha q} \|\mathcal{H}^{\beta} f\|_{L^p(\widetilde{B}_j)}^q \\
&\lesssim \sum_{j=0}^{\infty} 2^{j\alpha q} \left(\sum_{k=0}^{j} \lambda_k \|\mathcal{H}^{\beta}(a_k)\|_{L^p(\widetilde{B}_j)}\right)^q + \sum_{j=0}^{\infty} 2^{j\alpha q} \left(\sum_{k=j+1}^{\infty} \lambda_k \|\mathcal{H}^{\beta}(a_k)\|_{L^p(\widetilde{B}_j)}\right)^q \\
&:= I_1 + I_2.
\end{aligned}$$

Let we estimate I_1.
$$\begin{aligned}
\|\mathcal{H}^{\beta}(a_k)\|_{L^p(\widetilde{B}_j)} &\leq \left[\int_{\widetilde{B}_j} |x|^{(\beta-n)p} \left(\int_{C_k} \frac{|a_k(y)|}{|y|^{\beta}} dy\right)^p dx\right]^{\frac{1}{p}} \\
&\lesssim \|a_k\|_{L^p} 2^{k\beta + k\frac{n}{p'}} 2^{j(\beta-n) + j\frac{n}{p}} \\
&\lesssim 2^{-k\alpha + (j-k)\beta - (j-k)\frac{n}{p'}}.
\end{aligned}$$

If $0 < q \leq 1$, by inequality (1) we obtain
$$\begin{aligned}
I_1 &\lesssim \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \lambda_k 2^{(j-k)\left[\alpha+\beta-\frac{n}{p'}\right]}\right)^q \leq \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \lambda_k^q 2^{(j-k)\left[\alpha+\beta-\frac{n}{p'}\right]q}\right) \\
&= \sum_{k=0}^{\infty} \lambda_k^q \left(\sum_{j=k+1}^{\infty} 2^{(j-k)\left[\alpha+\beta-\frac{n}{p'}\right]q}\right) \lesssim \sum_{k=0}^{\infty} \lambda_k^q \sim \|f\|_{K_p^{\alpha,q}}^q.
\end{aligned}$$

If $1 < q < \infty$, by Hölder Inequality we have
$$\begin{aligned}
I_1 &\lesssim \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \lambda_k 2^{\frac{(j-k)}{q}\left[\alpha+\beta-\frac{n}{p'}\right]} 2^{\frac{(j-k)}{q'}\left[\alpha+\beta-\frac{n}{p'}\right]}\right)^q \\
&\leq \sum_{j=0}^{\infty} \left[\left(\sum_{k=0}^{j} \lambda_k^q 2^{(j-k)\left[\alpha+\beta-\frac{n}{p'}\right]}\right) \left(\sum_{k=0}^{j} 2^{(j-k)\left[\alpha+\beta-\frac{n}{p'}\right]}\right)^{q/q'}\right] \\
&\lesssim \sum_{k=0}^{\infty} \lambda_k^q \sim \|f\|_{K_p^{\alpha,q}}^q.
\end{aligned}$$

Next, let we estimate I_2. By Corollary 2 we obtain
$$I_2 \lesssim \sum_{j=0}^{\infty} 2^{j\alpha q} \left(\sum_{k=j+1}^{\infty} \lambda_k \|a_k\|_{L^p}\right)^q \lesssim \sum_{j=0}^{\infty} \left(\sum_{k=j+1}^{\infty} 2^{(j-k)\alpha}\right)^q.$$

Similarly, by divide two case $0 < q \leq 1$ and $1 < q < \infty$ when estimate I_1, we obtain

$$I_2 \lesssim \|f\|^q_{K_p^{\alpha,q}}$$

ii. Similar with proof of Theorem 1 Part ii.

4 Conclusion

In this article, we have proven the boundedness of Modified Hilbert–type operators on homogeneous Herz space and non–homogeneous Herz space. This is more general than previous Hilbert-type operator with parameter β. We also consider sufficient conditions for boundedness of the operators on homogeneous Herz Space and non–homogeneous Herz space.

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