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Preface

Dear Conference Participant,

Welcome to the International E-Conference on Mathematical Development and Applications (ICOMAA-2022) we-organized the fifth. The aim of our conferences is to bring together scientists and young researchers from all overthe world and their work on the fields of mathematics in a discussion environment. With this interaction, functional analysis, approach theory, differential equations and partial differential equations and the results of applications inthe field of Mathematics are discussed with our valuable academics, and in mathematical developments both science and young researchers are opened. We are happe to host many prominent experts from different countries who will present the state-of-the-art in real analysis, complex analysis, harmonic and non-harmonic analysis, operator theory and spectral analysis, applied analysis.

I would like express my gratitude to those who see and appreciate our efforts and innovative steps that we have made to improve our conference every year, to our dear invited speakers and to all our participants. I owe a debt of gratitude to the Scientific committee, organizing committee, local organizing committee and for their efforts throughout this conference series.

The conference brings together about 192 participants and 8 invited speakers from 25 countries (Algeria, Albania, Azerbaijan, Canada, China, Colombia, Cyprus, Czech Republic, Finland, Germany, Greece, India, Iran, Italy,Kuwait, Malaysia, Morocco, Pakistan, Qatar, Saudi Arabia, Thailand, Tunisia, Turkey, United Arab Emirates,USA, Uzbekistan, Yemen). More than 50% of our participants participated from abroad. This shows that the conference meets the criteria of being international.

The conference program represents the efforts of many people. I would like to express my gratitude to all membersof the scientific committee, external reviewers, sponsors and, honorary committee for their continued support to the ICOMAA. I also thank the invited speakers for presenting their talks on current researches. Also, the success of ICOMAA depends on the effort and talent of researchers in mathematics and its applications that have written and submitted papers on a variety of topics. So, I would like to sincerely thank all participants of ICOMAA-2022 for contributing to this great meeting in many different ways. I believe and hope that each of you will get the maximum benefit from the conference.

Prof. Dr. Yusuf ZEREN
Chairman
On behalf of the Organizing Committee

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Amply s-Supplemented Modules

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Abstract: In this work, all rings are associative with identity and all modules are unital left modules. Let M be an $R-$module. If every submodule of M which contains $SocM$ has ample supplements in M, then M is called an amply s-supplemented module. In this work, some properties of these modules are investigated.

Keywords: Essential Submodules, Small Submodules, Socle, Supplemented Modules.

1 Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. If every essential submodule of M has a supplement in M, then M is called an *essential supplemented* (or briefly, *e-supplemented*) module. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. If every essential submodule of M has ample supplements in M, then M is called an *amply essential supplemented* (or briefly, *amply e-supplemented*) module. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The sum of all simple submodules of an $R-$module M is called the *socle* of M and denoted by $SocM$. Let M be an $R-$module. It is defined the relation $'\beta^*{}'$ on the set of submodules of an $R-$module M by $X\beta^*Y$ if and only if $Y + K = M$ for every $K \leq M$ such that $X + K = M$ and $X + T = M$ for every $T \leq M$ such that $Y + T = M$. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More informations about (amply) supplemented modules are in [2, 6, 7]. More details about (amply) essential supplemented modules are in [4, 5]. The definition of β^* relation and some properties of this relation are in [1].

Lemma 1. *Let M be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

Proof: See [6, 17.3]. □

Lemma 2. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an epimorphism and $Kef \ll M$.*
(3) *If $K \ll M$, then $\frac{K+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*
(4) *If $L \leq M$ and $K \ll L$, then $K \ll M$.*
(5) *If $K_1, K_2, ..., K_n \ll M$, then $K_1 + K_2 + ... + K_n \ll M$.*
(6) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

Proof: See [2, 2.2] and [6, 19.3]. □

Lemma 3. *Let M be an $R-$module. The following assertions are hold.*

(1) *If* $L \ll M$, *then* $L \leq T$ *for every maximal submodule* T *of* M.
(2) $RadM = \sum_{L \ll M} L$.
(3) *Let* N *be an* $R-$*module and* $f : M \longrightarrow N$ *be an* $R-$*module homomorphism. Then* $f(RadM) \leq RadN$.
(4) *For* $K, L \leq M$, $\frac{RadK+L}{L} \leq Rad\frac{K+L}{L}$. *If* $L \leq RadK$, *then* $RadK/L \leq Rad(K/L)$.
(5) *If* $L \leq M$, *then* $RadL \leq RadM$.
(6) *For* $K, L \leq M$, $RadK + RadL \leq Rad(K + L)$.
(7) $Rx \ll M$ *for every* $x \in RadM$.

Proof: See [6]. □

Lemma 4. *Let* M *be an* R *-module. The following statements hold.*
(i) $SocM = \bigcap_{L \trianglelefteq M} L$.
(ii) *For* $K \leq M$, $SocK = K \cap SocM$.
(iii) $SocM \trianglelefteq M$ *if and only if* $SocK \neq 0$ *for every nonzero submodule* K *of* M.
(iv) *Let* N *be an* $R-$*module and* $f : M \longrightarrow N$ *be an* $R-$*module homomorphism. Then* $f(SocM) \subset Socf(M)$.
(v) *For* $K \leq M$, $(SocM + K)/K \subset Soc(M/K)$.
(vi) *If* $M = \bigoplus_{\Lambda} M_\lambda$, *then* $SocM = \bigoplus_{\Lambda} SocM_\lambda$.

Proof: See [6, 21.2]. □

Definition 1. *Let* M *be an* $R-$*module. If every* $U \leq M$ *with* $SocM \leq U$ *has a supplement in* M, *then* M *is called a socle supplemented (or briefly, s-supplemented) module. (See [3]).*

Definition 2. *Let* M *be an* $R-$*module and* $X \leq M$. *If* X *is a supplement of a submodule* U *of* M *with* $SocM \leq U$, *then* X *is called a s-supplement submodule in* M. *(See [3]).*

Lemma 5. *Let* M *be an socle supplemented module. Then every finitely* $M-$*generated* $R-$*module is socle supplemented. (See [3]).*

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Lemma 6. *Let* V *be a supplement of* U *in* M. *Then*
(1) *If* $W + V = M$ *for some* $W \leq U$, *then* V *is a supplement of* W *in* M.
(2) *If* M *is finitely generated, then* V *is also finitely generated.*
(3) *If* U *is a maximal submodule of* M, *then* V *is cyclic and* $U \cap V = RadV$ *is the unique maximal submodule of* V.
(4) *If* $K \ll M$, *then* V *is a supplement of* $U + K$ *in* M.
(5) *For* $K \ll M$, $K \cap V \ll V$ *and hence* $RadV = V \cap RadM$.
(6) *Let* $K \leq V$. *Then* $K \ll V$ *if and only if* $K \ll M$.
(7) *For* $L \leq U$, $\frac{V+L}{L}$ *is a supplement of* U/L *in* M/L.

Proof: See [6, 41.1]. □

Lemma 7. *Let* M *be an* $R-$*module.*
(1) *If* $M = U \oplus V$ *then* V *is a supplement of* U *in* M. *Also* U *is a supplement of* V *in* M.
(2) *For* $M_1, U \leq M$, *if* $M_1 + U$ *has a supplement in* M *and* M_1 *is supplemented, then* U *also has a supplement in* M.
(3) *Let* $M = M_1 + M_2$. *If* M_1 *and* M_2 *are supplemented, then* M *is also supplemented.*
(4) *Let* $M_i \leq M$ *for* $i = 1, 2, ..., n$. *If* M_i *is supplemented for every* $i = 1, 2, ..., n$, *then* $M_1 + M_2 + ... + M_n$ *is also supplemented.*
(5) *If* M *is supplemented, then* M/L *is supplemented for every* $L \leq M$.
(6) *If* M *is supplemented, then every homomorphic image of* M *is also supplemented.*
(7) *If* M *is supplemented, then* $M/RadM$ *is semisimple.*
(8) *Hollow and local modules are supplemented.*
(9) *If* M *is supplemented, then every finitely* $M-$*generated module is supplemented.*
(10) ${}_RR$ *is supplemented if and only if every finitely generated* $R-$*module is supplemented.*

Proof: See [6, 41.2]. □

Lemma 8. *Let* M *be an* $R-$*module.*
(1) *If* M *is supplemented, then* M *is essential supplemented.*
(2) *For* $M_1 \leq M$ *and* $U \trianglelefteq M$, *if* $M_1 + U$ *has a supplement in* M *and* M_1 *is essential supplemented, then* U *also has a supplement in* M.
(3) *Let* $M = M_1 + M_2$. *If* M_1 *and* M_2 *are essential supplemented, then* M *is also essential supplemented.*
(4) *Let* $M_i \leq M$ *for* $i = 1, 2, ..., n$. *If* M_i *is essential supplemented for every* $i = 1, 2, ..., n$, *then* $M_1 + M_2 + ... + M_n$ *is also essential supplemented.*
(5) *If* M *is essential supplemented, then* M/L *is essential supplemented for every* $L \leq M$.
(6) *If* M *is essential supplemented, then every homomorphic image of* M *is also essential supplemented.*
(7) *If* M *is essential supplemented, then* $M/RadM$ *have no proper essential submodules.*
(8) *Hollow and local modules are essential supplemented.*
(9) *If* M *is essential supplemented, then every finitely* $M-$*generated module is essential supplemented.*
(10) ${}_RR$ *is essential supplemented if and only if every finitely generated* $R-$*module is essential supplemented.*

Proof: See [4, 5]. ☐

Definition 3. *Let M be an $R-$module. If every submodule of M which contains $SocM$ has ample supplements in M, then M is called an amply socle supplemented (or briefly, amply s-supplemented) module.*

Proposition 1. *Every amply s-supplemented module is s-supplemented.*

Proof: Let M be an amply s-supplemented $R-$module and $SocM \leq U \leq M$. Since $U + M = M$, by the definition of amply s-supplemented module, U has a supplement $V \leq M$ in M. Hence M is s-supplemented, as desired. ☐

Proposition 2. *Every hollow module is amply s-supplemented.*

Proof: Clear from definitions. ☐

Proposition 3. *Every local module is amply s-supplemented.*

Proof: Clear from definitions. ☐

Proposition 4. *Every amply s-supplemented module is essential supplemented.*

Proof: Let M be an amply s-supplemented module and $U \trianglelefteq M$. Since $U \trianglelefteq M$, by Lemma 4, $SocM \leq U$. Since $U + M = M$, by the definition of amply s-supplemented module, U has a supplement $V \leq M$ in M. Hence M is essential supplemented, as desired. ☐

Proposition 5. *Let M be an $R-$module and $M_1 \leq M$ and $U \trianglelefteq M$. If $M_1 + U$ has a supplement in M and M_1 is amply s-supplemented, then U also has a supplement in M.*

Proof: Since M_1 is amply s-supplemented, by Proposition 4, M_1 is essential supplemented. Then by Lemma 8, U has a supplement in M. ☐

Proposition 6. *Let M be an $R-$module and $M_1, M_2, ..., M_n \leq M$ and $U \trianglelefteq M$. If $M_1 + M_2 + ... + M_n + U$ has a supplement in M and M_i is amply s-supplemented for every $i = 1, 2, ..., n$, then U also has a supplement in M.*

Proof: Clear from Proposition 5. ☐

Proposition 7. *Let $M = M_1 + M_2$. If M_1 and M_2 are amply s-supplemented, then M is essential supplemented.*

Proof: Since M_1 and M_2 are amply s-supplemented, by Proposition 4, M_1 and M_2 are essential supplemented. Then by Lemma 8, M is essential supplemented. ☐

Proposition 8. *Let M be an $R-$module and $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is amply s-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is essential supplemented.*

Proof: Since M_i is amply s-supplemented for every $i = 1, 2, ..., n$, by Proposition 4, M_i is essential supplemented. Then by Lemma 8, $M_1 + M_2 + ... + M_n$ is essential supplemented. ☐

Proposition 9. *Let M be an $R-$module and $M = M_1 + M_2 + ... + M_n$. If M_i is amply s-supplemented for every $i = 1, 2, ..., n$, then M is essential supplemented.*

Proof: Clear from Proposition 8. ☐

Proposition 10. *Let M be an $R-$module and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. If M_i is amply s-supplemented for every $i = 1, 2, ..., n$, then M is essential supplemented.*

Proof: Clear from Proposition 9. ☐

Proposition 11. *Let M be an $R-$module. If M is amply s-supplemented,, then $M^{(\Lambda)}$ is essential supplemented for every finite index set Λ.*

Proof: Clear from Proposition 10. ☐

Proposition 12. *Let M be an $R-$module. If M is amply s-supplemented, then M/L is essential supplemented for every $L \leq M$.*

Proof: Let $L \leq M$. Since M is amply s-supplemented, by Proposition 4, M is essential supplemented. Then by Lemma 8, M/L is essential supplemented. ☐

Corollary 1. *Let M be an amply s-supplemented $R-$module. Then every direct summand of M is essential supplemented.*

Proof: Let K be a direct summand of M and $M = K \oplus T$ with $T \leq M$. By Proposition 12, M/T is essential supplemented. Then by $M/T = (K+T)/T \cong K/(K \cap T) = K/0 \cong K$, K is essential supplemented, as desired. □

Proposition 13. *Let $f : M \rightarrow N$ be $R-$module epimorphism. If M is amply s-supplemented, then N is essential supplemented.*

Proof: Since M is amply s-supplemented, by Proposition 4, M is essential supplemented. Then by Lemma 8, N is essential supplemented. □

Proposition 14. *Let M be an $R-$module. If M is amply s-supplemented, then $M/RadM$ have no proper essential submodules.*

Proof: Since M is amply s-supplemented, by Proposition 4, M is essential supplemented. Then by Lemma 8, $M/RadM$ have no proper essential submodules. □

Proposition 15. *Let M be an $R-$module. If M is amply s-supplemented, then every finitely $M-$generated module is essential supplemented.*

Proof: Since M is amply s-supplemented, by Proposition 4, M is essential supplemented. Then by Lemma 8, every finitely $M-$generated module is essential supplemented. □

Proposition 16. *Let R be any ring. If $_RR$ is amply s-supplemented, then every finitely generated $R-$module is essential supplemented.*

Proof: Since $_RR$ is amply s-supplemented, by Proposition 4, $_RR$ is essential supplemented. Then by Lemma 8, every finitely generated $R-$module is essential supplemented. □

3 Conclusion

Amply s-supplemented modules are the special parts of supplemented modules.

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Cofinitely s-Supplemented Modules

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Abstract: In this work, every ring has unity and every module is unitary left module. Let M be an $R-$module. If every cofinite submodule of M which contains $SocM$ has a supplement in M, then M is called a cofinitely s-supplemented module. In this work some properties of these modules are investigated.

Keywords: Cofinite Submodules, Simple Modules, Socle, Supplemented Modules

1 Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. A submodule K of M is called a *cofinite* submodule of M if M/K is finitely generated. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. M is said to be *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. M is called an *essential supplemented* (briefly, *e-supplemented*) module if every essential submodule of M has a supplement in M. M is said to be *cofinitely essential supplemented* (briefly, *cofinitely e-supplemented*) if every cofinite essential submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. If every cofinite submodule of M has ample supplements in M, then M is called an *amply cofinitely supplemented* module. If every essential submodule of M has ample supplements in M, then M is called an *amply essential supplemented* (briefly, *amply e-supplemented*) module. M is said to be *amply cofinitely essential supplemented* (briefly, *amply cofinitely e-supplemented*) if every cofinite essential submodule of M has ample supplements in M.The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$.

More informations about (amply) supplemented modules are in [1, 3, 9, 10]. More informations about cofinitely supplemented modules are in [2]. More informations about (amply) essential supplemented modules are in [7, 8]. More informations about (amply) cofinitely essential supplemented modules are in [4, 5]. The definition of s-supplemented modules and some properties of them are in [6].

Lemma 1. *Let M be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

Proof: See [9, 17.3]. □

Lemma 2. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an epimorphism and $Kef \ll M$.*
(3) *If $K \ll M$, then $\frac{K+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*
(4) *If $L \leq M$ and $K \ll L$, then $K \ll M$.*
(5) *If $K_1, K_2, ..., K_n \ll M$, then $K_1 + K_2 + ... + K_n \ll M$.*
(6) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

Proof: See [3, 2.2] and [9, 19.3]. □

Lemma 3. *Let M be an $R-$module. The following assertions are hold.*
(1) *If $L \ll M$, then $L \leq T$ for every maximal submodule T of M.*
(2) $RadM = \sum\limits_{L \ll M} L$.
(3) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(RadM) \leq RadN$.*
(4) *For $K, L \leq M$, $\frac{RadK+L}{L} \leq Rad\frac{K+L}{L}$. If $L \leq RadK$, then $RadK/L \leq Rad(K/L)$.*
(5) *If $L \leq M$, then $RadL \leq RadM$.*
(6) *For $K, L \leq M$, $RadK + RadL \leq Rad(K + L)$.*
(7) *$Rx \ll M$ for every $x \in RadM$.*

Proof: See [9]. □

Lemma 4. *Let M be an R -module. The following statements hold.*
(i) $SocM = \bigcap\limits_{L \trianglelefteq M} L$.
(ii) *For $K \leq M$, $SocK = K \cap SocM$.*
(iii) *$SocM \trianglelefteq M$ if and only if $SocK \neq 0$ for every nonzero submodule K of M.*
(iv) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(SocM) \subset Socf(M)$.*
(v) *For $K \leq M$, $(SocM + K)/K \subset Soc(M/K)$.*
(vi) *If $M = \bigoplus\limits_{\Lambda} M_\lambda$, then $SocM = \bigoplus\limits_{\Lambda} SocM_\lambda$.*

Proof: See [9, 21.2]. □

Definition 1. *Let M be an $R-$module. If every $U \leq M$ with $SocM \leq U$ has a supplement in M, then M is called a socle supplemented (or briefly, s-supplemented) module. (See [6]).*

Definition 2. *Let M be an $R-$module and $X \leq M$. If X is a supplement of a submodule U of M with $SocM \leq U$, then X is called a s-supplement submodule in M. (See [6]).*

Lemma 5. *Let M be an socle supplemented module. Then every finitely $M-$generated $R-$module is socle supplemented. (See [6]).*

2 Cofinitely s-Supplemented Modules

Lemma 6. *Let V be a supplement of U in M. Then*
(1) *If $W + V = M$ for some $W \leq U$, then V is a supplement of W in M.*
(2) *If M is finitely generated, then V is also finitely generated.*
(3) *If U is a maximal submodule of M, then V is cyclic and $U \cap V = RadV$ is the unique maximal submodule of V.*
(4) *If $K \ll M$, then V is a supplement of $U + K$ in M.*
(5) *For $K \ll M$, $K \cap V \ll V$ and hence $RadV = V \cap RadM$.*
(6) *Let $K \leq V$. Then $K \ll V$ if and only if $K \ll M$.*
(7) *For $L \leq U$, $\frac{V+L}{L}$ is a supplement of U/L in M/L.*

Proof: See [9, 41.1]. □

Lemma 7. *Let M be an $R-$module.*
(1) *If M is supplemented, then M is cofinitely supplemented.*
(2) *If M is finitely generated and cofinitely supplemented, then M is supplemented.*
(3) *For $M_1 \leq M$ and U cofinite submodule of M, if $M_1 + U$ has a supplement in M and M_1 is cofinitely supplemented, then U also has a supplement in M.*
(4) *Let $M = \sum\limits_{i\in I} M_i$. If M_i is cofinitely supplemented for every $i \in I$, then M is also cofinitely supplemented.*
(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely supplemented.*
(6) *If M is cofinitely supplemented, then M/L is cofinitely supplemented for every $L \leq M$.*
(7) *If M is cofinitely supplemented, then every homomorphic image of M is also cofinitely supplemented.*
(8) *If M is cofinitely supplemented, then every cofinite submodule of $M/RadM$ is a direct summand of $M/RadM$.*
(9) *Hollow and local modules are cofinitely supplemented.*
(10) *If M is cofinitely supplemented, then every $M-$generated module is cofinitely supplemented.*
(11) *${}_RR$ is supplemented if and only if every generated $R-$module is cofinitely supplemented.*

Proof: See [2]. □

Lemma 8. *Let M be an $R-$module.*
(1) *If M is essential supplemented, then M is cofinitely essential supplemented.*
(2) *If M is supplemented, then M is cofinitely essential supplemented.*
(3) *If M is finitely generated and cofinitely essential supplemented, then M is essential supplemented.*
(4) *For $M_1 \leq M$ and U cofinite essential submodule of M, if $M_1 + U$ has a supplement in M and M_1 is cofinitely essential supplemented, then U also has a supplement in M.*
(5) *Let $M = \sum\limits_{i\in I} M_i$. If M_i is cofinitely essential supplemented for every $i \in I$, then M is also cofinitely essential supplemented.*

(6) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely essential supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely essential supplemented.*
(7) *If M is cofinitely essential supplemented, then M/L is cofinitely essential supplemented for every $L \leq M$.*
(8) *If M is cofinitely essential supplemented, then every homomorphic image of M is also cofinitely essential supplemented.*
(9) *If M is cofinitely essential supplemented, then $M/RadM$ have no proper essential submodules.*
(10) *Hollow and local modules are cofinitely essential supplemented.*
(11) *If M is cofinitely essential supplemented, then every $M-$generated module is cofinitely essential supplemented.*
(12) *${}_RR$ is essential supplemented if and only if every generated $R-$module is cofinitely essential supplemented.*

Proof: See [4, 7]. □

Definition 3. *Let M be an $R-$module. If every cofinite submodule U of M with $SocM \leq U$ has a supplement in M, then M is called a cofinitely socle supplemented (briefly, cofinitely s-supplemented) module.*

Proposition 1. *Let M be a finitely generated $R-$module. Then M is s-supplemented if and only if M is cofinitely s-supplemented.*

Proof: Clear since every submodule of M is cofinite. □

Proposition 2. *Every cofinitely s-supplemented module is cofinitely e-supplemented.*

Proof: Let U be a cofinite essential submodule of M. Since $U \trianglelefteq M$, by Lemma 4, $SocM \leq U$. Then U has a supplement in M. Hence U is cofinitely e-supplemented. □

Proposition 3. *Let M be a cofinitely s-supplemented module. Then $M/RadM$ have no proper cofinite essential submodules.*

Proof: Let $\frac{K}{RadM}$ be any cofinite essential submodule of $\frac{M}{RadM}$. By $\frac{M}{K} \cong \frac{M/RadM}{K/RadM}$, K is a cofinite submodule of M. Since $\frac{K}{RadM} \trianglelefteq \frac{M}{RadM}$, $K \trianglelefteq M$ and by Lemma 4, $SocM \leq K$. Since M is cofinitely s-supplemented, K has a supplement V in M. Then $M = K + V$ and $K \cap V \ll V$. Since $M = K + V$, $\frac{M}{RadM} = \frac{K}{RadM} + \frac{V+RadM}{RadM}$. Since $K \cap V \ll V$, by [9, 21.5], $K \cap V \leq RadM$. Then $\frac{K}{RadM} \cap \frac{V+RadM}{RadM} = \frac{K \cap V+RadM}{RadM} = 0$ and $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$. Since $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$ and $\frac{K}{RadM} \trianglelefteq \frac{M}{RadM}$, $\frac{K}{RadM} = \frac{M}{RadM}$. Hence $\frac{M}{RadM}$ have no proper cofinite essential submodules. □

Lemma 9. *Let M be an $R-$module, U be a cofinite submodule of M with $SocM \leq U$ and $M_1 \leq M$. If M_1 is cofinitely s-supplemented and $U + M_1$ has a supplement in M, then U has a supplement in M.*

Proof: Let X be a supplement of $U + M_1$ in M. Then $M = U + M_1 + X$ and $X \cap (U + M_1) \ll X$. Since U is a cofinite submodule of M and $\frac{M/U}{(U+X)/U} \cong \frac{M}{U+X} = \frac{M_1+U+X}{U+X} \cong \frac{M_1}{M_1 \cap (U+X)}$, $M_1 \cap (U + X)$ is a cofinite submodule of M_1. Since $SocM \leq U + X$, by Lemma 4, $SocM_1 = M_1 \cap SocM \leq (U + X) \cap M_1$. Then by M_1 being cofinitely s-supplemented, $(U + X) \cap M_1$ has a supplement Y in M_1. This case $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \ll Y$. Then $M = U + M_1 + X = U + X + (U + X) \cap M_1 + Y = U + X + Y$ and $U \cap (X + Y) \leq (U + X) \cap Y + (U + Y) \cap X \leq (U + M_1) \cap X + (U + X) \cap Y \ll X + Y$. Hence $X + Y$ is a supplement of U in M. □

Corollary 1. *Let U be a cofinite submodule of M with $SocM \leq U$ and $M_i \leq M$ for $i = 1, 2, ..., n$. If M_i is cofinitely s-supplemented for $i = 1, 2, ..., n$ and $U + M_1 + M_2 + ... + M_n$ has a supplement in M, then U has a supplement in M.*

Proof: Clear from Lemma 9. □

Lemma 10. *Any sum of cofinitely s-supplemented modules is cofinitely e-supplemented.*

Proof: Let U be a cofinite submodule of M with $SocM \leq U$ and $M = \sum_{\lambda \in \Lambda} M_\lambda$ for $M_\lambda \leq M$ and M_λ is cofinitely s-supplemented for every $\lambda \in \Lambda$. Since U is a cofinite submodule of M, then there exist $\lambda_1, \lambda_2, ..., \lambda_n \in \Lambda$ such that $M = U + M_{\lambda_1} + M_{\lambda_2} + ... + M_{\lambda_n}$. Then 0 is a supplement of $U + M_{\lambda_1} + M_{\lambda_2} + ... + M_{\lambda_n}$ in M. Since M_{λ_i} is cofinitely s-supplemented for $i = 1, 2, ..., n$, by Corollary 1, U has a supplement in M. Hence M is cofinitely s-supplemented. □

Lemma 11. *Every factor module a cofinitely s-supplemented module is cofinitely s-supplemented.*

Proof: Let M be a cofinitely s-supplemented $R-$module and $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K}$ be a cofinite submodule of $\frac{M}{K}$ with $Soc\frac{M}{K} \leq \frac{U}{K}$. By Lemma 4, $SocM \leq U$ and since M is cofinitely s-supplemented, U has a supplement V in M. Since $K \leq U$, by [9, 41.1 (7)], $\frac{V+K}{K}$ is a supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Hence $\frac{M}{K}$ is cofinitely s-supplemented. □

Corollary 2. *Every homomorphic image of a cofinitely s-supplemented module is cofinitely s-supplemented.*

Proof: Clear from Lemma 11. □

Lemma 12. *Let M be a cofinitely s-supplemented module. Then every $M-$generated $R-$module is cofinitely s-supplemented.*

Proof: Let N be a $M-$generated $R-$module. Then there exist an index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is cifinitely s-supplemented, by Lemma 10, $M^{(\Lambda)}$ is cofinitely s-supplemented. Then by Corollary 2, N is cofinitely s-supplemented. □

Proposition 4. *Let R be a ring. The following assertions are equivalent.*
(i) *$_RR$ is s-supplemented*
(ii) *$_RR$ is cofinitely s-supplemented.*
(iii) *$R^{(\Lambda)}$ is cofinitely s-supplemented for every index set Λ.*
(iv) *Every $R-$module is cofinitely s-supplemented.*

Proof: $(i) \Longleftrightarrow (ii)$ Clear from Proposition 1, since $_RR$ is finitely generated.
$(ii) \Longrightarrow (iii)$ Clear from Lemma 10.
$(iii) \Longrightarrow (iv)$ Clear from Corollary 2.
$(iv) \Longleftrightarrow (ii)$ Clear. □

3 Conclusion

Cofinitely s-supplemented module is a special part of e-supplemented module.

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On Absolute Matrix Summability Involving (ϕ, δ) Monotone Sequences

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Abstract: In this paper, a general theorem dealing with $|A, p_n; \beta|_k$ summability method of infinite series has been proved by using (ϕ, δ)- monotone sequences. This new theorem also includes several new results.

Keywords: Absolute matrix summability, Infinite series, Summability factors, (ϕ, δ) monotone sequences.

1 Introduction

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. A sequence (μ_n) is said to be (ϕ, δ)-monotone if and only if $\mu_n \to 0$, $\mu_n \geq 0$ ultimately and $\Delta\mu_n \geq -\delta_{n+1}$, where (δ_n) is sequence of non-negative numbers,(ϕ_n) is a positive monotone increasing sequence and $\sum \phi_n\delta_n < \infty$, [1]. Let $\sum a_n$ be an infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-m} = p_{-m} = 0, m \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $A = (A_n(s))$,where

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, \ldots$$

The series $\sum a_n$ is said to be summable $|A, p_n; \beta|_k$, $k \geq 1, \beta \geq 0$, if [2],

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta k+k-1} |\bar{\Delta}A_n(s)|^k < \infty,$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $\beta = 0$, then $|A, p_n; \beta|_k$ summability reduce to $|A, p_n|_k$ summability [3]. For $\beta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, $|A, p_n; \beta|_k$ summability reduce to $|\bar{N}, p_n|_k$ summability [4]. Also, if we take $\beta = 0$, $a_{nv} = \frac{p_v}{P_n}$, $p_n = 1$ for all values n, then $|A, p_n; \beta|_k$ summability reduce to $|C, 1|_k$summability [5].

2 Known Result

In [6], the following theorem dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series has been proved.

Theorem 1. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{1}$$

Suppose that there exist a sequence of numbers (μ_n) which is (ϕ, δ) monotone with $\sum \mu_n \Delta\phi_n$ is convergent. If the conditions

$$\sum_{n=1}^{m} n|\Delta^2 \mu_n|\phi_n = O(1) \quad as \quad m \to \infty, \tag{2}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(\phi_n) \quad as \quad m \to \infty, \tag{3}$$

where $t_n = \frac{1}{n+1}\sum_{v=0}^{n} va_v$, are satisfied then the series $\sum a_n\mu_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

If we take $\mu_n = \frac{2^{(-1)^n}}{n^4}$ and $\phi_n = \log n$, the conditions of Theorem 1 are satisfied. But the sequence (μ_n) does not satisfy the conditions of theorem Mazhar [7] on $|C, 1|_k$ summability factor.

Lemma 1. *[6] Under the conditions of Theorem 1, we get*

$$n\phi_n|\Delta\mu_n| = O(1) \quad as \quad n \to \infty. \tag{4}$$

Lemma 2. *[1] If the sequence (μ_n) is (ϕ, δ) monotone and $\sum \mu_n \Delta\phi_n$ converges, then*

$$\mu_n\phi_n = o(1) \quad as \quad n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} \phi_{n+1}|\Delta\mu_n| < \infty. \tag{6}$$

3 Main Result

There are many papers on absolute matrix summability [8]-[12]. The aim of this paper is to generalize Theorem 1 to $|A, p_n; \beta|_k$ summability. Before stating the main theorem we must first introduce some furthers notations;
Given a normal matrix $A = (a_{nv})$ be a normal matrix, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are given as follows.

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots \tag{8}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{9}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{10}$$

Now shall prove the following theorem.

Theorem 2. *Let $A=(a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0}=1, \quad n=0,1,..., \tag{11}$$

$$a_{n-1,v}\geq a_{nv} \quad for \quad n\geq v+1, \tag{12}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right), \tag{13}$$

$$\left|\hat{a}_{n,v+1}\right|=O\left(v\left|\Delta_v\hat{a}_{nv}\right|\right), \tag{14}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k}\left|\Delta_v(\hat{a}_{nv})\right|=O\left(\left(\frac{P_v}{p_v}\right)^{\beta k-1}\right) \quad as \quad m\to\infty, \tag{15}$$

where $\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}$. If all conditions of Theorem 1 *are satisfied with the condition* (3) *replaced by*

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta k-1}|t_n|^k=O(\phi_m) \quad as \quad m\to\infty \tag{16}$$

then the series $\sum a_n\mu_n$ is summable $|A,p_n;\beta|_k$, $k\geq 1$ and $\beta\geq 0$.

Proof: Let (Θ_n) denotes A-transform of the series $\sum a_n\mu_n$. Then, by (9) and (10), we have

$$\bar{\Delta}\Theta_n=\sum_{v=1}^{n}\frac{\hat{a}_{nv}\mu_v}{v}va_v.$$

By Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta}\Theta_n&=\sum_{v=1}^{n-1}\Delta_v\Big(\frac{\hat{a}_{nv}\mu_v}{v}\Big)\sum_{r=1}^{v}ra_r+\frac{\hat{a}_{nn}\mu_n}{n}\sum_{r=1}^{n}ra_r\\
&=\sum_{v=1}^{n-1}\Delta_v\Big(\frac{\hat{a}_{nv}\mu_v}{v}\Big)(v+1)t_v+\frac{\hat{a}_{nn}\mu_n}{n}(n+1)t_n\\
&=\frac{n+1}{n}a_{nn}\mu_nt_n+\sum_{v=1}^{n-1}\frac{v+1}{v}\Delta_v(\hat{a}_{nv})\mu_vt_v+\sum_{v=1}^{n-1}\frac{v+1}{v}\hat{a}_{n,v+1}\Delta\mu_vt_v+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\mu_{v+1}\frac{t_v}{v}\\
&=\Theta_{n,1}+\Theta_{n,2}+\Theta_{n,3}+\Theta_{n,4}.
\end{aligned}$$

To prove the Theorem 2, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}|\Theta_{n,r}|^k<\infty \quad for \quad r=1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|\Theta_{n,1}\right|^k&=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}a_{nn}^k|\mu_n|^k|t_n|^k\\
&=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta k-1}|\mu_n||t_n|^k\\
&=O(1)\sum_{n=1}^{m-1}\Delta|\mu_n|\sum_{r=1}^{n}\left(\frac{P_r}{p_r}\right)^{\beta k-1}|t_r|^k+O(1)|\mu_m|\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\beta k-1}|t_r|^k\\
&=O(1)\sum_{n=1}^{m-1}|\Delta\mu_n|\phi_{n+1}+O(1)|\mu_m|\phi_m=O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of hypotheses of the Theorem 2 and Lemma 2.
Now applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|\Theta_{n,2}\right|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_v||t_v|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_v|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_v|^k|t_v|^k\right)$$

$$= O(1)\sum_{v=1}^{m}|\mu_v||\mu_v|^{k-1}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k}|\Delta_v(\hat{a}_{nv})|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta k-1}|\mu_v||t_v|^k = O(1) \quad as \quad m\to\infty,$$

as in $\Theta_{n,1}$.

Now, since $v|\Delta\mu_v| = O(1/\phi_v) = O(1)$, by (4), we have that,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|\Theta_{n,3}\right|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\mu_v||t_v|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v||t_v|^k\right)\times\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v||t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\mu_v||t_v|^k\right)$$

$$= O(1)\sum_{v=1}^{m}v|\Delta\mu_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k}|\Delta_v(\hat{a}_{nv})|$$

$$= O(1)\sum_{v=1}^{m}v|\Delta\mu_v||t_v|^k\left(\frac{P_v}{p_v}\right)^{\beta k-1}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\mu_v|)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\beta k-1}|t_r|^k + O(1)m|\Delta\mu_m|\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\beta k-1}|t_r|^k$$

$$= O(1)\sum_{v=1}^{m-1}v|\Delta^2\mu_v|\phi_v + O(1)\sum_{v=1}^{m-1}|\Delta\mu_{v+1}|\phi_{v+1} + O(1)m|\Delta\mu_m|\phi_m$$

$$= O(1) \quad as \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 2, Lemma 1 and Lemma 2.
Finally, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|\Theta_{n,4}\right|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\mu_{v+1}|\frac{|t_v|}{v}\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_{v+1}||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_{v+1}|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\mu_{v+1}||t_v|^k\right) \\
&= O(1)\sum_{v=1}^{m}|\mu_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta k-1}|\mu_{v+1}||t_v|^k = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

as in $\Theta_{n,1}$.
Therefore, we obtain that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|\Theta_{n,r}\right|^k = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.$$

This completes the proof of theorem. □

4 Conclusion

If we take $\beta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ then we get Theorem 1. Also, if we take $\beta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values n, then we get theorem Mazhar [7] on $|C,1|_k$ summability. Moreover, if we take $\beta = 0$, then we get a new theorem involving $|A,p_n|_k$ summability.

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Hyper-Leonardo Numbers

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Abstract: In this paper, we introduce hyper-Leonardo numbers and present some combinatorial properties of these numbers using the Euler-Seidel symmetric algorithm. Additionally, we give the recurrence relations, summation formulas and generating function for the hyper-Leonardo numbers.

Keywords: Generating function, Leonardo sequence, Symmetric algorithm.

1 Introduction

Sequences of integers are widely used in scientific research. The most popular of these are the Fibonacci and Lucas sequences, which are defined by the following recurrence relations, respectively ($n \geq 1$) [1]:

$$F_{n+1} = F_n + F_{n-1} \quad \text{with} \quad F_0 = 0, \quad F_1 = 1, \tag{1}$$

and

$$L_{n+1} = L_n + L_{n-1} \quad \text{with} \quad L_0 = 2, \quad L_1 = 1. \tag{2}$$

The Fibonacci and Lucas numbers are generalized in many ways in the literature [2–6]. Dil and Mező [7] introduced hyper-Fibonacci and hyper-Lucas numbers as the generalizations of the Fibonacci and Lucas numbers, by the following formulas

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)} \quad \text{with} \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1 \tag{3}$$

and

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)} \quad \text{with} \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1, \tag{4}$$

where r is a positive integer, F_n and L_n are the Fibonacci and Lucas numbers, respectively. For $n \geq 1$ and $r \geq 1$, the authors obtained the following recurrence relations for the hyper-Fibonacci and hyper-Lucas numbers, respectively [7]:

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)} \tag{5}$$

and

$$L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}. \tag{6}$$

The first few values of the hyper-Fibonacci and hyper-Lucas numbers are as follows ($n \geq 0$) [8]:

$$F_n^{(1)} : 0, 1, 2, 4, 7, 12, 20, 33 \ldots, \quad F_n^{(2)} : 0, 1, 3, 7, 14, 26, 46, 79 \ldots, \tag{7}$$

$$L_n^{(1)} : 2, 3, 6, 10, 17, 28, 46, 75 \ldots, \quad L_n^{(2)} : 2, 5, 11, 21, 38, 66, 112, 187 \ldots. \tag{8}$$

BahÅ§i et al. [9] presented the summation formulas which give the relation between the hyper-Fibonacci and Fibonacci numbers, similarly the relation between the hyper-Lucas and Lucas numbers:

$$F_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} F_s \tag{9}$$

and

$$L_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} L_s \tag{10}$$

for $n \geq 1$ and $r \geq 1$.

Leonardo sequence, which has similar properties to the Fibonacci sequence, is defined by Catarino and Borges [10] as

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \qquad (n \geq 2) \tag{11}$$

with initial conditions $Le_0 = Le_1 = 1$. For $n \geq 2$, there is the following recurrence relation for the Leonardo numbers [10]:

$$Le_{n+1} = 2Le_n - Le_{n-2}. \tag{12}$$

The Binet formula and generating function for the Leonardo numbers are

$$Le_n = \frac{2\left(\alpha^{n+1} - \beta^{n+1}\right)}{\alpha - \beta} - 1, \tag{13}$$

and

$$g_{Le}(t) = \sum_{n=0}^{\infty} Le_n t^n = \frac{1 - t + t^2}{1 - 2t + t^3} \qquad \left(1 - 2t + t^3 \neq 0\right), \tag{14}$$

where α and β are the roots of the characteristic equation $\lambda^3 - 2\lambda^2 + 1 = 0$ [10]. The authors also obtained some relations among the Fibonacci, Lucas and Leonardo numbers such as

$$Le_n = 2F_{n+1} - 1 \tag{15}$$

and

$$Le_n = L_{n+2} - F_{n+2} - 1, \tag{16}$$

where $n \geq 0$ [10]. Furthermore, Alp and KoÃğer [11] gave the following equalities among the Fibonacci, Lucas and Leonardo numbers for $n \geq 1$:

$$Le_{n-1} + Le_{n+1} = 2L_{n+1} - 2 \tag{17}$$

and

$$Le_n + 2F_n = Le_{n+1}. \tag{18}$$

By the motivation of the above papers, we define hyper-Leonardo numbers as a generalization of the Leonardo numbers and present some properties of newly defined numbers such as the recurrence relations, summation formulas and generating function. In addition, we give some relations among the hyper-Fibonacci, hyper-Lucas and hyper-Leonardo numbers.

2 Main Results

Definition 2.1. The n-th hyper-Leonardo number $Le_n^{(r)}$ is defined as

$$Le_n^{(r)} = \sum_{s=0}^{n} Le_s^{(r-1)} \quad \text{with} \quad Le_n^{(0)} = Le_n, \quad Le_0^{(r)} = Le_0 \quad \text{and} \quad Le_1^{(r)} = r + 1, \tag{19}$$

where r is a positive integer and Le_n is the ordinary Leonardo number.

The first few values of the hyper-Leonardo numbers are as follows $n \geq 0$:

$$Le_n^{(1)} = 1, 2, 5, 10, 19, 34, 59, 100, 167, \ldots, \tag{20}$$

$$Le_n^{(2)} = 1, 3, 8, 18, 37, 71, 130, 230, 397, \ldots. \tag{21}$$

Definition 2.1 yields the recurrence relation

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_n^{(r-1)}, \tag{22}$$

where $n \geq 1$ and $r \geq 1$.

Theorem 1. *The generating function for the hyper-Leonardo numbers is*

$$g(r) = \sum_{n=0}^{\infty} Le_n^{(r)} t^n = \frac{Le_0 + t\left(Le_1 - 2Le_0\right) + t^2\left(Le_2 - 2Le_1\right)}{\left(1 - 2t + t^3\right)\left(1 - t\right)^r} = \frac{1 - t + t^2}{\left(1 - 2t + t^3\right)\left(1 - t\right)^r}. \tag{23}$$

Proof: We use the mathematical induction method on r. By using equation (14), we have

$$g(0) = \sum_{n=0}^{\infty} Le_n^{(0)} t^n = \frac{1-t+t^2}{(1-2t+t^3)(1-t)^0} = \sum_{n=0}^{\infty} Le_n t^n.$$

Thus, the result is true for $r = 0$. Let the result is true for $r = k$, then

$$g(k) = \sum_{n=0}^{\infty} Le_n^{(k)} t^n = \frac{1-t+t^2}{(1-2t+t^3)(1-t)^k}. \tag{24}$$

We must show that the result is true for $r = k+1$.

$$\begin{aligned} g(k+1) &= \sum_{n=0}^{\infty} Le_n^{(k+1)} t^n = Le_0^{(k+1)} + Le_1^{(k+1)} t + Le_2^{(k+1)} t^2 + Le_3^{(k+1)} t^3 + \ldots, \\ tg(k+1) &= Le_0^{(k+1)} t + Le_1^{(k+1)} t^2 + Le_2^{(k+1)} t^3 + \ldots. \end{aligned}$$

By subtracting the above equalities and considering equation (22), we have

$$\begin{aligned} (1-t)\, g(k+1) &= Le_0^{(k+1)} + \left(Le_1^{(k+1)} - Le_0^{(k+1)}\right) t + \left(Le_2^{(k+1)} - Le_1^{(k+1)}\right) t^2 \\ &\qquad + \left(Le_3^{(k+1)} - Le_2^{(k+1)}\right) t^3 + \ldots \\ &= Le_0^{(k)} + Le_1^{(k)} t + Le_2^{(k)} t^2 + Le_3^{(k)} t^3 + \ldots \\ &= \sum_{n=0}^{\infty} Le_n^{(r)} t^k \\ &= g(k). \end{aligned}$$

Since

$$g(k+1) = \frac{g(k)}{1-t},$$

the proof is completed. □

Theorem 2. *If $n \geq 1$ and $r \geq 1$, then there is the following summation formula which gives the relation between the hyper-Leonardo number and Leonardo number:*

$$Le_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} Le_s. \tag{25}$$

Proof: According to the Euler-Seidel algorithm in [12], the symmetric infinite matrix with entries a_n^r has the following recurrence relation:

$$a_n^0 = a_n, \quad a_0^n = a^n \quad (n \geq 0),$$

$$a_n^r = a_n^{r-1} + a_{n-1}^r \quad (n \geq 1, r \geq 1),$$

where (a_n) and (a^n) are two real initial sequences. The entries a_n^r have the following symmetric relation [7]:

$$a_n^r = \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} a_0^i + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} a_s^0. \tag{26}$$

For the case $a_n^r = Le_n^{(r)}$, equation (26) is of the form:

$$Le_n^{(r)} = \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} Le_0^{(i)} + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} Le_s^{(0)}. \tag{27}$$

Considering the initial conditions in Definition 2.1, we have

$$\begin{aligned} Le_n^{(r)} &= \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} Le_s \\ &= \sum_{i=0}^{r-1} \binom{n+r-i-2}{n-1} + \sum_{s=0}^{n-1} \binom{n+r-s-2}{r-1} Le_{s+1} \\ &= \sum_{k=0}^{r-1} \binom{n+k-1}{n-1} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} Le_{n-b}, \end{aligned}$$

where $k = r - i - 1$ and $b = n - s - 1$. The property of the combination in [13] is as follows:

$$\sum_{t=a}^{c} \binom{t}{a} = \binom{c+1}{a+1}. \tag{28}$$

Thus, equation (28) yields

$$\begin{aligned} Le_n^{(r)} &= \binom{n+r-1}{n} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} Le_{n-b} \\ &= \sum_{b=0}^{n} \binom{r+b-1}{r-1} Le_{n-b}, \\ &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} Le_s. \end{aligned}$$

□

Theorem 3. *If $n \geq 3$ and $r \geq 1$, then there is the recurrence relation for the hyper-Leonardo numbers:*

$$Le_n^{(r)} = 2Le_{n-1}^{(r)} - Le_{n-3}^{(r)} + \binom{n+r-3}{r-1} - \binom{n+r-2}{r-1} + \binom{n+r-1}{r-1}. \tag{29}$$

Proof: Considering Theorem 2 and equation (12), we have

$$\begin{aligned} Le_n^{(r)} &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} Le_s \\ &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} (2Le_{s-1} - Le_{s-3}) \\ &= 2\sum_{s=-1}^{n-1} \binom{n+r-(s+1)-1}{r-1} Le_s - \sum_{s=-3}^{n-3} \binom{n+r-(s+3)-1}{r-1} Le_s \\ &= 2\left(\sum_{s=0}^{n-1} \binom{(n-1)+r-s-1}{r-1} Le_s + \binom{n+r-1}{r-1} Le_{-1}\right) \\ &- \left(\sum_{s=0}^{n-3} \binom{(n-3)+r-s-1}{r-1} Le_s + \binom{n+r-3}{r-1} Le_{-1} + \binom{n+r-2}{r-1} Le_{-2} + \binom{n+r-1}{r-1} Le_{-3}\right) \\ &= 2Le_{n-1}^{(r)} - Le_{n-3}^{(r)} + \binom{n+r-3}{r-1} - \binom{n+r-2}{r-1} + \binom{n+r-1}{r-1}. \end{aligned}$$

□

Theorem 4. *If $n \geq 1$ and $r \geq 1$, then*

$$\sum_{s=0}^{r} Le_n^{(s)} = Le_{n+1}^{(r)} - 2F_n \tag{30}$$

is valid.

Proof: By using Theorem 2 and equation (28), we get

$$\begin{aligned} \sum_{s=1}^{r} Le_n^{(s)} &= \sum_{s=1}^{r} \left(\sum_{t=0}^{n} \binom{n+s-t-1}{s-1} Le_t\right) \\ &= \sum_{t=0}^{n} \left(Le_t \sum_{s=1}^{r} \binom{n+s-t-1}{s-1}\right) \\ &= \sum_{t=0}^{n} \binom{n+r-t}{r-1} Le_t \\ &= \sum_{t=0}^{n+1} \binom{n+r-t}{r-1} Le_t - Le_{n+1}. \end{aligned}$$

Then, equation (15) yields

$$\begin{aligned} \sum_{s=0}^{r} Le_n^{(s)} &= Le_{n+1}^{(r)} - (Le_{n-1} + 1) \\ &= Le_{n+1}^{(r)} - 2F_n. \end{aligned}$$

□

Theorem 5. *There are the following relations among the hyper-Leonardo, hyper-Fibonacci and hyper-Lucas numbers* $(r \geq 1)$*:*

(i) $Le_n^{(r)} = 2F_{n+1}^{(r)} - \binom{n+r}{r}, \quad (n \geq 1),$

(ii) $Le_n^{(r)} = L_{n+2}^{(r)} - F_{n+2}^{(r)} - 2\binom{n+r+1}{r-1} - \binom{n+r}{r}, \quad (n \geq 1),$

(iii) $2L_n^{(r)} = Le_{n-2}^{(r)} + Le_n^{(r)} - \binom{n+r-2}{r-1} + \binom{n+r-1}{r-1} + 2\binom{n+r}{r}, \quad (n \geq 2),$

(iv) $Le_n^{(r)} + 2F_n^{(r)} = Le_{n+1}^{(r)} - \binom{n+r}{r-1}, \quad (n \geq 1).$

Proof: Considering Theorem 2, equations (15), (16), (17) and (18), we get the proofs as follows:

(i)

$$\begin{aligned}
Le_n^{(r)} &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s \\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}2F_{s+1}-1 \\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}2F_{s+1} - \sum_{s=0}^{n}\binom{n+r-s-1}{r-1} \\
&= \sum_{s=1}^{n+1}\binom{n+r-(s-1)-1}{r-1}2F_s - \binom{n+r}{r} \\
&= 2F_{n+1}^{(r)} - \binom{n+r}{r},
\end{aligned}$$

(ii)

$$\begin{aligned}
Le_n^{(r)} &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s \\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\left(L_{s+2}-F_{s+2}-1\right) \\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_{s+2} - \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}F_{s+2} - \sum_{s=0}^{n}\binom{n+r-s-1}{r-1} \\
&= \sum_{s=2}^{n+2}\binom{n+r-(s-2)-1}{r-1}L_s - \sum_{s=2}^{n+2}\binom{n+r-(s-2)-1}{r-1}F_s - \binom{n+r}{r} \\
&= \sum_{s=0}^{n+2}\binom{(n+2)+r-s-1}{r-1}L_s - \binom{n+r+1}{r-1}L_0 - \binom{n+r}{r-1}L_1 \\
&\quad - \sum_{s=0}^{n+2}\binom{(n+2)+r-s-1}{r-1}F_s + \binom{n+r+1}{r-1}F_0 + \binom{n+r}{r-1}F_1 - \binom{n+r}{r} \\
&= L_{n+2}^{(r)} - F_{n+2}^{(r)} - 2\binom{n+r+1}{r-1} - \binom{n+r}{r},
\end{aligned}$$

(iii)

$$\begin{aligned}
2L_n^{(r)} &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}2L_s \\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\left(Le_{s-2}+Le_s+2\right) \\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_{s-2} + \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s + 2\sum_{s=0}^{n}\binom{n+r-s-1}{r-1} \\
&= \sum_{s=-2}^{n-1}\binom{n+r-(s+2)-1}{r-1}Le_s + \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s + 2\binom{n+r}{r} \\
&= \sum_{s=0}^{n-2}\binom{(n-2)+r-s-1}{r-1}Le_s + \binom{n+r-2}{r-1}Le_{-1} + \binom{n+r-1}{r-1}Le_{-2} + \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s + 2\binom{n+r}{r} \\
&= Le_{n-2}^{(r)} + Le_n^{(r)} - \binom{n+r-2}{r-1} + \binom{n+r-1}{r-1} + 2\binom{n+r}{r},
\end{aligned}$$

(iv)

$$\begin{aligned} Le_n^{(r)}+2F_n^{(r)} &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s+\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}2F_s \\ &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}(Le_s+2F_s) \\ &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_{s+1} \\ &= \sum_{s=1}^{n+1}\binom{n+r-(s-1)-1}{r-1}Le_s \\ &= \sum_{s=0}^{n+1}\binom{(n+1)+r-s-1}{r-1}Le_s-\binom{n+r}{r-1}Le_0 \\ &= Le_{n+1}^{(r)}-\binom{n+r}{r-1}. \end{aligned}$$

□

Theorem 6. *If $n \geq 0$ and $r = 1, 2$, then*

(i) $Le_n^{(1)} = Le_{n+2} - (n+2)$,
(ii) $Le_n^{(2)} = Le_{n+4} - \frac{1}{2}\left(n^2 + 7n + 16\right)$

are valid.

Proof: We use the mathematical induction method on n.

(i) The result is true for $n = 0$. That is,

$$Le_0^{(1)} = Le_2 - 2.$$

Let the result is true for $n = k$. Then,

$$Le_k^{(1)} = Le_{k+2} - (k+2). \tag{31}$$

We must show that the result is true for $n = k+1$. By using Definition 2.1 and recurrence relation in equation (22), we have

$$\begin{aligned} Le_{k+1}^{(1)} &= Le_k^{(1)} + Le_{k+1} \\ &= Le_{k+2} - (k+2) + Le_{k+1} \\ &= Le_{k+2} + Le_{k+1} + 1 - (k+3) \\ &= Le_{k+3} - (k+3), \end{aligned}$$

(ii) Since

$$Le_0^{(2)} = Le_4 - 8,$$

the result is true for $n = 0$. Assume the result is true for $n = k$. Then,

$$Le_k^{(2)} = Le_{k+4} - \frac{1}{2}\left(k^2 + 7k + 16\right) \tag{32}$$

is valid. For $n = k+1$, considering the assumption and (i), we get

$$\begin{aligned} Le_{k+1}^{(2)} &= Le_k^{(2)} + Le_{k+1}^{(1)} \\ &= Le_{k+4} - \tfrac{1}{2}\left(k^2 + 7k + 16\right) + Le_{k+3} - (k+3) + 1 - 1 \\ &= Le_{k+5} - \tfrac{1}{2}\left((k+1)^2 + 7(k+1) + 16\right). \end{aligned}$$

□

Theorem 7. *For $n \geq 0$ and $r = 1, 2$, the Binet formulas for the hyper-Leonardo numbers are:*

(i) $Le_n^{(1)} = \dfrac{2\left(\alpha^{n+3} - \beta^{n+3}\right)}{\alpha - \beta} - (n+3)$,
(ii) $Le_n^{(2)} = \dfrac{2\left(\alpha^{n+5} - \beta^{n+5}\right)}{\alpha - \beta} - \dfrac{1}{2}\left(n^2 + 7n + 18\right)$,

where α and β are the roots of the characteristic equation $\lambda^3 - 2\lambda^2 + 1 = 0$.

Proof: By using Theorem 6 and equation (13), we obtain desired results. □

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Some upper bounds of A-Berezin number inequalities

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Abstract: We study some new inequalities by using bounded function $\widetilde{U}$ and a positive bounded linear operator A, involving A-Berezin number inequalities and the A-Berezin norm for operators acting on the functional Hilbert space. In particular, for $U \in \mathcal{B}_A(\mathcal{H})$ we prove that

$$\mathrm{ber}_A^2(U) \leq \frac{1}{2}\left\|U^{\sharp_A}U + UU^{\sharp_A}\right\|_{A-\mathrm{Ber}} - \frac{1}{2}\eta_A(U),$$

where $\eta_A(U) := \inf_{\lambda \in Q}\left(\left\|U\widehat{k}_\lambda\right\|_A - \left\|U^{\sharp_A}\widehat{k}_\lambda\right\|_A\right)^2$ and $\mathrm{ber}_A(\cdot)$ is the Berezin number of operator U. Other connected issues are also covered.

Keywords: A-Berezin number, Berezin symbol, Positive operator, functional Hilbert space.

1 Introduction

We examine various issues concerning the Berezin symbols of bounded linear operators on Hilbert function spaces in this study.

Let Q be a region in the complex plane $\mathbb{C}$. A functional Hilbert space (f.H.s.) is a Hilbert space $\mathcal{H}$ of complex- valued functions on Q with the following properties: (i) if f and g are in $\mathcal{H}$ and if α and β are scalars, then $(\alpha f + \beta g)(\lambda) = \alpha f(\lambda) + \beta g(\lambda)$ for every $\lambda \in Q$, and (ii) for every $\lambda \in Q$, there exists a positive constant γ_λ, such that $|f(\lambda)| \leq \gamma_\lambda \|f\|$ for every $f \in \mathcal{H}$. Observe that if $\mathcal{H}$ is a f.H.s. over Q, then for each $\lambda \in Q$, the evaluation map $E_\lambda : \mathcal{H} \to \mathbb{C}$, $E_\lambda f : f(\lambda)$, is a bounded linear functional. Via the classical Riesz representation theorem in the functional analysis, there exists, for every $\lambda \in Q$, a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for every $f \in \mathcal{H}$. Further, we will denote the normalized reproducing kernel at λ as $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$. For more information on f.H.s. see [1].

For a given linear map $U : \mathcal{H} \to \mathcal{H}$, the Berezin symbol (or transform) $\widetilde{U}$, of U is defined by

$$\widetilde{U}(\lambda) = \left\langle U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle, \text{ for } \lambda \in Q.$$

The concept of the Berezin symbol was first introduced by Felix Alexandrovich Berezin, in 1972, see [5]. For the basic properties and facts on these new concepts, see [16, 17].

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for any $x \in H$ and we write a positive operator as $A \geq 0$. It is clear that a positive operator A induces a positive semidefinite sesquilinear form $\langle \cdot, \cdot\rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ defined by $\langle x, y\rangle_A = \langle Ax, y\rangle$ for all $x, y \in \mathcal{H}$. The semi-norm induced by $\langle \cdot, \cdot\rangle_A$ is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A}$, which also satisfies $\|x\|_A = \left\|A^{\frac{1}{2}}x\right\|$. An operator $V \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of an operator U if $\langle Ux, y\rangle_A = \langle x, Vy\rangle_A$ holds for all $x, y \in H$. The set of all operators in $\mathcal{B}(\mathcal{H})$ admitting A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas theorem, it holds that

$$\mathcal{B}_A(\mathcal{H}) = \left\{U \in \mathcal{B}(\mathcal{H}) : R\left(U^*A\right) \subseteq R(A)\right\},$$

where $R(U)$ and U^* are the range and adjoint of operator U, respectively. If $U \in \mathcal{B}_A(\mathcal{H})$, the Douglas solution of the equation $AX = U^*A$ is a distinguished A-adjoint operator of U, which is denoted by $U^{\sharp_A}$. The definitions and properties needed in this paper are shown in [6, 8, 9, 12, 18, 20, 21].

We can give the following definitions given in [12].

Definition 1. *(i) For $U \in \mathcal{B}_A(\mathcal{H})$, the A-Berezin set of $\left\langle U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A$ is defined by*

$$\mathrm{Ber}_A(U) = \left\{\left\langle U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A : \lambda \in Q\right\}.$$

(ii) A-Berezin number of operators $U \in \mathcal{B}_A(\mathcal{H})$ is defined by

$$\operatorname{ber}_A(U) = \sup_{\lambda \in Q} \left| \left\langle U\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A \right|.$$

(iii) A-Berezin norm of operators $U \in \mathcal{B}_A(\mathcal{H})$ is defined by

$$\|U\|_{A-\mathrm{Ber}} = \sup_{\lambda \in Q} \left\| AU\widehat{k}_\lambda \right\|_{\mathcal{H}}.$$

If $A = I$, we get the Berezin number. So, this concept generalizes the Berezin number of f.H.s. operators which have recently attracted the attention of many authors (for instance [3, 4, 10, 11, 13–15]).

In this paper, the structure is as follows. The literature review is covered in Section 1 of the introduction. The key findings are then demonstrated in Section 2. That is, the preliminary lemmas of this paper are shown. In Section 3, we present some upper bounds for A-Berezin radius of operators. In particular, for $U \in \mathcal{B}_A(\mathcal{H})$ we prove that

$$\operatorname{ber}_A^2(U) \leq \frac{1}{2} \left\| U^{\sharp_A}U + UU^{\sharp_A} \right\|_A - \frac{1}{2} \inf_{\lambda \in Q} \left(\left\| U\widehat{k}_\lambda \right\|_A - \left\| U^{\sharp_A}\widehat{k}_\lambda \right\|_A \right)^2$$

and

$$\operatorname{ber}_A^4(U) \leq \frac{3}{16} \left\| UU^{\sharp_A} + U^{\sharp_A}U \right\|_A^2 + \frac{1}{8} \left\| UU^{\sharp_A} + U^{\sharp_A}U \right\|_A \operatorname{ber}_A(U^2).$$

In addition, given the sum of two operators, we must develop an A-Berezin norm inequality and multiple A-Berezin number inequalities. Particularly, we demonstrate that

$$\|U+V\|_A^2 \leq \sqrt{\left\| \left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2 \right\|_{A-\mathrm{Ber}} + 2\operatorname{ber}_A^2(V^{\sharp_A}U)} + \|U\|_{A-\mathrm{Ber}} \|V\|_{A-\mathrm{Ber}} + \operatorname{ber}_A(V^{\sharp_A}U)$$

and

$$\operatorname{ber}_A^2(U+V) \leq \left\| U^{\sharp_A}U + V^{\sharp_A}V \right\|_{A-\mathrm{Ber}} + \frac{1}{2} \left\| U^{\sharp_A}U - V^{\sharp_A}V \right\|_{A-\mathrm{Ber}} 2\operatorname{ber}_A(V^{\sharp_A}U)$$

for $U, V \in \mathcal{B}_A(\mathcal{H})$.

2 Auxiliary Theorems

The following lemmas are required to validate the findings of this research (see, [2, 7, 18, 19]).

Lemma 1. *Let $x, y, z, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then, the following statements are supplied.*

$$\left|\langle x, y\rangle_A\right|^2 + \left|\langle x, z\rangle_A\right|^2 \leq \|x\|_A^2 \left(\max\left\{ \|y\|_A^2, \|z\|_A^2 \right\} + \left|\langle y, z\rangle_A\right| \right), \tag{1}$$

$$\left|\langle a, e\rangle_A \langle e, b\rangle_A\right| \leq \frac{1}{2} \left(\|a\|_A \|b\|_A + \left|\langle a, b\rangle_A\right| \right), \tag{2}$$

$$\left|\langle a, e\rangle_A \langle e, b\rangle_A\right|^2 \leq \frac{1}{4} \left(3\|a\|_A^2 \|b\|_A^2 + \|a\|_A \|b\|_A \left|\langle a, b\rangle_A\right| \right), \tag{3}$$

and

$$\left|\langle x, y\rangle_A\right|^2 + \left|\langle x, z\rangle_A\right|^2 \leq \|x\|_A^2 \sqrt{\left|\langle y, y\rangle_A\right|^2 + \left|\langle z, z\rangle_A\right|^2 + 2\left|\langle y, z\rangle_A\right|^2}. \tag{4}$$

Lemma 2. *Let $U \in \mathcal{B}_A(\mathcal{H})$. Then, $U = U^{\sharp_A}$ iff U is an A-adjoint operator and $R(U) \subseteq \overline{R(A)}$.*

Remark 1. $\left(U^{\sharp_A}U\right)^{\sharp_A} = U^{\sharp_A}U$ *may be deduced from Lemma* 2.

We have some basic inequalities that will assist us prove our results further down:

$$\sqrt{a}\sqrt{b} = \frac{1}{2}(a+b) - \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2, \ a, b \geq 0 \tag{5}$$

$$\max\{a, b\} = \frac{1}{2}(a + b + |a-b|), \tag{6}$$

and

$$(ab+cd)^2 \leq \left(a^2+c^2\right)\left(b^2+d^2\right), \ a, b, c, d \in \mathbb{R}. \tag{7}$$

3 Main result

Let $\mathcal{H}=\mathcal{H}(Q)$ be a f.H.s. with reproducing kernel $\widehat{k}_\lambda$. In this part, we'll look at potential upper boundaries for the A-Berezin radius.

Let us present our first lemma, which is required in our results.

Lemma 3. *Let $U\in\mathcal{B}_A(\mathcal{H})$. Then for any $\lambda,\mu\in Q$, we have*

$$\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle_A\right|^2\leq\sqrt{\left\langle U^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A}\sqrt{\left\langle UU^{\sharp_A}\widehat{k}_\mu,\widehat{k}_\mu\right\rangle_A}. \tag{8}$$

Proof: For $\lambda,\mu\in Q$, we can see that

$$\begin{aligned}\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle_A\right|^2&=\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle_A\right|\;\left|\left\langle\widehat{k}_\lambda,U^{\sharp_A}\widehat{k}_\mu\right\rangle_A\right|\\&=\left|\left\langle A^{\frac12}U\widehat{k}_\lambda,A^{\frac12}\widehat{k}_\mu\right\rangle\right|\left|\left\langle A^{\frac12}\widehat{k}_\lambda,A^{\frac12}U^{\sharp_A}\widehat{k}_\mu\right\rangle\right|\\&\leq\left\|U\widehat{k}_\lambda\right\|_A\left\|U^{\sharp_A}\widehat{k}_\mu\right\|_A\\&=\sqrt{\left\langle U^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A}\sqrt{\left\langle UU^{\sharp_A}\widehat{k}_\mu,\widehat{k}_\mu\right\rangle_A}.\end{aligned}$$

by applying the Cauchy-Schwarz inequality. As a consequence, the outcome is established. □

Now, let we give the first our theorem.

Theorem 1. *If $U\in\mathcal{B}_A(\mathcal{H})$, then we have*

$$\mathrm{ber}_A^2(U)\leq\frac12\left\|U^{\sharp_A}U+UU^{\sharp_A}\right\|_{A-\mathrm{Ber}}-\frac12\eta_A(U), \tag{9}$$

where $\eta_A(U):=\inf_{\lambda\in Q}\left(\left\|U\widehat{k}_{\mathcal{H},\lambda}\right\|_A-\left\|U^{\sharp_A}\widehat{k}_\lambda\right\|_A\right)^2$.

Proof: Let $\lambda\in Q$ be arbitrary. By setting $a=\left\langle U^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A$ and $b=\left\langle UU^{\sharp_A}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A$ in (5), we get

$$\begin{aligned}\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|^2&\leq\sqrt{\left\langle U^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A}\sqrt{\left\langle UU^{\sharp_A}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A}\\&\text{(by the inequality (8))}\\&=\frac12\left(\left\langle U^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A+\left\langle UU^{\sharp_A}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right)\\&\quad-\frac12\left(\sqrt{\left\langle U^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A}-\sqrt{\left\langle UU^{\sharp_A}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A}\right)^2\\&\text{(by the inequality (5))}\\&=\frac12\left\langle\left(U^{\sharp_A}U+UU^{\sharp_A}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A-\frac12\left(\left\|U\widehat{k}_\lambda\right\|_A-\left\|U^{\sharp_A}\widehat{k}_\lambda\right\|_A\right)^2\\&\leq\frac12\left\|U^{\sharp_A}U+UU^{\sharp_A}\right\|_{A-\mathrm{Ber}}-\frac12\inf_{\lambda\in Q}\left(\left\|U\widehat{k}_\lambda\right\|_A-\left\|U^{\sharp_A}\widehat{k}_\lambda\right\|_A\right)^2\end{aligned}$$

and

$$\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|^2\leq\frac12\left\|U^{\sharp_A}U+UU^{\sharp_A}\right\|_{A-\mathrm{Ber}}-\frac12\eta_A(U).$$

Therefore, taking the supremum over $\lambda\in Q$ in the above inequality we deduce that

$$\mathrm{ber}_A^2(U)\leq\frac12\left\|U^{\sharp_A}U+UU^{\sharp_A}\right\|_{A-\mathrm{Ber}}-\frac12\eta_A(U).$$

We get the desired inequality (9). □

Theorem 2. *If $U,V\in\mathcal{B}_A(\mathcal{H})$, then we have*

$$\mathrm{ber}_A^4\left(V^{\sharp_A}U\right)\leq\frac{3}{16}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}}^2+\frac18\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}}\mathrm{ber}_A\left(V^{\sharp_A}VU^{\sharp_A}U\right).$$

Proof: The following inequality holds for any $\lambda \in Q$:

$$\begin{aligned}
&4\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left\langle S\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2 \\
&\leq 3\left\|T\widehat{k}_\lambda\right\|_A^2 \left\|S^{\sharp_A}\widehat{k}_\lambda\right\|_A^2 + \left\|T\widehat{k}_\lambda\right\|_A \left\|S^{\sharp_A}\widehat{k}_\lambda\right\|_A \left|\left\langle T\widehat{k}_\lambda, S^{\sharp_A}\widehat{k}_\lambda\right\rangle_A\right| \\
&\text{(by the inequality (3))} \\
&= 3\left(\sqrt{\left\langle T^{\sharp_A}T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left\langle SS^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}\right)^2 \\
&+ \sqrt{\left\langle T^{\sharp_A}T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left\langle SS^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}\left|\left\langle ST\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right| \\
&\leq \frac{3}{4}\left(\left\langle T^{\sharp_A}T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A + \left\langle SS^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right)^2 \\
&+ \frac{1}{2}\left(\left\langle T^{\sharp_A}T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A + \left\langle SS^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right)\left|\left\langle ST\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right| \\
&\text{(by the AM-GM inequality)} \\
&= \frac{3}{4}\left\langle\left(T^{\sharp_A}T + SS^{\sharp_A}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A^2 + \frac{1}{2}\left\langle\left(T^{\sharp_A}T + SS^{\sharp_A}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left|\left\langle ST\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|.
\end{aligned}$$

So we have

$$\begin{aligned}
&\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left\langle S\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2 \\
&\leq \frac{3}{16}\left\langle\left(T^{\sharp_A}T + SS^{\sharp_A}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A^2 + \frac{1}{8}\left\langle\left(T^{\sharp_A}T + SS^{\sharp_A}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left|\left\langle ST\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|.
\end{aligned}$$

Let $T = U^{\sharp_A}U$ and $S = V^{\sharp_A}V$ in the above inequality. By Remark 1, we have$(U^{\sharp_A}U)^{\sharp_A} = U^{\sharp_A}U$ and $(V^{\sharp_A}V)^{\sharp_A} = V^{\sharp_A}V$. The conclusion may then be clearly recognised as

$$\begin{aligned}
&\left|\left\langle U^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left\langle V^{\sharp_A}V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2 \\
&\leq \frac{3}{16}\left\langle\left[\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right]\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A^2 \\
&+ \frac{1}{8}\left\langle\left[\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right]\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left|\left\langle V^{\sharp_A}VU^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right| \\
&\leq \frac{3}{16}\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_A^2 + \frac{1}{8}\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_A \operatorname{ber}_A\left(V^{\sharp_A}VU^{\sharp_A}U\right).
\end{aligned}$$

Furthermore, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^4 = \left|\left\langle U\widehat{k}_\lambda, V\widehat{k}_\lambda\right\rangle_A\right|^4 &\leq \left\|U\widehat{k}_\lambda\right\|_A^4 \left\|V\widehat{k}_\lambda\right\|_A^4 \\
&= \left\langle U\widehat{k}_\lambda, U\widehat{k}_\lambda\right\rangle_A^2 \left\langle V\widehat{k}_\lambda, V\widehat{k}_\lambda\right\rangle_A^2 \\
&= \left\langle U^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A^2 \left\langle V^{\sharp_A}V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A^2 \\
&= \left|\left\langle U^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \left\langle V^{\sharp_A}V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2.
\end{aligned}$$

Taking the supremum over $\lambda \in Q$, we may conclude that

$$\operatorname{ber}_A^4\left(V^{\sharp_A}U\right) \leq \frac{3}{16}\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}}^2 + \frac{1}{8}\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}} \operatorname{ber}_A\left(V^{\sharp_A}VU^{\sharp_A}U\right).$$

□

Remark 2. *It should be noted that Theorem* 2 *is more precise. Indeed;*

$$\begin{aligned}
\operatorname{ber}_A^2\left(V^{\sharp_A}U\right) &\leq \sqrt{\frac{3}{16}\left\|(U^{\sharp_A}U)^2 + (V^{\sharp_A}V)^2\right\|_{A-\mathrm{Ber}}^2 + \frac{1}{8}\left\|(U^{\sharp_A}U)^2 + (V^{\sharp_A}V)^2\right\|_{A-\mathrm{Ber}} \operatorname{ber}_A\left(V^{\sharp_A}VU^{\sharp_A}U\right)} \\
&\leq \frac{1}{2}\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{ber}}. \qquad (10)
\end{aligned}$$

Proof: To demonstrate this claim, we first conclude that

$$\text{ber}_A\left(V^{\sharp_A}VU^{\sharp_A}U\right) \leq \frac{1}{2}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}. \tag{11}$$

Let $\lambda \in Q$ be arbitrary. Since $\left(U^{\sharp_A}U\right)^{\sharp_A} = U^{\sharp_A}U$ and $\left(V^{\sharp_A}V\right)^{\sharp_A} = V^{\sharp_A}V$, through making use of inequality of Cauchy-Schwarz inequality, we have obtained

$$\begin{aligned}
\left|\left\langle V^{\sharp_A}VU^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right| &= \left|\left\langle U^{\sharp_A}U\widehat{k}_\lambda, V^{\sharp_A}V\widehat{k}_\lambda\right\rangle_A\right| \\
&\leq \left\|U^{\sharp_A}U\widehat{k}_\lambda\right\|_{A-\text{Ber}}\left\|V^{\sharp_A}V\widehat{k}_\lambda\right\|_{A-\text{Ber}} \\
&= \sqrt{\left\langle U^{\sharp_A}U\widehat{k}_\lambda, U^{\sharp_A}U\widehat{k}_\lambda\right\rangle_A\left\langle V^{\sharp_A}V\widehat{k}_\lambda, V^{\sharp_A}V\widehat{k}_\lambda\right\rangle_A} \\
&\leq \frac{1}{2}\left\langle\left[\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right]\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A \\
&\leq \frac{1}{2}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}.
\end{aligned}$$

Taking the supremum over $\lambda \in Q$, we have

$$\text{ber}_A\left(V^{\sharp_A}VU^{\sharp_A}U\right) \leq \frac{1}{2}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}$$

and the inequality (11). As a result, we obtain the desirable the inequality (10)

$$\begin{aligned}
\text{ber}_A^4\left(V^{\sharp_A}U\right) &\leq \frac{3}{16}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}^2 + \frac{1}{8}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}\text{ber}_A\left(V^{\sharp_A}VU^{\sharp_A}U\right) \\
&\leq \frac{3}{16}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}^2 + \frac{1}{16}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}^2 \\
&= \frac{1}{4}\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}}^2.
\end{aligned}$$

□

In [14], for $U, V \in \mathcal{B}(\mathcal{H})$ and $r = 1$, Huban et al. prove that

$$\text{ber}^2\left(V^*U\right) \leq \frac{1}{2}\left\|\left(U^*U\right)^2+\left(V^*V\right)^2\right\|. \tag{12}$$

Remark 2 shows that Theorem 2 is a refinement of the inequality (12) if taking $A = I$.

We now refine the triangle inequality for the A-operator semi-norm and show various A-Berezin radius inequalities for the sum of two operators.

Theorem 3. *If $U, V \in \mathcal{B}_A(\mathcal{H})$, then we have*

$$\text{ber}_A^2(U+V) \leq \sqrt{\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\text{Ber}} + 2\text{ber}_A^2\left(V^{\sharp_A}U\right)} + \|U\|_{A-\text{Ber}}\|V\|_{A-\text{Ber}} + \text{ber}_A\left(V^{\sharp_A}U\right).$$

Proof: For every $\lambda, \mu \in Q$, we have

$$\begin{aligned}
&\left|\left\langle (U+V)\,\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|^2 \\
&\le \left(\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| + \left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\right)^2 \\
&= \left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|^2 + \left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|^2 + 2\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| \\
&\le \sqrt{\left|\left\langle U\widehat{k}_\lambda, U\widehat{k}_\lambda \right\rangle_A\right|^2 + \left|\left\langle V\widehat{k}_\lambda, V\widehat{k}_\lambda \right\rangle_A\right|^2 + 2\left|\left\langle U\widehat{k}_\lambda, V\widehat{k}_\lambda \right\rangle_A\right|^2} \\
&+ 2\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| \\
&\text{(by the inequality (4))} \\
&\le \left(\left\|U^{\sharp_A}U\widehat{k}_\lambda\right\|_A^2 + \left\|V^{\sharp_A}V\widehat{k}_\lambda\right\|_A^2 + 2\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A\right|^2\right)^{\frac{1}{2}} + 2\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| \\
&\le \left(\left\langle U^{\sharp_A}U\widehat{k}_\lambda, U^{\sharp_A}U\widehat{k}_\lambda \right\rangle_A + \left\langle V^{\sharp_A}V\widehat{k}_\lambda, V^{\sharp_A}V\widehat{k}_\lambda \right\rangle_A + 2\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A\right|^2\right)^{\frac{1}{2}} \\
&+ 2\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| \\
&= \left(\left\langle \left(U^{\sharp_A}U\right)^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A + \left\langle \left(V^{\sharp_A}V\right)^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A + 2\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A\right|^2\right)^{\frac{1}{2}} \\
&+ 2\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| \left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| \\
&\left(\text{since } \left(U^{\sharp_A}U\right)^{\sharp_A} = U^{\sharp_A}U \text{ and } \left(V^{\sharp_A}V\right)^{\sharp_A} = V^{\sharp_A}V\right) \\
&\le \sqrt{\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}} + 2\mathrm{ber}_A^2\left(V^{\sharp_A}U\right)} + 2\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|. \qquad (13)
\end{aligned}$$

Putting $a = U\widehat{k}_\lambda$, $e = \widehat{k}_\mu$ and $b = V\widehat{k}_\lambda$ in the inequality (3) we have

$$\begin{aligned}
\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right| &= \left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|\left|\left\langle \widehat{k}_\mu, V\widehat{k}_\lambda \right\rangle_A\right| \\
&\le \frac{1}{2}\left\|U\widehat{k}_\lambda\right\|_A\left\|V\widehat{k}_\lambda\right\|_A + \frac{1}{2}\left|\left\langle U\widehat{k}_\lambda, V\widehat{k}_\lambda \right\rangle_A\right| \\
&= \frac{1}{2}\|U\|_{A-\mathrm{Ber}}\|V\|_{A-\mathrm{Ber}} + \frac{1}{2}\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A\right|. \qquad (14)
\end{aligned}$$

Now, on making use of the inequalities (13) and (14), we get the inequality

$$\begin{aligned}
\left|\left\langle (U+V)\,\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|^2 \le& \sqrt{\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}} + 2\mathrm{ber}_A^2\left(V^{\sharp_A}U\right)} \\
&+ \|U\|_{A-\mathrm{Ber}}\|V\|_{A-\mathrm{Ber}} + \left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A\right|.
\end{aligned}$$

Therefore, taking the supremum over $\lambda \in Q$ with $\lambda = \mu$, we deduce

$$\begin{aligned}
\sup_{\lambda\in Q}\left|\left\langle (U+V)\,\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle_A\right|^2 \le& \sqrt{\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}} + 2\mathrm{ber}_A^2\left(V^{\sharp_A}U\right)} \\
&+ \|U\|_{A-\mathrm{Ber}}\|V\|_{A-\mathrm{Ber}} + \sup_{\lambda\in Q}\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_A\right|
\end{aligned}$$

and

$$\mathrm{ber}_A^2(U+V) \le \sqrt{\left\|\left(U^{\sharp_A}U\right)^2 + \left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}} + 2\mathrm{ber}_A^2\left(V^{\sharp_A}U\right)} + \|U\|_{A-\mathrm{Ber}}\|V\|_{A-\mathrm{Ber}} + \mathrm{ber}_A\left(V^{\sharp_A}U\right).$$

as desired. □

The inequality in ([13, Theorem 2.12]) states that

$$\mathrm{ber}^2(U+V) \le \mathrm{ber}^2(U) + \mathrm{ber}^2(V) + \|U\|_{\mathrm{Ber}}\|V\|_{\mathrm{Ber}} + \mathrm{ber}\left(V^*U\right)$$

This shows that if we consider $A = I$, then the upper bound obtained in Theorem 3 is better than that obtained in ([13, Theorem 2.12]).

Also, by trangle inequality we will obtain

$$\begin{aligned}
&\sqrt{\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}}+2\mathrm{ber}_A^2\left(V^{\sharp_A}U\right)}\\
&\le\sqrt{\left\|\left(U^{\sharp_A}U\right)^2\right\|_{A-\mathrm{Ber}}+\left\|\left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}}+2\left\|V^{\sharp_A}\right\|_{A-\mathrm{Ber}}^2\|U\|_{A-\mathrm{Ber}}^2}\\
&=\sqrt{\|U\|_{A-\mathrm{Ber}}^4+\|V\|_{A-\mathrm{Ber}}^4+2\|V\|_{A-\mathrm{Ber}}^2\|U\|_{A-\mathrm{Ber}}^2}\\
&=\|U\|_{A-\mathrm{Ber}}^2+\|V\|_{A-\mathrm{Ber}}^2.
\end{aligned}$$

Theorem 3 is sharper than triangle inequality. Since

$$\sqrt{\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}}+2\mathrm{ber}_A^2\left(V^{\sharp_A}U\right)}\le\|U\|_{A-\mathrm{Ber}}^2+\|V\|_{A-\mathrm{Ber}}^2,$$

we get

$$\begin{aligned}
\|U+V\|_{A-\mathrm{Ber}}^2&\le\sqrt{\left\|\left(U^{\sharp_A}U\right)^2+\left(V^{\sharp_A}V\right)^2\right\|_{A-\mathrm{Ber}}+2\mathrm{ber}_A^2\left(V^{\sharp_A}U\right)}+\|U\|_{A-\mathrm{Ber}}\|V\|_{A-\mathrm{Ber}}+\mathrm{ber}_A\left(V^{\sharp_A}U\right)\\
&\le\|U\|_{A-\mathrm{Ber}}^2+\|V\|_{A-\mathrm{Ber}}^2+\|U\|_{A-\mathrm{Ber}}\|V\|_{A-\mathrm{Ber}}+\left\|V^{\sharp_A}\right\|_{A-\mathrm{Ber}}\|U\|_{A-\mathrm{Ber}}\\
&=\left(\|U\|_{A-\mathrm{Ber}}+\|V\|_{A-\mathrm{Ber}}\right)^2.
\end{aligned}$$

Theorem 4. *If $U,V\in\mathcal{B}_A(\mathcal{H})$, then we have*

$$\mathrm{ber}_A^2(U+V)\le\left\|U^{\sharp_A}U+V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}+\frac{1}{2}\left\|U^{\sharp_A}U-V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}+2\mathrm{ber}_A\left(V^{\sharp_A}U\right).$$

Proof: Let $\lambda\in Q$ be arbitrary. By putting $y=U\widehat{k}_\lambda$, $x=\widehat{k}_\mu$ and $z=V\widehat{k}_\lambda$ in the inequality (1) and $a=\left\langle U\widehat{k}_\lambda,U\widehat{k}_\lambda\right\rangle_A$ and $b=\left\langle V\widehat{k}_\lambda,V\widehat{k}_\lambda\right\rangle_A$ in (6) we have

$$\begin{aligned}
&\left|\left\langle(U+V)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|^2\\
&\le\left(\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|+\left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\right)^2\\
&=\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|^2+\left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|^2+2\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\ \left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\\
&\le\max\left\{\left\|U\widehat{k}_\lambda\right\|_{A-\mathrm{Ber}}^2,\left\|V\widehat{k}_\lambda\right\|_{A-\mathrm{Ber}}^2\right\}+\left|\left\langle U\widehat{k}_\lambda,V\widehat{k}_\lambda\right\rangle_A\right|+2\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\\
&=\frac{1}{2}\left(\left\langle U\widehat{k}_\lambda,U\widehat{k}_\lambda\right\rangle_A+\left\langle V\widehat{k}_\lambda,V\widehat{k}_\lambda\right\rangle_A+\left|\left\langle U\widehat{k}_\lambda,U\widehat{k}_\lambda\right\rangle_A-\left\langle V\widehat{k}_\lambda,V\widehat{k}_\lambda\right\rangle_A\right|\right)\\
&+\left|\left\langle U\widehat{k}_\lambda,V\widehat{k}_\lambda\right\rangle_A\right|+2\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\ \left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\\
&=\frac{1}{2}\left(\left\langle\left(U^{\sharp_A}U+V^{\sharp_A}V\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\left|\left\langle\left(U^{\sharp_A}U-V^{\sharp_A}V\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\right)\\
&+\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|+2\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\\
&\le\frac{1}{2}\left(\left\|U^{\sharp_A}U+V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}+\left\|U^{\sharp_A}U-V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}\right)\\
&+\mathrm{ber}_A\left(V^{\sharp_A}U\right)+2\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|.
\end{aligned}$$

On the other hand, we can see from the inequality in (2) that

$$\begin{aligned}
2\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\ \left|\left\langle V\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|&=2\left|\left\langle U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\ \left|\left\langle\widehat{k}_\lambda,V\widehat{k}_\lambda\right\rangle_A\right|\\
&\le\left\|U\widehat{k}_\lambda\right\|_{A-\mathrm{Ber}}\left\|V\widehat{k}_\lambda\right\|_{A-\mathrm{Ber}}+\left|\left\langle U\widehat{k}_\lambda,V\widehat{k}_\lambda\right\rangle_A\right|\\
&\le\frac{1}{2}\left\langle\left(U^{\sharp_A}U+V^{\sharp_A}V\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A+\left|\left\langle V^{\sharp_A}U\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle_A\right|\\
&\le\frac{1}{2}\left\|U^{\sharp_A}U+V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}+\mathrm{ber}_A\left(V^{\sharp_A}U\right).
\end{aligned}$$

So, we obtain

$$\mathrm{ber}_A^2(U+V) \leq \left\|U^{\sharp_A}U + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}} + \frac{1}{2}\left\|U^{\sharp_A}U - V^{\sharp_A}V\right\|_{A-\mathrm{Ber}} + 2\mathrm{ber}_A\left(V^{\sharp_A}U\right)$$

by taking the supremum over $\lambda \in Q$ and combining the preceding two inequalities. □

Recall that the operator $U^{\sharp_A}$ is said to be A-normal if $U^{\sharp_A}U = UU^{\sharp_A}$.

Remark 3. *If $V = U^{\sharp_A}$ is an A-normal operator, then*

$$\mathrm{ber}_A^2\left(U + U^{\sharp_A}\right) = \left\|U + U^{\sharp_A}\right\|_{A-\mathrm{Ber}}^2 \leq 2\left(\|U\|_{A-\mathrm{Ber}}^2 + \mathrm{ber}_A\left(U^2\right)\right)$$
$$\leq \left(\|U\|_{A-\mathrm{Ber}} + \left\|U^{\sharp_A}\right\|_{A-\mathrm{Ber}}\right)^2.$$

Theorem 5. *If $U, V \in \mathcal{B}_A(\mathcal{H})$, then we have*

$$\mathrm{ber}_A^2(U+V)$$
$$\leq 2\min\left\{\left\|U^{\sharp_A}U + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}^{1/2}\left\|UU^{\sharp_A} + VV^{\sharp_A}\right\|_{A-\mathrm{Ber}}^{1/2}, \left\|U^{\sharp_A}U + VV^{\sharp_A}\right\|_{A-\mathrm{Ber}}^{1/2}\left\|UU^{\sharp_A} + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}^{1/2}\right\}.$$

Proof: Let $\lambda \in Q$ be arbitrary. By putting $a = \left\langle U^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A$, $b = \left\langle UU^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A$, $c = \left\langle V^{\sharp_A}V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A$ and $d = \left\langle VV^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A$ in the inequality (7) and using the convexity of $f(U) = U^2$ we have

$$\left|\left\langle (U+V)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2 \leq \left(\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right| + \left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|\right)^2$$
$$\leq 2\left(\left|\left\langle U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2 + \left|\left\langle V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2\right)$$
$$\leq 2\sqrt{\left\langle U^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}\sqrt{\left\langle UU^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}$$
$$+ 2\sqrt{\left\langle V^{\sharp_A}V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}\sqrt{\left\langle VV^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}$$

(by the inequality (8))

$$\leq 2\sqrt{\left\langle U^{\sharp_A}U\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A + \left\langle V^{\sharp_A}V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}$$
$$\sqrt{\left\langle UU^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A + \left\langle VV^{\sharp_A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}$$

(by the inequality (7))

$$= 2\sqrt{\left\langle \left(U^{\sharp_A}U + V^{\sharp_A}V\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}\sqrt{\left\langle \left(UU^{\sharp_A} + VV^{\sharp_A}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A}$$
$$\leq 2\left\|U^{\sharp_A}U + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}^{1/2}\left\|UU^{\sharp_A} + VV^{\sharp_A}\right\|_{A-\mathrm{Ber}}^{1/2}$$

and

$$\left|\left\langle (U+V)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2 \leq 2\left\|U^{\sharp_A}U + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}^{1/2}\left\|UU^{\sharp_A} + VV^{\sharp_A}\right\|_{A-\mathrm{Ber}}^{1/2}. \tag{15}$$

We may also get

$$\left|\left\langle (U+V)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_A\right|^2 \leq 2\left\|U^{\sharp_A}U + VV^{\sharp_A}\right\|_{A-\mathrm{Ber}}^{1/2}\left\|UU^{\sharp_A} + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}^{1/2} \tag{16}$$

by changing the values of a, b, c, d in the preceding proof.

Taking the supremum over $\lambda \in Q$ in the inequalities (15) and (16), we have

$$\mathrm{ber}_A^2(U+V) \leq 2\left\|U^{\sharp_A}U + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}^{1/2}\left\|UU^{\sharp_A} + VV^{\sharp_A}\right\|_{A-\mathrm{Ber}}^{1/2}$$

and

$$\mathrm{ber}_A^2(U+V) \leq 2\left\|U^{\sharp_A}U + VV^{\sharp_A}\right\|_{A-\mathrm{Ber}}^{1/2}\left\|UU^{\sharp_A} + V^{\sharp_A}V\right\|_{A-\mathrm{Ber}}^{1/2},$$

hence, we get the desired result. □

In Theorem 5, if $V = U$ and $V = U^{\sharp_A}$, the inequalities

$$\mathrm{ber}_A^2(U) \leq \min\left\{\|U\|_{A-\mathrm{Ber}}^2, \frac{1}{2}\left\|U^{\sharp_A}U + UU^{\sharp_A}\right\|_{A-\mathrm{Ber}}\right\} = \frac{1}{2}\left\|U^{\sharp_A}U + UU^{\sharp_A}\right\|_{A-\mathrm{Ber}}$$

and

$$\mathrm{ber}_A^2\left(U + U^{\sharp_A}\right) = \left\|U + U^{\sharp_A}\right\|_{A-\mathrm{Ber}}^2 \leq 2\left\|U^{\sharp_A}U + UU^{\sharp_A}\right\|_{A-\mathrm{Ber}} \leq \left(\|U\|_{A-\mathrm{Ber}} + \left\|U^{\sharp_A}\right\|_{A-\mathrm{Ber}}\right)^2$$

are established.

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SPECTRAL ANALYSIS OF A BOUNDARY VALUE PROBLEM WITH A SPECTRAL PARAMETER IN BOUNDARY CONDITIONS

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Abstract: In this study, we consider on a finite interval Sturm-Liouville problem with nonlinear dependence on the spectral parameter in the boundary conditions. The properties of the eigenvalues of the boundary value problem are investigated. The oscillation properties of the eigenfunctions of the boundary value problem are given.
Keywords: Eigenvalue, Non-Linear Spectral Parameter, Oscillation Theorem.

1 Introduction

The boundary value problem with spectral parameter appearing in the boundary conditions for Sturm-Liouville operator arises from a varied assortment of physical problems and other applied problems such as the vibration problem of string with one end is fixed is examined in [1], the study of heat conduction is handled in [2], boundary condition not containing linear spectral parameter is completely given in [3], [4].

In this paper, we are concerned with the boundary value problem by the differential equation

$$-u'' + q(x)u = \mu^2 u, \quad 0 < x < 1, \tag{1}$$

and the boundary conditions

$$u'(0) + \alpha\mu^2 u(0) = 0, \tag{2}$$

$$u'(1) + (\beta_0 + \beta_1\mu + \beta_2\mu^2)u(1) = 0. \tag{3}$$

Here μ is spectral parameter, $q(x)$ is a nonnegative continuous function on the interval $[0, 1]$ and α , β_i $(i = 0, 1, 2)$ are real constants and the following conditions are satisfied:

$$\alpha > 0, \quad \beta_0 > 0, \quad \beta_1 \neq 0, \quad \beta_2 < 0. \tag{4}$$

2 Main Results

2.1 The Eigenvalues of the Boundary Value Problems

Lemma 1. *All eigenvalues of the boundary value problem (1)-(3) are real and different from zero.*

Proof: Let μ_0 be an eigenvalue of the boundary value problem (1)-(3) and $u_0(x)$ is a eigenfunction corresponding to the eigenvalue. Multiplying both side of the following equality by the the function $\overline{u_0(x)}$

$$-u_0''(x) + q(x)u_0(x) = \mu_0^2 u_0(x),$$

and we the obtained the idendity by x from 0 to 1:

$$-\int_0^1 u_0''(x)\overline{u_0(x)}dx + \int_0^1 q(x)|u_0(x)|^2 dx = \mu_0^2 \int_0^1 |u_0(x)|^2 dx.$$

By using the formula of integration by part

$$u_0'(0)\overline{u_0(0)} - u_0'(1)\overline{u_0(1)} + \int_0^1 (|u_0'(x)|^2 + q(x)|u_0(x)|^2)dx = \mu_0^2 \int_0^1 |u_0(x)|^2 dx. \quad (5)$$

From $(2) - (3)$, we get

$$P\mu_0^2 - Q\mu_0 - R = 0 \quad (6)$$

where

$$P = \int_0^1 |u_0(x)|^2 dx + \alpha|u_0(0)|^2 - \beta_2|u_0(1)|^2,$$
$$Q = \beta_1|u_0(1)|^2,$$
$$R = \beta_0|u_0(1)|^2 + \int_0^1 (|u_0'(x)|^2 + q(x)|u_0(x)|^2)dx.$$

From the quadratic equation (6), we have

$$\mu_0 = \frac{Q \mp \sqrt{Q^2 + 4PR}}{2P}.$$

By (4) , $P > 0$, $R > 0$ and $Q \neq 0$; therefore, $Q^2 + 4PR > 0$. Consequently, the equation (6) has only real roots. Lemma 1 is proven. □

Similarly to Theorem 1.1. in [[5], p.14], we can prove that there is a unique solution to equation (1) satisfying the initial conditions

$$\varphi(0, \mu) = 1, \quad \varphi'(0, \mu) = -\alpha\mu^2,$$

where at every fixed $x \in [0, 1]$, the function $\varphi(x, \mu)$ is a entire function of the argument μ.

The eigenvalues of the boundary value problem (1)-(3) are zeros of the entire function

$$\phi(\mu) = \varphi'(1, \mu) + (\beta_0 + \beta_1\mu + \beta_2\mu^2)\varphi(1, \mu) = 0,$$

and its zeros form at most countable set without finite limit points . The eigenvalues are zeros of the function $\phi(x, \lambda)$. It is obtained from Lemma 1, this function does not convert to zero for non-real μ. Thus, the zeros constitute an at most countable set without finite limit points.

Lemma 2. *All eigenvalues of the boundary value problem (1)-(3) are simple.*

Proof: To prove the Lemma, let's show that the zeros of the function $\phi(\mu)$ are simple. Assume that contrary i.e. $\mu = \mu'$ be double zeros of the function $\phi(\mu)$. Therefore, $\phi(\mu') = 0$ and $\dot{\phi}(\mu') = 0$ (the dot '·' denoting the derivative with respect to μ), Since $\phi(x, \mu)$ is solution of the equation (1) , for $\mu \neq \lambda$, the relations below are valid:

$$-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda^2\varphi(x, \lambda),$$
$$-\varphi''(x, \mu) + q(x)\varphi(x, \mu) = \mu^2\varphi(x, \mu).$$

Multiplying the first equation by $\varphi(x, \mu)$ and the second one by $\varphi(x, \lambda)$ and adding together, we have

$$-\varphi''(x, \lambda)\varphi(x, \mu) + \varphi''(x, \mu)\varphi(x, \lambda) = (\lambda^2 - \mu^2)\varphi(x, \lambda)\varphi(x, \mu),$$
$$\frac{d}{dx}(\varphi(x, \lambda)\varphi'(x, \mu) - \varphi'(x, \lambda)\varphi^(x, \mu)) = (\lambda^2 - \mu^2)\varphi(x, \lambda)\varphi(x, \mu).$$

Integrating it from 0 to 1 and using boundary condition (2, 3) we obtain

$$\frac{\varphi(1, \lambda)\varphi'(1, \mu) - \varphi(1, \mu)\varphi'(1, \lambda)}{\lambda - \mu} = (\lambda + \mu)\left\{\alpha + \int_0^1 \varphi(x, \mu)\varphi(x, \lambda)dx\right\}.$$

For $\mu = \mu'$ as $\lambda \to \mu$, we get

$$\varphi'(1, \mu')\dot{\varphi}(1, \mu') - \varphi(1, \mu')\dot{\varphi}(1, \mu') = 2\mu'[\alpha + \int_0^1 \varphi^2(x, \mu)dx] \quad (7)$$

On the other hand, taking into account $\phi(\mu') = 0$, $\dot{\phi}(\mu') = 0$,

$$\varphi'(1, \mu') = -(\beta_0 + \beta_1\mu' + \beta_2\mu')\varphi(1, \mu'^2),$$

and

$$\dot{\varphi}'(1,\mu') = -(\beta_1 + 2\beta_2\mu')\varphi(1,\mu') - (\beta_0 + \beta_1\mu' + \beta_2\mu'^2)\dot{\varphi}(1,\mu').$$

Substituting tle last equations into (7), we have

$$\beta_1\varphi^2(1,\mu') = 2\mu'\left[\int_0^1 \varphi^2(x,\mu')dx - \beta_2\varphi^2(x,\mu') + \alpha\right] \tag{8}$$

It is shown that in In Lemma 1, if μ is an eigenvalue of the boundary value problem (1)-(3), the following square equation is provided:

$$P\mu'^2 - Q\mu' - R = 0, \tag{9}$$

where

$$\begin{aligned} P &= \int_0^1 \varphi^2(x,\mu')dx - \beta_2\varphi^2(x,\mu') + \alpha \\ Q &= \beta_1\varphi^2(x,\mu')dx, \\ R &= \beta_0\varphi^2(x,\mu') + \int_0^1 \varphi'^2(x,\mu')dx + \int_0^1 q(x)\varphi^2(x,\mu'). \end{aligned}$$

From 8, 9 and last equations, we get

$$\mu' = -\frac{Q}{2P},$$

where $P > 0$. From here

$$Q^2 = -4PR,$$

is obtained. The relation gives us a contradiction for $P > 0$, $R > 0$. Lemma 2 is proved.

□

2.2 Oscillatory Properties of the Eigenfunctions

Lemma 3. *Let $u(x)$ is a solution to the equation*

$$u'' + p(x)u = 0, \tag{10}$$

with initial condition

$$u(0) = 1, \quad u'(0) = -\alpha\mu'^2, \tag{11}$$

and $v(x)$ is a solution to the equation

$$v'' + r(x)v = 0, \tag{12}$$

with initial condition

$$v(0) = 1, \quad v'(0) = -\alpha\mu''^2. \tag{13}$$

Assume that

$$p(x) < r(x), \quad x \in [0,1],$$

$u(x)$ has m zeros then $v(x)$ has at least m zeros in same interval. Additionally kth zero of $v(x)$ is less than kth zero of $u(x)$.

Proof: Firstly, we consider the condition $\mu'' > \mu' \geq 0$. Denote by x_1 the zeros of $u(x)$ nearest to 0. By Sturm's Theorem [5], it suffices to prove that $v(x)$ has at least one zero inside the interval $[0, x_1]$. Suppose the contrary. It is obvious that we have the inequalities $u(x) > 0$ and $v(x) > 0$ for $0 \leq x \leq x_1$. Since $u(x_1) = 0$, the function $u(x)$ decreases in a neighborhood of x_1. Consequently, $u'(x_1) \leq 0$. Integrating the identity $\frac{d}{dx}(u'v - uv') = (r(x) - p(x))u(x)v(x)$ from 0 to x_1, we obtain

$$u'(x_1)v(x_1) - \alpha(\mu''^2 - \mu'^2) = \int_0^1 (r(x) - p(x))u(x)v(x)dx. \tag{14}$$

Since $r(x) > p(x)$, $u(x) > 0$ and $v(x) > 0$ in the interval $(0, x_1)$, the right hand side of the last equality is positive. Moreover, $u'(x_1)v(x_1) \leq 0$. Since $\alpha > 0$ and $\mu''^2 > \mu'^2 \geq 0$, we have $\alpha(\mu''^2 - \mu'^2) \geq 0$. Thus, the left hand side of 14 is negative; it gives us a contradiction. Similarly, the case $\mu''^2 < \mu'^2 \leq 0$ is treated . Lemma is proven. □

Theorem 1. *The set of eigenvalues of the boundary value problem (1)-(3) consists of infinitely decreasing sequance of negative eigenvalues $\{\mu_{-n}\}_{n=1}^{\infty}$ and infinitely increasing sequence of positive eigenvalues $\{\mu_n\}_{n=1}^{\infty}$*

$$\cdots < \mu_{-n} < \mu_{-n+1} < \cdots < \mu_{-2} < \mu_{-1} < \mu_1 < \mu_2 < \cdots < \mu_{n-1} < \mu_n < \cdots$$

Besides. there exist such numbers $n_*, n^* \in N$ and $k_*, k^* \in N \cup \{0\}$ that eigenfunctions corresponding to eigenvalues $\mu_{-n}(n \geq n_*)$ and $\mu_n(n \geq n^*)$ have zeros respectively $(n + k_* - n_*)$ and $(n + k^* - n^*)$ in the interval $(0, 1)$.

Proof: The proof of this theorem are corallaries of Lemma 1, Lemma 2, Lemma 3, and Theorem 1. and Theorem 4.1. of paper [6]. □

3 Conclusion

In this paper, spectral properties for Sturm-Liouville operator with spectral parameter contained in the conditions were studied. Some properties of these eigenvalues and the oscillation property of the eigenfunction were examined.

The boundary value problems involving spectral parameters in boundary conditions often encountered in physical applications. By examining such the boundary value problems, the vibration of a weighted bar can be its mathematical expression can be simplified so that it can be worked on. Therefore, this topic is interesting and current for authors.

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Comparison of Finite Difference Method and Residual Power Series Method for Time Fractional Diffusion Problems

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Abstract: In this study, we present a comparison of numerical solutions of time fractional diffusion problems obtained from finite difference method and residual power series method. We review the corresponding methods and investigate their numerical solutions of time fractional diffusion problems. Moreover, absolute and relative absolute errors between exact solution and numerical solutions for $\alpha = 1$ are obtained to compare effectiveness of the methods. It is illustrated that the numerical solutions of time fractional diffusion problems established by these methods are sufficiently accurate even if they have different error and convergence rates.

Keywords: Finite difference method, Residual power series method,Time fractional diffusion equation.
AMS Mathematics Subject Classification : 35R11, 26A33, 65M06.

1 Introduction

Fractional differential equations (FDEs) plays a significant role in a wide spectrum of physical and engineering problems. Since, modelling with FDEs yields much more better results, compare to ordinary differential equations [1–7]. For instance, the modelling the behaviour of slower diffusive matter by time fractional differential equations (TFDEs) gives more realistic results [8]. There are various methods to establish the solution of TFDEs. Analytically diverse algorithms such as Homotopy analysis method, Adomian decomposition method, Laplace transform method are developed and investigated to accomplish the closed form solutions of TFDEs [9–12]. However, specific types of FDEs are solved by these methods. Moreover, numerical algorithms such as combination of finite elements method [13], residual power series method (RPSM) [14–17] and implicit and explicit finite difference method (FDM) [18–22] are employed to acquire the truncated solutions of TFDEs. FDM is a common method to construct the numerical solutions of differential equations [22].

In this paper, we tackle time FDEs below in Caputo sense by using implicit FDM and RPSM :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k(t)\frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < l, 0 < t < T, 0 < \alpha \leqslant 1 \tag{1}$$

subject to the initial and boundary conditions

$$u(x,0) = \varphi(x), 0 \leqslant x \leqslant l, \tag{2}$$

$$u(0,t) = \mu_1(t), u(1,t) = \mu_2(t), 0 \leqslant t \leqslant T. \tag{3}$$

in which $k(t)$ and α denote the diffusion coefficient and the order of time fractional derivative, respectively.

Existence of solution of general form of problem (1-3) is considered in [24].

In this study, approximate solutions obtained using RPSM and FDM are compared and it is concluded that RPSM is a more effective method than FDM for the investigated problem.

2 Preliminaries

Basic notions and features of fractional derivatives are presented in this section [17, 18].
Definition 1. The Riemann-Liouville fractional integral of order α $(\alpha \geqslant 0)$ is given as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,\ \alpha > 0,\ x > 0, \tag{4}$$

$$J^0 f(x) = f(x). \tag{5}$$

Definition 2. The Liouville-Caputo fractional derivative of order α is given as

$$D^{\alpha} f(x) = J^{n-\alpha} D^n f(x) = \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t)dt,\ n-1 < \alpha < n,\ x > 0, \tag{6}$$

where D^n denotes the ordinary derivative of order n.
Definition 3. The α^{th} order derivative of $u(x,t)$ in Liouville-Caputo sense is given as

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} \frac{\partial^n u(x,\xi)}{\partial t^n} d\xi,\ n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n} \qquad ,\alpha = n \in N. \end{cases} \tag{7}$$

Definition 4. The power series expansions of the function $f(t)$ about $t = t_0$

$$\sum_{k=0}^{\infty} \sum_{l=0}^{n-1} f_k (t-t_0)^{k\alpha+l},\ 0 \leqslant n-1 < \alpha \leqslant n,\ t_0 \leqslant t < t_0 + R \tag{8}$$

where f_k are the coefficients, is called multiple fractional power series about $t = t_0$.

3 Finite Difference Method

The expression

$$D_t^{\alpha} u_{i,j} \cong \sigma_{\alpha,\tau} \sum_{n=1}^{j} w_n^{\alpha} (u_{i,j-n+1} - u_{i,j-n}) \tag{9}$$

where

$$\sigma_{\alpha,\tau} = \frac{1}{\Gamma(2-\alpha)\tau^{\alpha}},\ \ w_n^{\alpha} = n^{1-\alpha} - (n-1)^{1-\alpha}. \tag{10}$$

gives an approximation of fractional derivative in Caputo sense (7). By using the advantage of Eq. (9), the discretization of problem (1)-(3) leads to the following

$$\sigma_{\alpha,\tau} \sum_{n=1}^{j} w_n^{\alpha} (u_{i,j-n+1} - u_{i,j-n}) = k_j \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \\ 1 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant m-1, \tag{11}$$

$$u_{i,0} = \varphi(x_i), 0 \leqslant i \leqslant n, \tag{12}$$

$$u_{1,j} = \mu_1(t_j), u_{n,j} = \mu_2(t_j), 1 \leqslant j \leqslant m \tag{13}$$

where $t_j = j\tau, x_i = ih$ with step lengths τ and h on time and space coordinate and $u_{i,j}$ represents approximate value of $u(x_i, t_j)$ and $k_j = k(t_j)$.
The problem (11)-(13) is the implicit finite difference approximation of the problem (1)-(3) at the grid points i, j. In order to construct problem (11)-(13) the domain of the problem $[0,T] \times [0,l]$ is partitioned uniformly. The rearranging the Eq. (11) we get the following the tridiagonal linear system

$$a_i u_{i-1,j} + c_i u_{i,j} + b_i u_{i+1,j} = f_i, \tag{14}$$

where
$a_i = \frac{\tau k_j}{h^2}, c_i = -\frac{2\tau k_j}{h^2} - 1, b_i = \frac{\tau k_j}{h^2}, f_i = -u_{i,j-1} - \sigma_{\alpha,\tau} \sum\limits_{n=2}^{j} w_n^{\alpha} (u_{i,j-n+1} - u_{i,j-n})$.
in order to establish the solution of this tridiagonal system, Thomas algorithm is employed [9].

4 Residual power series method (RPSM)

The form of the solutions Eqs. (1)-(2) for obtained by RPSM is as follows:

$$u(x,t)=\sum_{k=0}^{\infty}f_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},x\in I,0\leqslant t<R \tag{15}$$

The truncated series $u_m(x,t)$ is defined as follows :

$$u_m(x,t)=\sum_{k=0}^{m}f_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},x\in I,0\leqslant t<R \tag{16}$$

Based on RPSM, the first approximation solution of $u(x,t)$ is taken as the initial condition

$$u_0(x,t)=f_0(x)=u(x,0) \tag{17}$$

Therefore, Eq. (16) can be rearranged as follows

$$u_m(x,t)=f_0(x)+\sum_{k=2}^{m}f_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},\ 0<\alpha\leqslant 1,\quad x\in I,\quad 0\leqslant t,k=2,3,... \tag{18}$$

As a result, the residual function of Eq.(1) becomes

$$Res_0(x,t)=D_t^{\alpha}u_0-k(t)u_{0xx}, \tag{19}$$

$$Res_m(x,t)=D_t^{\alpha}u_m-k(t)u_{mxx} \tag{20}$$

is the residual function for m^{th} truncated function. From [20, 22–24], by taking $Res(x,t)=0$ and $D_t^{k\alpha}Res_m(x,0)=0,\ k=0,1,2,...,m,m=1,2,3,...$ the numerical solution is constructed. In order to determine the coefficients $f_k(x),k=2,3,...,m$ in Eq. (18), we utilize the operator $D_t^{(m-1)\alpha}$ for $m=1,2,3,...$ at $t=0$ in Eq.(20) we get

$$D_t^{(m-1)\alpha}Res_m(x,0)=0,0<\alpha\leq 1,m=1,2,3,... \tag{21}$$

Based on the above theory, the coefficients $f_1(x)$ is acquired by utilizing the following residual function

$$Res_1(x,t)=f_1-k(t)\Big(f_0''(x)+f_1''(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\Big) \tag{22}$$

In Eq. (21) which yields

$$f_1(x)=k(0)f_0''(x). \tag{23}$$

Similarly, to other coefficients $f_2(x)$ is obtained as

$$f_2(x)=D_t^{\alpha}(k(t)u_{2xx}(x,t))\mid_{t=0}. \tag{24}$$

In the same manner, the other coefficients are obtained

$$f_3(x)=D_t^{\alpha}D_t^{\alpha}(k(t)u_{3xx}(x,t))\mid_{t=0} \tag{25}$$

$$f_4(x)=D_t^{\alpha}D_t^{\alpha}D_t^{\alpha}(k(t)u_{4xx}(x,t))\mid_{t=0} \tag{26}$$

and so on.

5 Illustrative Examples

Some numerical examples are presented to illustrate the implementation of FDM and RPSM in this section.

Example 1. Take the following problem

$$\frac{\partial^{\alpha}u(x,t)}{\partial t}=\Big(1+6\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\Big)\frac{\partial^2u(x,t)}{\partial x^2},0<x<1,0<t<1,0<\alpha\leqslant 1 \tag{27}$$

subject to the initial and boundary conditions

$$u(x,0)=exp(2x),0\leqslant x\leqslant 1, \tag{28}$$

$$u(0,t)=exp(4t^3+4t),u(1,t)=exp(4t^3+4t+2),0\leqslant t\leqslant 1. \tag{29}$$

$u(x,t)=exp(4t^3+4t+2x)$ represents the analytical solution of the problem (27)-(29) for $\alpha=1$.

	$\alpha = 1$			$\alpha = 0.9$		$\alpha = 0.7$	
x	Exact	FDM	RPSM	FDM	RPSM	FDM	RPSM
0	2980.95799	2980.95799	2980.95767	2980.95799	2981.11669	2980.95799	2981.35986
0.1	3640.95031	3648.24308	3640.94992	4203.94673	3641.14414	4721.34561	3641.44115
0.2	4447.06675	4460.69202	4447.06627	5503.71549	4447.30350	6479.74391	4447.66627
0.3	5431.65959	5450.64826	5431.65901	6903.97278	5431.94876	8262.85434	5432.39185
0.4	6634.24401	6657.51375	6634.24330	8430.27121	6634.59720	10077.47454	6635.13839
0.5	8103.08393	8129.31605	8103.08306	10110.47615	8103.51532	11930.52477	8104.17633
0.6	9897.12906	9924.61820	9897.12800	11975.27705	9897.65596	13829.07484	9898.46332
0.7	12088.38073	12114.84726	12088.37944	14080.81507	12089.02429	15780.37163	12090.01040
0.8	14764.78157	14787.13398	14764.77999	16398.98786	14765.56761	17791.86732	14766.77205
0.9	18033.74493	18047.77654	18033.74300	19038.79162	18034.70500	19871.24844	18036.17611
1	22026.46579	22026.46579	22026.46345	22026.46579	22027.63843	22026.46579	22029.43525

Table 1 Comparison of FDM with $m = n = 1000$ and RPSM with 60 iteration at $T = 1$ for various values of α in Ex.1.

Taking time and space steps $m = 40$ and $n = 40$,respectively yields to numerical results presented in Figs. 1-3 for FDM and RPSM.
It is clear from Table 1 that improved the approximation is establish for the sufficiently large number of nodes and numerical solutions and for $\alpha = 1$ get closer to the exact solution uniformly.

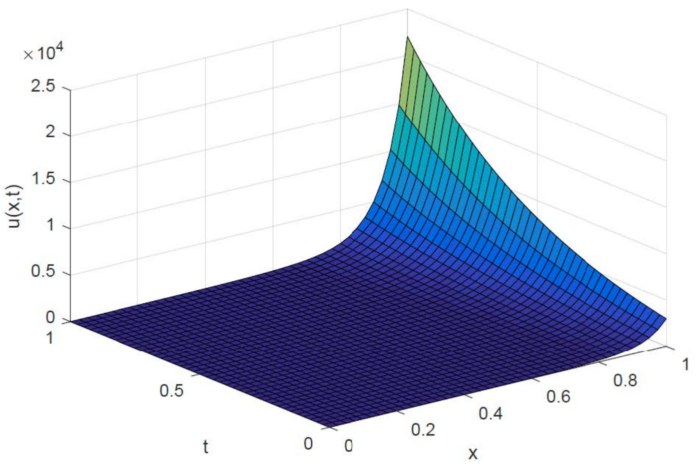

Fig. 1: The graphics of exact solution for $m = n = 40$ and $\alpha = 1$.

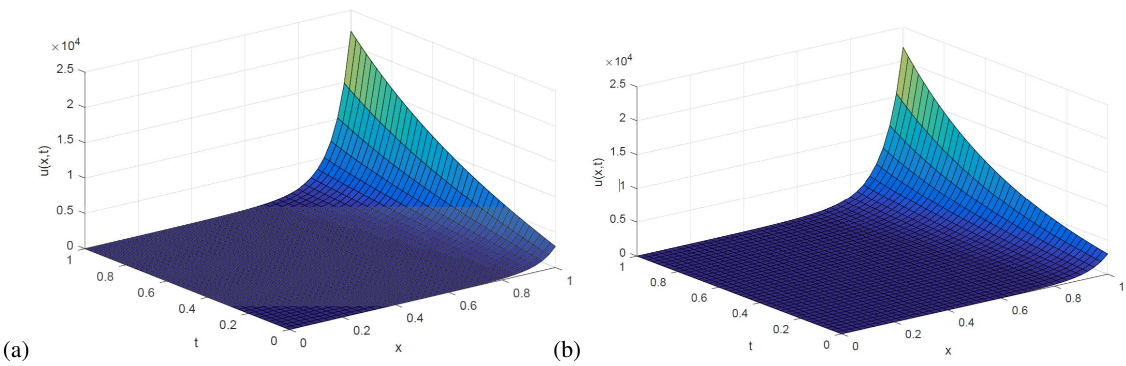

Fig. 2: (a) The graphics of FDM solution for $m = n = 40$ and $\alpha = 0.9$. (b) The graphics of RPSM solution for $\alpha = 0.9$.

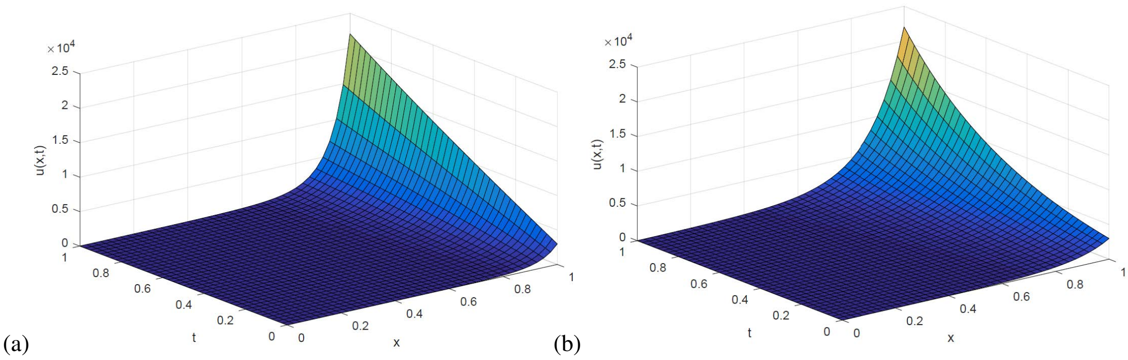

Fig. 3: (a) The graphics of FDM solution for $m = n = 40$ and $\alpha = 0.5$. (b) The graphics of RPSM solution for $\alpha = 0.5$

6 Conclusion

The numerical solutions of time FDEs in Caputo sense are accomplished through FDM and RPSM. Utilizing FDM leads to a linear system which is in the form of tridiagonal matrix. Moreover, RPSM is utilized to establish the series solution by determining the unknown coefficients through the residual function. The implementation these two methods are presented in detail and the obtained the numerical solution are analyzed by comparing them. Based on the theory and illustrative examples we conclude that the solution of RPSM is better than the solution of FDM.

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Kernel principal component analysis based on interval-valued Fermatean fuzzy sets aproach with medical decision-making application

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Abstract: This presentation offers the group decision-making problem related to the hip prosthesis material selection in an interval-valued Fermatean fuzzy environment. For hip prosthesis material selection problems, the advantage of interval-valued Fermatean fuzzy sets is that they can reasonably express the evaluation information given by decision-makers through both qualitative and quantitative aspects. However, if the dimension and nonlinear relationship of the decision data keep growing, the traditional decisionâĂŘmaking methods will fail. For nonlinear features, we construct the intervalâĂŘvalued Fermatean fuzzy language kernel principal component analysis model to reduce the dimensionality. The proposed method will be used in material selection in distinct implementations, exclusively in biomedical applications where the prosthesis materials should have similar characteristics to human tissues. Since biomedical materials are used in various parts of the human body for many different purposes, in this study, material selection will be made using the method presented for the femoral component of the hip joint prosthesis for orthopedists and practitioners who will choose biomaterials.

Keywords: group decision-making, hip prosthesis, interval-valued Fermatean fuzzy environment, kernel principal component analysis.

1 Introduction

In general, uncertainty is the situation in which a given event may have different consequences and there is no information about the probabilities of those consequences. Therefore, uncertainty is a very important notion for the DM process. It is not easy to know the probabilities of events happening in real-life. Therefore, the DM process occurs under uncertainty. Fuzzy logic theory [1] proposes a strong logical inference structure in the face of uncertain and imprecise knowledge. Fuzzy logic theory gives computers the ability to process people's linguistic data and work using people's experiences. While gaining this ability, it uses symbolic expressions instead of numerical expressions. These symbolic expressions are called fuzzy sets(FS). It is understood that the elements of fuzzy sets are decision variables containing probability states. Instead of probability values of possibilities, fuzzy sets arise by assigning membership degrees to each of them objectively.

In an FS F, the degree of belonging and the degree of not belonging to a set, respectively, are expressed as ζ_F and $1 - \zeta_F$. Hence $\zeta_F + 1 - \zeta_F = 1$. While it was thought that FS could explain all the uncertainties when it first emerged, it was later understood that it was insufficient for some real-world problems. The new set developed by Atanassov [2], called intuitionistic fuzzy set(IFS), was a solution for problems where FS fell short. IFS consisted of non-membership degree(ND) together with membership degree(MD) and satisfies the condition $MD + ND = 1$. Yager [3] defined the Pythagorean fuzzy set (PFS) as a more general and comprehensive set than IFS. In PFS, the condition $MD^2 + ND^2 = 1$ is satisfied. There is an extensive diversity of studies on FS, IFS, and PFS such as [4]-[18].

Yager [19] introduced the q-step orthopair fuzzy set. The basic rule in this set theory is that the sum of MD with ND should not be greater than 1. Based on this idea, Senapati & Yager [20] introduced the Fermatean fuzzy set(FFS) and examined its basic features. In the FFS, the MD and ND fulfill $0 \leq m_A^3 + n_A^3 \leq 1$. FFS, which is included in the literature as a new concept, gives better results than the IFS [2], PFS [21], [18] in defining uncertainties. In [22], Fermatean arithmetic means, division, and subtraction which are new transactions for FFS, are defined and some of their properties are examined. In [23], new weighted aggregated operators related to FFSs are defined. Further, the TOPSIS method has been applied to FFS. In addition, Senapati $\&$ Yager [20], the TOPSIS method was applied to FFS. Shahzadi and Akram [24] offered a new decision support algorithm concerning the FFSS and defined the new aggregated operators. Garg et al. [25] new FFS type aggregated operators have been defined. In a study by Donghai et al [26], the notion of Fermatean fuzzy linguistic term sets is offered. Operations, score, and accuracy functions belonging to these sets were given. In [27], a new similarity measure related to Fermatean fuzzy linguistic term sets is constructed.

In a real decision-making environment, affected by the preferences and thinking of DMs, some evaluation information cannot be effectively expressed only with numerical values. On the basis of this, Zadeh proposed linguistic variables (LVs). The LVs can express personal preferences in vague language terms and use qualitative variables to describe the uncertainty of decision information. After the method was proposed, scholars combined the LVs with various FSs and used them in multi-attribute decision-making (MADM). At the same time, to effectively solve some practical problems, scholars have proposed a large number of methods such as TOPSIS, TODIM, VIKOR, and so forth and applied them in the fuzzy decision-making environment.

Xian et al constructed the intervalâĂŘvalued Pythagorean fuzzy language principal component analysis (IVPFLâĂŘPCA) model to reduce the dimensions of the DMs and attribute variables, respectively. However, the IVPFLâĂŘPCA model ignored the nonlinear relationship of the data, and the principal component analysis (PCA) method often produced undesirable results when dealing with nonlinear problems. For non-linear data problems, Vapnik proposed the kernel space theory and used a suitable kernel function as a non-linear mapping. Then nonseparable data are non-linearly mapped to a highâĂŘdimensional feature space to convert into linearly solvable data. With the development of this theory, scholars have developed a PCA method to solve nonlinear data, called the kernel principal component analysis (KPCA) method. This method combined the nonlinear kernel function with PCA, performed linear PCA on the data in the high-dimensional feature space of the mapping, and selected the principal components (PCs) of the feature space for data dimensionality reduction.

In this presentation,considering the decision data may have a nonlinear relationship, we want to reduce the dimensionality in some ways before making an effective decision. Based on these analyses and the advantages of IVFFLSs, we first use IVFFLSs to describe the decisionâĂŘmaking information more reasonably. Then for highâĂŘdimensional decision data, we newly define a distance measure between IVFFLS (intervalâĂŘvalued Fermatean fuzzy linguistic weighted Euclidean distance [IVFFLWD]) to construct the IVFFLâĂŘKPCA model. Based on this model, we will get the decision data after dimensionality reduction and reasonable weight vectors of the attribute and DMs by cumulative contribution rate (CCR). Finally, the optimal decisionâĂŘmaking plan is selected according to the TOPSIS evaluation method. In summary, an IVFFLâĂŘKPCA model based on the TOPSIS method is proposed in this paper for solving the biomaterial selection problem in the highâĂŘdimensional IVFFL environment.

2 Preliminaries

Now, some fundamental information that will be used in the study will be given.

Definition 1. *Let $S : \{s_i : i = 0, 1, \cdots, g\}$ be a linguistic term set with continuous finite subsets, where $g+1$ is an odd number, and $s_i \in S$ is the possible value of a language variable. Let s_i, s_j be any two language term sets in S, satisfying the following properties:*

i. if $i > j$, then $s_i > s_j$,
ii. the negation operator: $neg(s_i) = s_i$, $j = g - 1$.

Definition 2. *For $\mathcal{X} = \{x_1, x_2, \cdots, x_n\}$, if*

$$S = \{(x, \zeta_S(x), \eta_S(x)) : x \in \mathcal{X}\}$$

satisfies the following conditions, then the set S is called FFS:

$$\rho_S, \tau_S \in [0,1], \quad 0 \le \zeta_S^3 + \eta_S^3 \le 1.$$

$\theta_S = (1 - \eta_S^3 + \zeta_S^3)^{1/3}$ *shows the hesitation degree.*

Definition 3. *Let $\mathcal{X} = \{x_1, x_2, \cdots, x_n\}$ be a non-empty finite set of the universe, the S is a continuous set of language term set, then the IVFFLS is defined as*

$$\bar{A} \quad = \quad \{\big(x, (s_\theta(x), (\bar{\zeta}_{\bar{A}}(x), \bar{\eta}_{\bar{A}}(x)))\big) : x \in X\},$$

where $s_\theta(x) \in S$, $\bar{\zeta}_{\bar{A}}(x) = [\zeta^L_{\bar{A}}(x), \zeta^U_{\bar{A}}(x)]$, $\bar{\eta}_{\bar{A}}(x) = [\eta^L_{\bar{A}}(x), \eta^U_{\bar{A}}(x)]$, respectively, represent the degree of interval-valued membership and interval-valued non-membership of the element $x \in X$ for the interval-valued Fermatean fuzzy language variables (IVFFLVs). Then, it satisfied the following conditions:

i. $\bar{\zeta}_{\bar{A}}(x) \subseteq [0,1]$, $\bar{\eta}_{\bar{A}}(x) \subseteq [0,1]$,
ii. $(\zeta^U_{\bar{A}}(x))^3 + (\eta^U_{\bar{A}}(x))^3 \le 1$, for all $x \in X$.

The hesitancy degree of each element $x \in X$ $h_{\bar{A}}(x) = \left(\sqrt[3]{\left(1 - (\zeta^U_{\bar{A}}(x))^3 - (\eta^U_{\bar{A}}(x))^3\right)}, \sqrt[3]{\left(1 - (\zeta^L_{\bar{A}}(x))^3 - (\eta^L_{\bar{A}}(x))^3\right)}\right)$.

Given a set of data (with nonzero mean) x_k for all $k = 1, 2, , m$, where $x_k \in R^{\mathbb{N}}$. The KPCA method first maps the original data to the feature space F through a nonlinear mapping $\Phi : R^{\mathbb{N}} \to F$, where F is composed of $\Phi(x_1), \Phi(x_2), \cdots, \Phi(x_m)$. Then KPCA is to discuss the covariance matrix of the mapped data set $\Phi(x_i)$ $(i = 1, 2, , m)$ in F.

Let $\mathbf{X}^T = [\Phi(x_1), \Phi(x_2), \cdots, \Phi(x_m)]$. We suppose $\Phi(x_i)$ is a given set of data with non-zero mean, then the covariance matrix C on the linear feature space F can be expressed as follows:

$$C = \frac{1}{m}\sum_{i=1}^{m}\left(\Phi(x_i)-\bar{\Phi}\right)\left(\Phi(x_i)-\bar{\Phi}\right)^T, \tag{1}$$

$$\bar{\Phi} = \frac{\Phi(x_1)+\Phi(x_2)+\cdots+\Phi(x_m)}{m} = \frac{1}{m}\mathbf{X}^T\mathbf{1}_{x\times 1}. \tag{2}$$

Since

$$\sum_{i=1}^{m}\Phi(x_i)\bar{\Phi}^T = m\bar{\Phi}\bar{\Phi}^T,$$

$$\sum_{i=1}^{m}\Phi(x_i)\bar{\Phi}^T = \mathbf{X}^T\mathbf{X}.$$

Thus, Equation (1) can be split and written as

$$C = \frac{1}{m}\sum_{i=1}^{m}\left(\Phi(x_i)-\bar{\Phi}\right)\left(\Phi(x_i)-\bar{\Phi}\right)^T = \frac{1}{m}\mathbf{X}^T\mathbf{X}-\Phi(x_i)\bar{\Phi}^T. \tag{3}$$

From Equation (2), we can obtain

$$\bar{\Phi}\bar{\Phi}^T = \frac{1}{m}\mathbf{X}^T\mathbf{1}_{m\times 1}\cdot\left(\frac{1}{m}\mathbf{X}^T\mathbf{1}_{m\times 1}\right)^T = \frac{1}{m^2}\mathbf{X}^T.\mathbf{1}_{m\times m}.\mathbf{X}.$$

Since $\mathbf{X}^T.\mathbf{X}$ in Equation (3) is unkonwn, the eigenvectors and eigenvalues of the covariance matrix C cannot be calculated. Now, let us defined $m\times m$ square kernel matrix K by $\kappa(x_i,x_j)=\Phi(x_i)^T.\Phi(x_j)$ such that

$$K=\mathbf{X}\mathbf{X}^T=\begin{bmatrix}\kappa(x_1,x_1) & \cdots & \kappa(x_1,x_m)\\ \vdots & \ddots & \vdots\\ \kappa(x_m,x_1) & \cdots & \kappa(x_m,x_m)\end{bmatrix}$$

Obviously, the kernel matrix K is a symmetric matrix and can be calculated. So we can arrive at

$$\left(\frac{1}{m}K-\frac{1}{m^2\mathbf{1}_{m\times m}K}\right)\alpha=\lambda\alpha\rightarrow\left(\frac{1}{m}\mathbf{I}-\frac{1}{m^2\mathbf{1}_{m\times m}}\right)K\alpha=\lambda\alpha \tag{4}$$

Among them, λ and α in Equation (4) are obtainable. Multiply both sides of Equation (4) by $\mathbf{X}^T$ at the same time, we have

$$\mathbf{X}^T\left(\frac{1}{m}\mathbf{X}\mathbf{X}^T-\frac{1}{m^2\mathbf{1}_{m\times m}\mathbf{X}\mathbf{X}^T}\right)\alpha=\lambda\mathbf{X}^T\alpha\rightarrow\left(\frac{1}{m}\mathbf{X}\mathbf{X}^T-\frac{1}{m^2\mathbf{1}_{m\times m}}\mathbf{X}\mathbf{X}^T\right)\mathbf{X}^T\alpha=\lambda(\mathbf{X}^T\alpha) \tag{5}$$

then, $\mathbf{X}^T\alpha$ and λ are the eigenvectors and eigenvalues of the covariance matrix C, which can solve from Equation (5). Besides, we also need to normalize the $\mathbf{X}^T\alpha$, that is $\xi=(\mathbf{X}^T\alpha)/\sqrt{\alpha^TK\alpha}$. Assuming that the first p max eigenvalues are $\lambda_1,\lambda_2,\cdots,\lambda_p$, the CCR can be calculated according to the following equation:

$$CCR = (\sum_{i=1}^{p}\lambda_i)/(\sum_{i=1}^{m}\lambda_i).$$

If the CCR exceeds 80%, we can retain the first p eigenvectors of the covariance matrix C, there are $\mathbf{V}=[\xi_1,\xi_2,\cdots,\xi_p]=\mathbf{X}^T\mathbf{U}$, where $\mathbf{U}=[(\alpha_1/\sqrt{\alpha_1^TK\alpha_1}),(\alpha_2/\sqrt{\alpha_2^TK\alpha_2}),\cdots,(\alpha_p/\sqrt{\alpha_p^TK\alpha_p})]$ is known.

For PC $\mathbf{y}$ extraction, we can compute projections of the data $\Phi(x)$ onto the eigenvectors $\mathbf{V}$ in F according to

$$\mathbf{y} = \mathbf{V}^T(\Phi(x)-\bar{\Phi})=\mathbf{U}^T\left(\begin{bmatrix}\kappa(x_1,x)\\ \kappa(x_2,x)\\ \vdots\\ \kappa(x_m,x)\end{bmatrix}-\frac{1}{m}K.\mathbf{1}_{m\times m}\right).$$

In addition, we can also get more detailed results:

$$Y = \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_m^T \end{bmatrix} = \left(K - \frac{1}{m}\mathbf{1}_{m\times m}K \right)\mathbf{U}. \tag{6}$$

3 Interval Valued Fermatean Fuzzy Language âĂŘ Kernel Principal Component Analysis Model

In the IVFFL-KPCA model, the kernel function used in this presentation is the commonly Gaussian radial kernel function

$$K(x_1, x_2) = exp\left(-\frac{|x_1 - x_2|^2}{2\sigma^2} \right).$$

The Euclidean distance between IVFFLSs will be used in this kernel. Therefore, this presentation first defined the interval-valued Fermatean fuzzy linguistic Euclidean and weighted Euclidean distance measures (IVFFLD, IVFFLWD).

3.1 New linguistic distance measure

Definition 4. *Let $\widehat{P} = \{\widehat{P}_1, \widehat{P}_2, \cdots, \widehat{P}_n\}$ and $\widehat{Q} = \{\widehat{Q}_1, \widehat{Q}_2, \cdots, \widehat{Q}_n\}$ be any two IVFFLSs, and S is a linguistic term set, where $\widehat{P}_j = (s_\theta(\widehat{P}_j), [\zeta^L_{\widehat{P}_j}, \zeta^U_{\widehat{P}_j}], [\eta^L_{\widehat{P}_j}, \eta^U_{\widehat{P}_j}])$, $\widehat{Q}_j = (s_\theta(\widehat{Q}_j), [\zeta^L_{\widehat{Q}_j}, \zeta^U_{\widehat{Q}_j}], [\eta^L_{\widehat{Q}_j}, \eta^U_{\widehat{Q}_j}])$ and $S = \{s_i : i \in [0, g]\}$. The $IVFFLD(\widehat{P}, \widehat{Q})$ can be defined as*

$$IVFFLD(\widehat{P}, \widehat{Q}) = \sqrt{\frac{1}{4ng^6} \sum_{j=1}^{n} \left\{ \left| \left(\theta_{\widehat{P}_j} \zeta^L_{\widehat{P}_j}\right)^3 - \left(\theta_{\widehat{Q}_j} \zeta^L_{\widehat{Q}_j}\right)^3 \right|^2 + \left| \left(\theta_{\widehat{P}_j} \zeta^U_{\widehat{P}_j}\right)^3 - \left(\theta_{\widehat{Q}_j} \zeta^U_{\widehat{Q}_j}\right)^3 \right|^2 + \left| \left(\theta_{\widehat{P}_j} \eta^L_{\widehat{P}_j}\right)^3 - \left(\theta_{\widehat{Q}_j} \eta^L_{\widehat{Q}_j}\right)^3 \right|^2 + \left| \left(\theta_{\widehat{P}_j} \eta^U_{\widehat{P}_j}\right)^3 - \left(\theta_{\widehat{Q}_j} \eta^U_{\widehat{Q}_j}\right)^3 \right|^2 \right\}}.$$

Theorem 1. *If $\widehat{P}, \widehat{Q}$ are two IVFFLSs in X, the $IVFFLD(\widehat{P}, \widehat{Q})$ has the following properties:*

i. $0 \leq IVFFLD(\widehat{P}, \widehat{Q}) \leq 1$,
ii. $IVFFLD(\widehat{P}, \widehat{Q}) = IVFFLD(\widehat{Q}, \widehat{P})$.

In addition, we suppose that the weight of each $x_j \in X (j = 1, 2, \cdots, n)$ is ω_j. Then the weighted Euclidean distance measure $IVFFLWD(\widehat{P}, \widehat{Q})$ is defined as

$$IVFFLWD(\hat{P}, \hat{Q}) = \sqrt{\frac{1}{4ng^6} \sum_{j=1}^{n} \omega_j \left\{ \begin{array}{l} \left| \left(\theta_{\hat{P}_j}(x_j) \zeta^L_{\hat{P}_j}(x_j)\right)^3 - \left(\theta_{\hat{Q}_j}(x_j) \zeta^L_{\hat{Q}_j}(x_j)\right)^3 \right|^2 \\ + \left| \left(\theta_{\hat{P}_j}(x_j) \zeta^U_{\hat{P}_j}(x_j)\right)^3 - \left(\theta_{\hat{Q}_j}(x_j) \zeta^U_{\hat{Q}_j}(x_j)\right)^3 \right|^2 \\ + \left| \left(\theta_{\hat{P}_j}(x_j) \eta^L_{\hat{P}_j}(x_j)\right)^3 - \left(\theta_{\hat{Q}_j}(x_j) \eta^L_{\hat{Q}_j}(x_j)\right)^3 \right|^2 \\ + \left| \left(\theta_{\hat{P}_j}(x_j) \eta^U_{\hat{P}_j}(x_j)\right)^3 - \left(\theta_{\hat{Q}_j}(x_j) \eta^U_{\hat{Q}_j}(x_j)\right)^3 \right|^2 \end{array} \right\}}.$$

3.2 New Model

Suppose that there is an EDM problem in the IVFFL environment. Let $M = \{M_1, M_2, \cdots, M_m\}$ be the set of m hip prothesis materials. $K = \{K_1, K_2, \cdots, K_n\}$ be the set of n attributes(criteria of biomaterials) and $D = \{D_1, D_2, \cdots, D_l\}$ be the set of l DMs.

For attributes, $IVFFLKPCA$**:**
Let $\widehat{R}^{(k)} = (\widehat{r}^k_{ij})_{m\times n}$ $(k = 1, 2, \cdots, l)$ denote that each DM D_k $(k = 1, 2, \cdots, l)$ gives her/his IVFFL evaluation matrices, where $\widehat{r}^k_{ij} = (s^k_{\theta_{ij}}, [\zeta^{L(k)}_{ij}, \zeta^{U(k)}_{ij}], [\eta^{L(k)}_{ij}, \eta^{U(k)}_{ij}])$ is the evaluation information of A_i with respect to C_j based on language variable. Then $s^k_{\theta_{ij}}$ is the language evaluation value of A_i with respect to C_j given by the D_k based on the language term set $S = \{S_1, S_2, \cdots, S_G\}$. The evaluation matrix $\widehat{R}^{(k)}$ $(k = 1, 2, \cdots, l)$ given by each D_k $(k = 1, 2, \cdots, l)$ is aggregated into a comprehensive evaluation matrix $\widehat{R}$ of dimensionally $m \times n$ Therefore, all evaluation information is contained in the matrix $\widehat{R}$.

$$\widehat{R}^{(k)} = \begin{bmatrix} \widehat{r}^k_{11} & \widehat{r}^k_{12} & \cdots & \widehat{r}^k_{1n} \\ \widehat{r}^k_{21} & \widehat{r}^k_{22} & \cdots & \widehat{r}^k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{r}^k_{m1} & \widehat{r}^k_{m2} & \cdots & \widehat{r}^k_{mn} \end{bmatrix} \quad and \quad \widehat{R} \begin{bmatrix} \widehat{R}^{(1)} \\ \widehat{R}^{(2)} \\ \cdots \\ \widehat{R}^{(l)} \end{bmatrix} \quad k = 1, 2, \cdots, l.$$

Calculated eigenvalues λ^C and eigenvectors α from Equation XXX, and λ^C is arranged from the largest to the smallest. According to Equation YYY, the first p PCs are retained, and the CCR of the corresponding eigenvalues $\lambda^C_1, \lambda^C_2, \cdots, \lambda^C_p$ are exceeded 80%. Then we calculated the dimensionality-reduced evaluation matrix $\dot{R}$ with dimensionality $lm \times p$ by Equation ZZZ.

$$\dot{R}^{(k)} = \begin{bmatrix} \dot{r}^k_{11} & \dot{r}^k_{12} & \cdots & \dot{r}^k_{1n} \\ \dot{r}^k_{21} & \dot{r}^k_{22} & \cdots & \dot{r}^k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \dot{r}^k_{m1} & \dot{r}^k_{m2} & \cdots & \dot{r}^k_{mn} \end{bmatrix} \quad and \quad \dot{R} \begin{bmatrix} \dot{R}^{(1)} \\ \dot{R}^{(2)} \\ \cdots \\ \dot{R}^{(l)} \end{bmatrix} \quad k = 1, 2, \cdots, l.$$

Since the eigenvalues λ^C described the amount of information contained in the direction of the corresponding eigenvectors, the weight of each PC can be reasonably expressed as

$$\omega^C_s = \frac{\lambda^C_s}{\sum_{j=1}^{p} \lambda^C_j}, \quad s = 1, 2, \cdots, p,$$

therefore, the weight of attributes after dimensionality reduction is $\omega_C = (\omega^C_1, \omega^C_2, \cdots, \omega^C_p)$.

For Decision-makers, $IVFFLKPCA$**:**

On the basis of the matrix $\dot{R}^{(k)} = (\dot{r}^k_{ij})_{m\times p}$ $(k = 1, 2, \cdots, l)$, the evaluation information given by DMs can be written as $D^{(k)} = (\dot{r}^k_{11}, \dot{r}^k_{12}, \cdots, \dot{r}^k_{mp})^T$. Then, the evaluation matrix $\dot{R}$ is converted to $D = \{D^{(1)}, D^{(2)}, \cdots, D^{(l)}\}$, where each column can be regarded as a sample of the DMs. The dimension of the matrix D is $mp \times l$.

$$D = \{D^{(1)}, D^{(2)}, \cdots, D^{(l)}\} = \begin{bmatrix} \dot{r}^1_{11} & \cdots & \dot{r}^1_{1p} & \cdots & \dot{r}^1_{m1} & \cdots & \dot{r}^1_{mp} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dot{r}^l_{11} & \cdots & \dot{r}^l_{1p} & \cdots & \dot{r}^l_{m1} & \cdots & \dot{r}^l_{mp} \end{bmatrix}$$

In the same way, we used Equation XXX to find the eigenvalues λ^D and eigenvector $\alpha^{'}$, and λ^D is arranged from large to small. By Equation YYY, the CCR is more than 80% to retain the first q PCs. Then, we calculated the dimensionality-reduced evaluation matrix $\dot{D}$ $(k = 1, 2, \cdots, q; i = 1, 2, \cdots, m; j = 1, 2, \cdots, p)$ by Equation ZZZ, and the dimension of matrix $\dot{D}$ is $mq \times p$ can be obtained after two dimensionality reductions.

$$\dot{D} = \{\dot{D}^{(1)}, \dot{D}^{(2)}, \cdots, \dot{D}^{(q)}\} = \begin{bmatrix} \dot{r}^1_{11} & \cdots & \dot{r}^1_{1p} & \cdots & \dot{r}^1_{m1} & \cdots & \dot{r}^1_{mp} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dot{r}^q_{11} & \cdots & \dot{r}^q_{1p} & \cdots & \dot{r}^q_{m1} & \cdots & \dot{r}^q_{mp} \end{bmatrix}$$

$$\ddot{R}^{(k)} = \begin{bmatrix} \ddot{r}^k_{11} & \ddot{r}^k_{12} & \cdots & \ddot{r}^k_{1p} \\ \ddot{r}^k_{21} & \ddot{r}^k_{22} & \cdots & \ddot{r}^k_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \ddot{r}^k_{m1} & \ddot{r}^k_{m2} & \cdots & \ddot{r}^k_{mp} \end{bmatrix} \quad and \quad \ddot{R} \begin{bmatrix} \ddot{R}^{(1)} \\ \ddot{R}^{(2)} \\ \cdots \\ \ddot{R}^{(q)} \end{bmatrix} \quad k = 1, 2, \cdots, q.$$

In addition, since the eigenvalues λ^D represented the information content of the DMs, the weight of each PC can be expressed as

$$\omega^D_t \quad = \quad \frac{\lambda^D_t}{\sum_{k=1}^{q} \lambda^D_k}, \quad t = 1, 2, \cdots, q,$$

therefore, after dimensionality reduction, the weight of each Dm is $\omega_D = (\omega^D_1, \omega^D_2, \cdots, \omega^D_q)$.

4 New Model based on the TOPSIS method

4.1 New Model with the TOPSIS

Let the matrix $\ddot{R}^{(k)} = (\ddot{r}^k_{ij})_{m\times p}$ $(k = 1, 2, \cdots, q)$ be the standardized evaluation matrix after two IVFFL-KPCA models as shown as Equation TTT, and the attributes weight vector is $\omega_C = (\omega^C_1, \omega^C_2, \cdots, \omega^C_p)$, and the DMs vector is $\omega_D = (\omega^D_1, \omega^D_2, \cdots, \omega^D_q)$.

First, we calculated the weighted evaluation matrix $\ddot{R}^{'} = (\ddot{r}^{'}_{ij})_{m\times p}$ according to the weight vector $\omega_D = (\omega^D_1, \omega^D_2, \cdots, \omega^D_q)$. The weighted evaluation information is given as below:

$$\ddot{r}^{'}_{ij} \quad = \quad \sum_{k=1}^{q} \omega^D_k \ddot{r}^k_{ij} \quad (i = 1, 2, \cdots, m; j = 1, 2, \cdots, p).$$

Then, the decision steps of IVFFL-KPCA model based on the TOPSIS method are summarized as below:

Step 1: Constructing the attributes weighted evaluation matrix $Z = (z_{ij})_{m\times p}$ through the attribute weighted vector $\omega_C = (\omega^C_1, \omega^C_2, \cdots, \omega^C_p)$, where $z_{ij} = \omega^C_j .r^{'}_{ij}$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, p)$.

Step 2: Determining the positive and negative ideal solution for each evaluation attribute:

$$z^+ = \left\{z^+_1, z^+_2, \cdots, z^+_p\right\} \quad z^- = \left\{z^-_1, z^-_2, \cdots, z^-_p\right\}.$$

Step 3: Computing the Euclidean distance measure D^+_i and D^-_i of each emergency plan Z_i from the ideal solutions z^+ and z^-, respectively.

$$D^+_i = \sqrt{\sum_{j=1}^{p} (z_{ij} - z^+)^3}, \quad D^-_i = \sqrt{\sum_{j=1}^{p} (z_{ij} - z^-)^3} \quad i = 1, 2, \cdots, m.$$

Step 4: According to calculate distance, the relative closeness $\xi(Z_i)$ $(i = 1, 2, \cdots, m)$ of each emergency plan is constructed as

$$\xi(Z_i) = \frac{D^-_i}{D^+_i + D^-_i}, \quad i = 1, 2, \cdots, m.$$

Step 5: Ranking all $\xi(Z_i)$ $(i = 1, 2, \cdots, m)$ and choosing the optimal alternative. For greater the $\xi(Z_i)$, the better the alternative Z_i is.

Thus, according to the relative closeness $\xi(Z_i)$ $(i = 1, 2, \cdots, m)$, we can determine the ranking order of all emergency plans and select the best implementation plan from a set of emergency plans.

4.2 Decision Steps

To solve emergency group decision-making problems, the new model with TOPSIS will be implemented as follows:

Step 1: DM D_k $(k = 1, 2, \cdots, l)$ give evaluation indicator C_j $(j = 1, 2, \cdots, n)$ of each emergency plan A_i $(i = 1, 2, \cdots, m)$. A comprehensive evaluation matrix $\bar{R}^{(k)} = (r^k_{ij})_{m\times n}$ $(k = 1, 2, \cdots, l)$ is obtained.

Step 2: Utilize the IVFFL-KPCA (attributes) model to reduce the dimensionality of the C_j in the matrix $\bar{R}$. Normalize the reduced-dimensional matrix to obtain $\dot{R}$. Moreover, the weighted vector of attributes can be obtained as $\omega_C = (\omega^C_1, \omega^C_2, \cdots, \omega^C_p)$.

Step 3: Transform the matrix $\dot{R}$ into the matrix D. On the basis of the sample matrix D, the dimension of matrix D_k are reduced according to the IVFFL-KPCA (DMs) model, and the matrix $\dot{D}$ is obtained. Besides, the weight vector of each DM can be obtained as $\omega_D = (\omega^D_1, \omega^D_2, \cdots, \omega^D_q)$.

Step 4: Arrange and standardize the evaluation matrix $\dot{D}$ after two dimensionality reductions,and the matrix $\ddot{R}$ can be obtained.

Step 5: Calculate the weighted comprehensive evaluation matrix $R^{'}$ by Equation DDD based on the weighted vector of DMs in Step 3.

Step 6: Use the TOPSIS method to select the best emergency solution.

Table 1 Decision matrix for hip joint prosthesis material selection [28]

Materials / Criteria	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
$SS316$	10	7	517	350	8	8	200	8	1
$SS317$	9	7	630	415	10	8.5	200	8	1.1
$SS321$	9	7	610	410	10	8	200	7.9	1.1
$SS347$	9	7	650	430	10	8.4	200	8	1.2
$CCA(castable)$	10	9	655	425	2	10	238	8.3	3.7
$CCA(wrought)$	10	9	896	600	10	10	242	9.1	4
$Puretitanium$	8	10	550	315	7	8	110	4.5	1.7
$Ti-6A1-4V$	8	10	985	490	7	8.3	124	4.4	1.9
$Epoxy(70\%glass)$	7	7	680	200	3	7	22	2.1	3
$Epoxy(63\%carbon)$	7	7	560	170	3	7.5	56	1.6	10
$Epoxy(62\%aramid)$	7	7	430	130	3	7.5	29	1.4	5

Table 2 Linguistic variables for criteria and DMs

Linguistic variables	FFNs
L_1	[0.90, 0.20, 0.641]
L_2	[0.85, 0.50, 0.638]
L_3	[0.70, 0.65, 0.725]
L_4	[0.40, 0.75, 0.800]
L_5	[0.20, 0.90, 0.641]

Table 3 Linguistic variables for alternatives

Linguistic variables	FFNs
LA_1	[1.00, 0.00, 0.00]
LA_2	[0.90, 0.20, 0.641]
LA_3	[0.85, 0.30, 0.710]
LA_4	[0.75, 0.40, 0.800]
LA_5	[0.70, 0.45, 0.825]
LA_6	[0.65, 0.55, 0.822]
LA_7	[0.55, 0.70, 0.789]
LA_8	[0.40, 0.80, 0.751]
LA_9	[0.30, 0.85, 0.710]
LA_{10}	[0.10, 0.95, 0.521]

5 Application

Material selection will be made for the femoral component of the hip joint prosthesis using the method presented in the previous section. There are different methods in the literature regarding the selection of this biomedical material. The hip joint basically consists of two parts, the femoral head, and the acetabulum, and is an important load-bearing joint in the human body. The hip joint is formed by the insertion of the femoral head into the socket called the acetabulum in the pelvis bone. Movements of the femoral head within the acetabulum allow the leg to make inwardoutward, anterior backward, and circular movements. Since the main task of the hip joint is load-bearing, it is a joint that must have a sufficient range of motion and sufficient stability. Hip arthroplasty is a surgical procedure to replace or renew the damaged joint in people whose hip joint is severely calcified (osteoarthritis) or damaged. Hip replacement is the best treatment option in cases of severe pain, limitation of movement, and shortness that prevent activities of daily living. The use of implants with appropriate material and design features increases the success of hip prosthesis application. Today, many prostheses with different materials and design features have been developed. The design and material properties of the selected implant should allow the prosthesis to be simple, manufacturable, inexpensive, reliable, and long-lasting. The complexity in materials choice for hip replacement prostheses is that the design requires many distinct essential characteristics which are very difficult to devise in only one material.

The materials set as alternatives [14]

$$\begin{aligned} Z &= \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, \} \\ &= \left\{ \begin{matrix} \text{SS 316}, & \text{SS 317}, & \text{SS 321}, & \text{SS 347}, & \textit{CCA(castable)}, \\ \textit{CCA(wrought)}, & \textit{Pure titanium}, & Ti-6A1-4V, & \textit{Epoxy(70\% glass)}, & \\ \textit{Epoxy(63\% carbon)}, & \textit{Epoxy(62\% aramid)} & & & \end{matrix} \right\} \end{aligned}$$

The criteria set [14]

$$\begin{aligned} K &= \{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9\} \\ &= \left\{ \begin{matrix} \textit{Tissue tolerance}, & \textit{Corrosion resistance}, & \textit{Tensile strenght(MPa)}, & \textit{Fatigue strenght(MPa)}, & \\ \textit{Toughness}, & \textit{Wear resistance}, & \textit{EElastic modulus(GPa)}, & \textit{Density}(g/cm^3), & \textit{Cost} \end{matrix} \right\} \end{aligned}$$

where $K_1 - K_6$ values are max, K_7, K_8 are target value, K_9 is cost value.

To select the best solution, 10 DMs to evaluate 11 biomaterials. The values in Table 2 consist of the linguistic terms used by DMs in relative importance rating and the criteria considered, where Very Important(L_1), Important(L_2), Medium(L_3), Unimportant(L_4), Very Unimportant(L_5). The values in Table 3 represent the linguistic variables for the relative importance rating of the alternatives, where Extremely preferable(LA_1), Very very preferable (LA_2), Very preferable (LA_3), Preferable (LA_4), Medium preferable (LA_5), Medium (LA_6), Medium Un-preferable (LA_7), Un-preferable (LA_8), Very Un-preferable (LA_9), Very very Un-preferable (LA_{10}).

The decision steps to select the best solution are obtained as below:

Step 1: DMs evaluate the attributes of each emergency solution using IVFFLNs and construct the decision information matrix $\bar{R}^{(k)} = (\bar{r}_{ij}^{k})_{11\times 9}$ $(k = 1, 2, \cdots, 10)$.

Calculate the evaluation matrix $\dot{R}$ and the converted matrix D ($\omega_D = (0.4471, 0.3511, 0.2107)$). Further, the DMs' weighted matrix R' is obtained. That is, the matrix $\bar{R}$ is reduced to the matrix R', and the best solution is selected using the TOPSI method. If $\omega_C = (0.7642, 0.1397, 0.0852)$, we can get the attributes weighted matrix Z.

$$Z = \begin{bmatrix} -0.3112 & -0.0533 & -0.0326 \\ -0.1881 & -0.0849 & -0.0381 \\ -0.0552 & -0.0506 & 0.1189 \\ -0.0073 & -0.0017 & -0.0517 \\ 0.4642 & 0.2154 & 0.0335 \end{bmatrix}$$

Table 4 Eigenvalues and CCRs of the attribute PCs

	Y_1	Y_2	Y_3
Eigenvalues	0.000526005	0.000019873	0.000057401
CCR	0.514660059	0.50672662	0.618291415

The positive ideal solution Z^+ and negative ideal solution Z^- of the matrix Z are obtained as $Z^+ = \{0.3662, 0.3052, 0.1167\}$ and $Z^- = \{-0.2909, -0.1047, -0.1167\}$.

Calculate the normalized Euclidean distance $D(Z_i, Z^+)$ and $D(Z_i, Z^-)$ with each Z_i. Furthermore, compute the relative closeness $\xi(Z_i)$. The larger the value of the relative closeness $\xi(Z_i)$ is from the positive ideal plan Z^+. Thus, the ranking of biomaterials is $Z_{11} > Z_{10}Z_1 > Z_4 > Z_3 > Z_2 > Z_7 > Z_6 > Z_5 > Z_9 > Z_8$.

6 Conclusion

In this study, we have been given linguistic term sets, IVFFLSs, KPCA and TOPSIS methods. Then, we studied EGDM method, where the evaluation information of DMs was expressed as IVFFLNs. To solve the EDM problem of incomplete information, high-dimensional data, and non-linear separability between information, this paper proposed an IVFFL-KPCA model based on the TOPSIS method.

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Fermatean Fuzzy Entropy Measure

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Abstract: In this presentation, the definition of fermatean fuzzy soft sets and some its properties are introduced. The fermatean fuzzy soft sets are a parameterized family of fermatean fuzzy soft sets. Fermatean fuzzy soft sets are a generalization of soft sets. Particularly, the basic properties and operations of fermatean soft sets are given. The notion of entropy measure is defined for the fermatean fuzzy soft sets. Further, we propose an algorithm to solve the decision-making problem. Finally, an illustrative example is discussed to prove that they can be effectively used to solve problems with uncertainties.

Keywords: Decision-making, entropy, fermatean fuzzy set, fermatean fuzzy soft set, soft set.

1 Introduction

Uncertainty is a crucial concept for decision-making problems. It is not easy to make precise decisions in life since each information contains vagueness, uncertainty, imprecision. Fuzzy Set(FS) Theory, Zadeh's [1] pioneering work, proposed a membership function to solve problems such as vagueness, uncertainty, imprecision, and this function took value in the range of [0,1]. FS Theory had solved many problems in practice, but there was no membership function in real life, which only includes acceptances. Rejection is as important as acceptance in real life. Atanassov [2] clarified this problem and posed the Intuitionistic Fuzzy Set(IFS) Theory using the membership function as well as the non-membership function. In IFS, the sum of membership and non-membership grades is 1. This condition is also a limitation for solutions of vagueness, uncertainty, imprecision. Yager [3] has presented a solution to this situation and suggested Pythagorean Fuzzy Sets(PFS). PFS is more comprehensive than IFS because it uses the condition that the sum of the squares of membership and non-membership grades is equal to or less than 1. PFS is also a particular case of the Neutrosophic Set initiated by Smarandache [4]. There are many studies in the literature on FS, IFS, and PFS theories [5]-[29]. Despite all the possible solutions, these theories have limitations. How to set the membership function in each particular object and the deficiencies in considering the parametrization tool can be given as examples of these limitations. These limitations handicap decision-makers from making a correct decision during the analysis.

A new method, called Soft Set, was proposed by Molodtsov [30], in which the preferences for each alternative were given in distinct parameters, and thus a solution was found for the limitations expressed. Immediately after the occurrence of SS theory, Fuzzy Soft Sets [19] and Intuitionistic Fuzzy Soft Set(IFSS) [20] were defined and their various properties were studied [31], [32]. Pythagorean Fuzzy Soft Set(PFSS) is defined by Peng et al [21]. PFSS is a natural generalization of IFSS and is a parameterized family of PFSs. In [33]-[37], the main features of PFSS were examined and applied to various areas such as medical diagnosis, selection of a team of workers for business, stock exchange investment problem. The benefit of these extended theories is that they are capable of simplifying the characterization of real-life cases with the help of their parameterized feature.

Entropy is an important concepts in generalized set theory. The entropy quantifies the degree of vagueness and Zadeh [38] introduced fuzzy entropy. The entropy of a system directly proportional to the irregularity. Thus one can identify that which one is more stable if the entropy of each system is given. The axiomatic definition of entropy is proposed by De Luca and Termini [39]. The concept of entropy is particularly notable as it is applied across physics, information theory, mathematics, and many other branches of science and engineering. Originally defined by Rudolph Clausius in 1865, entropy, is a measure in thermodynamics of the unavailability of a system's energy to do work, also a measure of disorder;the higher the entropy the greater the disorder.The concept of information entropy was first introduced by Shannon [40]. In information theory, entropy is a measure of the uncertainty associated with a random variable. Shannon's entropy represents an absolute limit on the best possible lossless compression of any communication, under certain constraints: treating messages to be encoded as a sequence of independent and identically-distributed random variables, Shannon's source coding theorem shows that, in the limit, the average length of the shortest possible representation to encode the messages in a given alphabet is their entropy divided by the logarithm of the number of symbols in the target alphabet. According to information entropy, the number and quality of the information at hand is the most important determinant of the accuracy and reliability of the decision to be made in a decision-making problem [41].

The Fermatean fuzzy set(ffs)was initiated by Senepati and Yager [42]. In the (ffs), the membership and non-membership degrees fulfill the condition $0 \leq m_A^3 + n_A^3 \leq 1$. (ffs), which is included in the literature as a new concept, gives better results than the intuitionistic fuzzy set(ifs) and Pythagorean fuzzy set(Pfs) in defining uncertainties. For example $0.9 + 0.6 > 1$, $0.9^2 + 0.6^2 > 1$ and $0.9^3 + 0.6^3 < 1$. Some properties, score and accuracy functions of (ffs)s are given in [42]. Further, the TOPSIS method, which is frequently used in Multi Criteria Decision Making(MCDM) problems, has been applied to (ffs). In addition, Senepathy and Yager [42], the TOPSIS method, which is frequently used in MCDM problems, has been applied to (ffs). As a continuation of this work, Senapati ve Yager [43] investigated several new operations, subtraction, division, and Fermatean arithmetic mean operations over (ffs)s and employed Fermatean fuzzy weighted product model to solve MCDM problems. In [44], new aggregation operators belonging to (ffs) have been defined, and properties related to these operators have been examined. In study of Donghai and et al [45], the concept of Fermatean fuzzy linguistic term sets(fflt) is offered. Operations, score, and accuracy functions belonging to these sets were given. In [46], a new similarity measure related to (fflt)s is constructed. The new measurement is a combination of Euclidean distance measure and cosine similarity measure.

In this paper, we suggested the concept of Fermatean fuzzy soft set(ffss). In Section 3, the fundamental properties of (ffss) such as fermatean fuzzy soft subset, "AND","OR" operators, union, interseciton and complement of (ffss) is studied. Section 4 devoted the entropy of FFSS. In Section 5, for decision-making problems, new algorithm is established and illustrative example is given.

2 Preliminaries

Let Σ be set of parameters and Υ be the universal set. A pair (Φ, Σ) is called a soft set(ss) over Υ, where Φ is a mapping $\Phi : \Sigma \to P_{ss}(\Upsilon)$. In other word, the soft set is a parameterized family of subsets of the set Υ [30].

Let (Φ, A) and (Ψ, B) be two soft sets over a common universe Υ. Then,

i. $(\Phi, A) \subseteq (\Psi, B)$ if $A \subseteq B$ and $\Phi(\epsilon) \subseteq \Psi(\epsilon), \forall \epsilon \in A$.
ii. If $(\Phi, A)\tilde{\subseteq}(\Psi, B)$ and $(\Psi, \tilde{B}) \subseteq (\Phi, A)$, then it is said to be soft equal.
iii. Complement: $(\Phi, A)^t = (\Phi^t, A)$, where $\Phi^t : A \to P(\Upsilon)$ is a mapping given by $\Phi^t(\epsilon) = \{\Phi(\epsilon)\}^t = \Upsilon - \Phi(\epsilon), \forall \epsilon \in A$.
iv. The union of two soft sets (Φ, A) and (Ψ, B) over the common universe Υ is a soft set (Ω, C), denoted by $(\Phi, A)\tilde{\cup}(\Psi, B) = (\Omega, C)$, where $C = A \cup B$ and for each $\epsilon \in C$,

$$\Omega(\epsilon) = \begin{cases} \Phi(\epsilon) & , \quad e \in A - B \\ \Psi(\epsilon) & , \quad e \in B - A \\ \Phi(\epsilon) \cup \Psi(\epsilon) & , \quad \epsilon \in A \cap B. \end{cases}$$

v. The intersection of two soft sets (Φ, A) and (Ψ, B) over the common universe Υ is a soft set (Ω, C), denoted by $(\Phi, A)\tilde{\cap}(\Psi, B) = (\Upsilon, C)$, where $C = A \cup B$, and $\Upsilon(\epsilon) = \Phi(\epsilon) \cap \Psi(\epsilon), \forall \epsilon \in C$ [31].

Let $P_{fss}(\Upsilon)$ be a set of all fuzzy subset of Υ. A pair (Φ, A) is called a fuzzy soft set(fss) over Υ, if $\Phi : A \to P_{fss}(\Upsilon)$ [19].

Let $P_{ifs}(\Upsilon)$ denotes the intuitionistic fuzzy power set of Υ and $A \subset \Sigma$. A pair (Φ, A) is called an intuitionistic fuzzy soft set(ifss) over Υ, where Φ is a mapping given by, $\Phi : A \to P_{ifs}(\Upsilon)$ [20].

An (ifss) is a parameterized family of intuitionistic fuzzy subsets of Υ. A fuzzy soft set is a special case of an ifss, because when all the intuitionistic fuzzy subsets of Υ degenerate into fuzzy soft sets, the corresponding (ifss)s degenerate into (fss)s.

A (Pfs) in Υ is given by, $P = \{< a, m_p(a), n_p(a) >: a \in \Upsilon\}$ where, $m_p : \Upsilon \to [0, 1]$ denotes the degree of membership and $n_p : \Upsilon \to [0, 1]$ denotes the degree of nonmembership of the element $a \in \Upsilon$ to the set P with the condition that $0 \leq m_p^2(a) + n_p^2(a) \leq 1$. The degree of indeterminacy is denoted by $\pi_p(a) = \sqrt{1 - (m_p^2(a) + n_p^2(a))}$ [26].

The Pythagorean fuzzy soft sets(Pfss) is defind as the pair (Φ, Σ) where, $\Phi : \Sigma \to P_{Pfs}(\Upsilon)$ and $P_{Pfs}(\Upsilon)$ is the set of all Pythagorean fuzzy subsets of Υ.

The (ffs) Φ in Υ is an object having the form $\Phi = \{< a, m_\Phi(a), n_\Phi(a) >: a \in \Upsilon\}$, where $m_p : \Upsilon \to [0, 1]$ and $n_p : \Upsilon \to [0, 1]$, including the condition $0 \leq m_\Phi^3(a) + n_\Phi^3(a) \leq 1$ [42].

For any ffs Φ and $a \in \Upsilon$, $\pi_p(a) = \sqrt[3]{1 - (m_p^3(a) + n_p^3(a))}$ is identified as the degree of indeterminacy of a to Φ.

Theorem 1. *[42] The set of Fermatean membership grades is larger than the set of Pythagorean membership grades and intuitionistic membership grades.*

3 Fermatean Fuzzy Soft Sets

Definition 1. *Let Υ be a set, Σ be a parameter set. Denote $P_{ffs}(\Upsilon)$ the whole of all (ffs)s on Υ. Let $A \subseteq \Sigma$. (Φ, A) is fermatean fuzzy soft set (ffss) over Υ, where $\Phi : A \to P_{ffs}(\Upsilon)$.*

A (ffs) on Υ is a family of parameters formed by some fermatean fuzzy subsets on Υ. For any parameter $\epsilon \in A$, $\Phi(\epsilon)$ is a (ffss) associated with ϵ of Υ. Then, it is called fermatean fuzzy value set of parameter ϵ. $\Phi(\epsilon)$ can be written as an (ffs) such that

$$\Phi(\epsilon) = \{< a, m_{\Phi(\epsilon)}(a), n_{\Phi(\epsilon)}(a) >: a \in \Upsilon\},$$

where $m_{\Phi(\epsilon)}(a)$ and $n_{\Phi(\epsilon)}(a)$ are the membership and non-membership functions, respectively. The condition $(m_{\Phi(\epsilon)}(a))^3 + (n_{\Phi(\epsilon)}(a))^3 \leq 1$ holds. Then, $n_{\Phi(\epsilon)}(a) = \sqrt[3]{1 - (m_{\Phi(\epsilon)}(a))^3}$ for all a.

Example 1. *Let* $\Upsilon = \{a_1 = HepatitisC, a_2 = influenzaA(H1N1), a_3 = norovirus\}$ *represent the infectious diseases and* $A \subset \Sigma = \{p_1 = headache, p_2 = temperature, p_3 = nausea\}$ *is a parameter set. Then the (ffss)* $\Phi(A)$ *as follows (Table 1):*

$$\Phi(p_1) = \{< a_1, 0.6, 0.9) >, < a_2, 0.8, 0.7) >, < a_3, 0.8, 0.9) >\}$$
$$\Phi(p_2) = \{< a_1, 0.7, 0.9) >, < a_2, 0.9, 0.5) >, < a_3, 0.8, 0.8) >\}$$
$$\Phi(p_3) = \{< a_1, 0.8, 0.7) >, < a_2, 0.8, 0.9) >, < a_3, 0.9, 0.6) >\}$$

Table 1 $\Phi(A)$

Υ/Σ	p_1	p_2	p_3
a_1	(0.6, 0.9)	(0.8, 0.7)	(0.8, 0.9)
a_2	(0.7, 0.9)	(0.9, 0.5)	(0.8, 0.8)
a_3	(0.8, 0.7)	(0.8, 0.9)	(0.9, 0.6)

Definition 2. *Let* $(\Phi, A), (\Psi, B)$ *be two (ffss)s, for* $A, B \subset \Sigma$. (Φ, A) *is called a (ffs) subset of* (Ψ, B) *if*

1. $A \subseteq B$,
2. *For all* $\epsilon \in A$, (Φ, A) *is fermatean fuzzy subset of* (Ψ, B), *that is, for all* $a \in \Upsilon$ *and* $\epsilon \in A$, $m_A(x) \geq m_B(x)$ *and* $n_A(x) \leq n_B(x)$.

fermatean fuzzy soft subset is denoted by $(\Psi, B)\hat{\subset}(\Phi, A)$.

Example 2. *Choose* $B = \{p_1 = headache\}$. *Then, (ffss)* (Ψ, B) *is defined as follows:*

$$\Phi(p_1) = \{< a_1, 0.6, 0.8 >, < a_2, 0.6, 0.8) >, < a_3, 0.7, 0.9) >\}$$

Then, $(\Psi, B)\hat{\subset}(\Phi, A)$.

Definition 3. *For the (ffss)s* (Φ, A) *and* (Ψ, B), *if* $(\psi, B)\hat{\subset}(\Phi, A)$ *and* $(\Phi, A)\hat{\subset}(\Psi, B)$, *then* $(\Psi, B)\hat{=}(\Phi, A)$, *that is,* (ψ, B) *and* (Φ, A) *are Fermatean fuzzy soft equals.*

Definition 4. *Let* (Φ, A) *be the (ffss) on* Υ. *The complement of* (Φ, A), *denoted by* $(\Phi, A)^t$, *is defined by* $(\Phi, A)^t = (\Phi^t)$, *where* $\Phi^t : A \rightarrow P_{ffs}(\Upsilon)$ *is a mapping given by* $\Phi^t(\epsilon) = (\Phi(\epsilon))^t$ *for every* $\epsilon \in A$.

Since $(\Phi^t(\epsilon))^t$ is equal to the $\Phi(\epsilon)$, we get $((\Phi, A)^t)^t = (\Phi, A)$.

Example 3. *Let the (ffss)* (Ψ, B) *on* Υ *be as defined in Example 2. If*

$$\Psi(p_1) = \{< a_1, 0.6, 0.8) >, < a_2, 0.5, 0.7) >, < a_3, 0.7, 0.6) >\},$$

then

$$\Psi^t(p_1) = \{< a_1, 0.8, 0.6) >, < a_2, 0.7, 0.5) >, < a_3, 0.6, 0.7) >\}.$$

Definition 5. *For the (ffss)s* (Φ, A) *and* (Ψ, B), (Φ, A) *AND* (Ψ, B) *is denoted as* $(\Phi, A) \wedge (\Psi, B) = (\Phi, (A \times B))$, $(\Phi, (\alpha, \beta)) = (\Phi, \alpha) \cap (\Psi, \beta)$, $\forall \alpha, \beta \in A \times B$. *That is,* $(\Phi, (\alpha, \beta))(a) =< a, \min(m_\alpha(a), m_\beta(a)), \max(n_\alpha(a), n_\beta(a)) >$, *for all* $(\alpha, \beta) \in A \times B$ *and* $a \in \Upsilon$.

Definition 6. *For the (ffss)s* (Φ, A) *and* (Ψ, B), (Φ, A) *OR* (Ψ, B) *is denoted as* $(\Phi, A) \vee (\Psi, B) = (\tau, A \times B)$, $(\tau, (\alpha, \beta)) = (\Phi, \alpha) \cup (\psi, \beta)$, $\forall \alpha, \beta \in A \times B$. *That is,* $(\tau, (\alpha, \beta)) = (\max(m_\alpha, m_\beta), \min(n_\alpha, n_\beta))$, *for all* $(\alpha, \beta) \in A \times B$ *and* $a \in \Upsilon$.

Example 4. *Choose* $B = \{p_1, p_2\}$. *Then, (ffss)* (Ψ, B) *is defined as follows:*

$$\Psi(p_1) = \{< a_1, 0.6, 0.8) >, < a_2, 0.4, 0.8) >, < a_3, 0.8, 0.5) >\}$$
$$\Psi(p_2) = \{< a_1, 0.6, 0.7) >, < a_2, 0.6, 0.4) >, < a_3, 0.9, 0.4) >\}$$

Then, $(\Psi, B)\hat{\subset}(\Phi, A)$. *The AND and OR operations of* (Φ, A) *in Examples 2, 4 and* (Ψ, B) *are shown in Table 2 and 3.*

Theorem 2. *Choose the (ffss)s* (Φ, A) *and* (Ψ, B). *Then,*

- *(i.)* $((\Phi, A) \wedge (\Psi, B))^t = (\Phi, A)^t \vee (\Psi, B))^t$
- *(ii.)* $((\Phi, A) \vee (\Psi, B))^t = (\Phi, A))^t \wedge (\Psi, B))^t$

Table 2 $(\Phi, A) \wedge (\Psi, B)$

Υ/Σ	(p_1, p_1)	(p_1, p_2)	(p_2, p_1)	(p_2, p_2)	(p_3, p_1)	(p_3, p_2)
a_1	(0.6, 0.9)	(0.6, 0.9)	(0.6, 0.8)	(0.6, 0.7)	(0.6, 0.9)	(0.4, 0.9)
a_2	(0.4, 0.9)	(0.6, 0.9)	(0.4, 0.8)	(0.6, 0.5)	(0.4, 0.8)	(0.6, 0.8)
a_3	(0.8, 0.7)	(0.8, 0.7)	(0.8, 0.9)	(0.8, 0.9)	(0.8, 0.6)	(0.9, 0.6)

Table 3 $(\Phi, A) \vee (\Psi, B)$

Υ/Σ	(p_1, p_1)	(p_1, p_2)	(p_2, p_1)	(p_2, p_2)	(p_3, p_1)	(p_3, p_2)
a_1	(0.6, 0.8)	(0.6, 0.7)	(0.8, 0.7)	(0.8, 0.7)	(0.8, 0.8)	(0.8, 0.7)
a_2	(0.7, 0.8)	(0.7, 0.4)	(0.9, 0.5)	(0.9, 0.4)	(0.8, 0.8)	(0.8, 0.4)
a_3	(0.8, 0.5)	(0.9, 0.4)	(0.8, 0.5)	(0.9, 0.4)	(0.9, 0.5)	(0.9, 0.4)

Definition 7. *Let (Φ, A) and (Ψ, B) be the (ffss)s on Υ. If $C = A \cup B$ and $\forall \epsilon \in C$, the union (Γ, C) of (Φ, A) and (Ψ, B) is defined as*

$$\Gamma(\epsilon) = \begin{cases} \Phi(\epsilon) & , \quad \epsilon \in A - B \\ \Psi(\epsilon) & , \quad \epsilon \in B - A \\ \Phi(\epsilon) \cup \Psi(\epsilon) & , \quad \epsilon \in A \cap B. \end{cases}$$

That is, $\forall \epsilon \in A \cap B$, we have $\Phi(\epsilon) \cup \Psi(\epsilon) =< a, \max(m_{\Phi(\epsilon)}(a), m_{\Psi(\epsilon)}(a)), \min(n_{\Phi(\epsilon)}(a), n_{\Psi(\epsilon)}(a)) >: a \in \Upsilon$. This relation is denoted by $(\Phi, A) \cup (\Psi, B) = (\Gamma, C)$.

Theorem 3. *The union (Γ, C) of the (ffss)s (Φ, A) and (Ψ, B) is a (ffss).*

Definition 8. *Let (Φ, A) and (Ψ, B) be the (ffss)s on U. If $C = A \cup B$ and $\forall \epsilon \in C$, the intersection (Γ, C) of (Φ, A) and (Ψ, B) is defined as*

$$\Gamma(\epsilon) = \begin{cases} \Phi(\epsilon) & , \quad \epsilon \in A - B \\ \Psi(\epsilon) & , \quad \epsilon \in B - A \\ \Phi(\epsilon) \cap \Psi(\epsilon) & , \quad \epsilon \in A \cap B. \end{cases}$$

That is, $\forall \epsilon \in A \cap B$, we have $\Phi(\epsilon) \cap \Psi(\epsilon) =< a, \min(m_{\Phi(\epsilon)}(a), m_{\Psi(\epsilon)}(a)), \max(n_{\Phi(\epsilon)}(a), n_{\Psi(\epsilon)}(a)) >: a \in \Upsilon$. This relation is denoted by $(\Phi, A) \cap (\Psi, B) = (\Gamma, C)$.

Theorem 4. *The intersection (Γ, C) of the (ffss)s (Φ, A) and (Ψ, B) is a (ffss).*

Theorem 5. *For the (ffss)s (Φ, A), (Ψ, B) and (Γ, C),*

- *i.* $(\Phi, A) \cup (\Phi, A) = (\Phi, A)$
- *ii.* $(\Phi, A) \cap (\Phi, A) = (\Phi, A)$
- *iii.* $(\Phi, A) \cup (\Psi, B) = (\Psi, B) \cup (\Phi, A)$
- *iv.* $(\Phi, A) \cap (\Psi, B) = (\Psi, B) \cap (\Phi, A)$
- *v.* $((\Phi, A) \cup (\Psi, B)) \cup (\Gamma, C) = (\Phi, A) \cup ((\Psi, B) \cup (\Gamma, C))$
- *vi.* $((\Phi, A) \cap (\Psi, B)) \cap (\Gamma, C) = (\Phi, A) \cap ((\Psi, B) \cap (\Gamma, C))$.

Theorem 6. *For the (ffss)s (Φ, A) and (Ψ, B),*

- *i.* $((\Phi, A) \cap (\Psi, B))^t = (\Phi, A)^t \cup (\Psi, B)^t$
- *ii.* $((\Phi, A) \cup (\Psi, B))^t = (\Phi, A))^t \cap (\Psi, B))^t$

Example 5. *Take the infectious diseases set $\Upsilon = \{a_1, a_2, a_3, a_4, a_5\} = \{HepatitisC, Crimean - CongoHemorrhagicFever(CCHF), influenza$ Select $\Sigma = \{p_1, p_2, p_3, p_4, p_5\} = \{headache, temperature, nausea, vomiting, anorexia\}$ symptom set as parameter set. Assume that (Φ, A), the complement of (Φ, A), (Ψ, B), and (Ω, C) are four (ffss)s over Υ given by $A = \{p_1, p_2\}$, $B = \{p_1, p_2, p_4\}$ and $C = \{p_1, p_3, p_4\}$ defined as*

Table 4 (Φ, A)

Υ/Σ	p_1	p_2
a_1	(0.64, 0.88)	(0.81, 0.72)
a_2	(0.73, 0.79)	(0.94, 0.53)
a_3	(0.85, 0.59)	(0.92, 0.49)
a_4	(0.83, 0.67)	(0.67, 0.85)

The operations $(\Phi, A) \cup (\Omega, C)$, $(\Phi, A) \cap (\Omega, C)$, $(\Phi, A) \wedge (\Omega, C)$, $(\Phi, A) \vee (\Omega, C)$ are given in Tables 8-11.

Table 5 (Φ^t, A)

Υ/Σ	p_1	p_2
a_1	(0.88, 0.64)	(0.72, 0.81)
a_2	(0.79, 0.73)	(0.53, 0.94)
a_3	(0.59, 0.85)	(0.49, 0.92)
a_4	(0.67, 0.83)	(0.85, 0.67)

Table 6 (Ψ, B)

	p_1	p_2	p_4
a_1	(0.82, 0.73)	(0.92, 0.57)	(0.85, 0.67)
a_2	(0.66, 0.78)	(0.75, 0.62)	(0.54, 0.91)
a_3	(0.84, 0.49)	(0.72, 0.39)	(0.71, 0.81)
a_4	(0.43, 0.87)	(0.67, 0.59)	(0.76, 0.37)

Table 7 (Ω, C)

	p_1	p_3	p_4
a_1	(0.44, 0.95)	(0.57, 0.69)	(0.86, 0.59)
a_2	(0.56, 0.81)	(0.68, 0.69)	(0.79, 0.38)
a_3	(0.68, 0.56)	(0.92, 0.35)	(0.72, 0.65)
a_4	(0.63, 0.76)	(0.84, 0.37)	(0.95, 0.29)

[htb!]
Table 8 $(\Phi, A) \cup (\Omega, C)$

	p_1	p_2	p_3	p_4
a_1	(0.64, 0.88)	(0.81, 0.72)	(0.57, 0.69)	(0.86, 0.59)
a_2	(0.73, 0.81)	(0.94, 0.53)	(0.68, 0.69)	(0.79, 0.38)
a_3	(0.85, 0.56)	(0.92, 0.49)	(0.92, 0.35)	(0.72, 0.65)
a_4	(0.83, 0.67)	(0.67, 0.85)	(0.84, 0.37)	(0.95, 0.29)

Table 9 $(\Phi, A) \cap (\Omega, C)$

	p_1	p_2	p_3	p_4
a_1	(0.44, 0.95)	(0.81, 0.72)	(0.57, 0.69)	(0.86, 0.59)
a_2	(0.56, 0.81)	(0.94, 0.53)	(0.68, 0.69)	(0.79, 0.38)
a_3	(0.68, 0.59)	(0.92, 0.49)	(0.92, 0.35)	(0.72, 0.65)
a_4	(0.63, 0.76)	(0.67, 0.85)	(0.84, 0.37)	(0.95, 0.29)

Table 10 $(\Phi, A) \wedge (\Psi, B)$

	(p_1, p_1)	(p_1, p_2)	(p_1, p_4)	(p_2, p_1)	(p_2, p_2)	(p_2, p_4)
a_1	(0.64, 0.88)	(0.64, 0.88)	(0.64, 0.88)	(0.81, 0.73)	(0.81, 0.72)	(0.81, 0.72)
a_2	(0.66, 0.79)	(0.73, 0.79)	(0.54, 0.91)	(0.66, 0.78)	(0.75, 0.62)	(0.54, 0.91)
a_3	(0.84, 0.59)	(0.72, 0.59)	(0.71, 0.81)	(0.84, 0.49)	(0.72, 0.49)	(0.71, 0.81)
a_4	(0.43, 0.87)	(0.67, 0.67)	(0.76, 0.67)	(0.43, 0.87)	(0.67, 0.85)	(0.67, 0.85)

Table 11 $(\Phi, A) \vee (\Psi, B)$

	(p_1, p_1)	(p_1, p_2)	(p_1, p_4)	(p_2, p_1)	(p_2, p_2)	(p_2, p_4)
a_1	(0.82, 0.73)	(0.92, 0.57)	(0.85, 0.67)	(0.82, 0.72)	(0.92, 0.57)	(0.85, 0.67)
a_2	(0.73, 0.78)	(0.75, 0.62)	(0.73, 0.79)	(0.94, 0.53)	(0.94, 0.53)	(0.94, 0.53)
a_3	(0.85, 0.49)	(0.85, 0.39)	(0.85, 0.59)	(0.92, 0.49)	(0.92, 0.39)	(0.92, 0.49)
a_4	(0.83, 0.67)	(0.83, 0.59)	(0.83, 0.37)	(0.67, 0.85)	(0.67, 0.59)	(0.76, 0.37)

4 Entropy Measure

Entropy is an essential tool to measure uncertain information. If the entropy is less the uncertainty is also less thus one can quickly identify that which one is the more stable information. (ffss) being a more generalized structure, it is able to represent an information in which other existing structures fail. Thus introducing the measure for entropy is important in the current scenario. In this section, by introducing some definitions and results, the expression for entropy and distance measure for (ffss)s are obtained and illustrated with examples.

Definition 9. *Choose the two (ffss)s (Φ, A) and (Ψ, B). Define the relation (Φ, A) is less than or equal to (Ψ, B), denoted as $(\Phi, A) \preceq (\Psi, B)$ if, for all $a \in \Upsilon$ and $\epsilon \in \Sigma$, $m_{\Phi(\epsilon)}(a) \leq m_{\Psi(\epsilon)}(a)$ and $n_{\Phi(\epsilon)}(a) \leq n_{\Psi(\epsilon)}(a)$.*

The following definition is about a mapping which maps every (ffss) to a fuzzy soft set. It is also shown that the collection of images of (ffss)s with $\alpha \in [0,1]$ and with the relation $\subseteq$ is a totally ordered family of fuzzy soft sets.

Definition 10. *For $\alpha \in [0,1]$, the mapping $f_\alpha : (ffss)(\Upsilon) \to (fss)(\Upsilon)$ is defined as $f_\alpha(\Phi(P)) = \Phi_\alpha(P)$, for every (ffss) $\Phi(P)$ with membership value $m_{\Phi(p)}$ and non-membership value $n_{\Phi(p)}$ and $\Phi_\alpha(p) = f_\alpha(\Phi_p)$ and,*

$$f_\alpha(\Phi_p) =< a, m^3_{\Phi(p)}(a) + \alpha.\pi^3_{\Phi(p)}(a), 1 - m^3_{\Phi(p)}(a) - \alpha.\pi^3_{\Phi(p)}(a) : a \in \Upsilon > . \tag{1}$$

Thus the map f_α assign every (ffss)s to a fuzzy soft set. The operator f_α is an extension of [47]. That is, the operator f_α is assign to a (ffss) to a fuzzy soft set, however, the operator f_α given in [47] is to assign a (ffss) to a fuzzy set.

Now,we give a example related to this definition:

Example 6. *Let $\alpha = 0.8$, and $\Phi(P) = [a_{ij}] = \begin{pmatrix} (0.8, 0.7) & (0.7, 0.4) \\ (0.5, 0.8 & (0.9, 0.6) \end{pmatrix}$. Then we can compute*

$$\Phi_\alpha(p_1) = f_\alpha[(u_1, 0.8, 0.7), (u_2, 0.7, 0.4)] = \{(u_1, 0.628, 0.372), (u_2, 0.8174, 0.1826)\}$$
$$\Phi_\alpha(p_2) = f_\alpha[(u_1, 0.5, 0.8), (u_2, 0.9, 0.6)] = \{(u_1, 0.4154, 0.5846), (u_2, 0.773, 0.227)\}$$

Thus fuzzy soft set is obtained and is represented by matrix $\begin{pmatrix} (0.628, 0.372) & (0.8174, 0.1826) \\ (0.4154, 0.5846 & (0.773, 0.227) \end{pmatrix}$

Theorem 7. *For any $\alpha, \beta \in [0,1]$ and $p, \bar{p} \in (ffss)(\Upsilon)$, the following statements are true.*

- *i. If $\alpha \le \beta$, then $f_\alpha(p) \subset f_\beta(p)$.*
- *ii. If $p \subset \bar{p}$, then $f_\alpha(p) \subset f_\alpha(\bar{p})$.*
- *iii. $f_\alpha(f_\beta(p)) = f_\beta(p)$*
- *iv. $(f_\alpha(p^t))^t = f_{1-\alpha}(p)$.*

It is understood from this theorem that the $(\{p_\alpha\}_{\alpha \in [0,1]}, \subseteq)$ is totally ordered family of fuzzy soft sets, for $p = (\Phi, \Sigma) \in (ffss)(\Upsilon)$.

Definition 11. *A real function $D : (ffss)(\Upsilon) \to \mathbb{R}^+$ is called a fermatean fuzzy soft entropy on $(ffss)(\Upsilon)$, if E has the following properties:*

i. $D(p) = 0$ if and only if $(fss)(\Upsilon)$
ii. Let $p = (\Phi, D) = [a_{ij}]_{m \times n}$, $D(p) = mn$ if and only if $m_{\Phi(\epsilon)}(a) = n_{\Phi(\epsilon)}(a) = 0$, $\forall \epsilon \in D$, $\forall a \in \Upsilon$.
iii. $E(p) = E(p^t)$, $p \in (ffss)(\Upsilon)$
iv. If $p \preceq \bar{p}$, then $D(p) \ge D(\bar{p})$, where $(\Phi, D) = p$ and $(\Psi, D) = \bar{p}$.

When the entropy value is minimum, ie zero, it is understood that the (Pfss) has degenerated into a (ss).

Theorem 8. *Entropy of (ffss) p is maximum if and only if $p = (\Phi, D) = [a_{ij}]_{m \times n} = [0]_{m \times n}$ i.e., $m_{\Phi(e_j)}(u_i) = n_{\Phi(e_j)}(u_i) = 0$, $\forall e_j \in D$, $u_i \in \Upsilon$ where $0 \le i \le m$ and $0 \le j \le n$ and $p \in (fss)(\Upsilon)$.*

Our goal is give an expression which allows us to create entropies for (ffss)s. It is the same as the approach to crete entropies for (fss)s, let us take the following set given $K = \{(a, b) \in [0,1] \times [0,1] : a^3 + b^3 \le 1\}$ and with it let us construct $\Theta_K : K \to [0,1]$, which satisfies the following conditions:

i. $\Theta_K(a, b) = 1$ if and only if $(a, b) = (0, 1)$ or $(a, b) = (1, 0)$
ii. $\Theta_K(a, b) = 0$ if and only if $a = b = 0$
iii. $\Theta_K(a, b) = \Theta_K(b, a)$
iv. If $a \le a^{'}$ and $b \le b^{'}$ then $\Theta_K(a, b) \le \Theta_K(a^{'}, b\Sigma^{'})$.

Theorem 9. *Let $D : P_{ffss}(\Upsilon) \to \mathbb{R}^+$ and $p = (\Phi, D) = [a_{ij}]_{m \times n} \in P_{ffss}(\Upsilon)$. If $D(p) = \sum_{j=1}^{n} \sum_{i=1}^{m} [1 - (\Theta_K(m_{\Phi(\epsilon_j)}(u_i), n_{\Phi(\epsilon_j)}(u_i)))]$ where Θ_K satisfies the conditions (i)-(iv) given above then D is a (ffss) entropy.*

Example 7. $D(p) = \sum_{j=1}^{n} \sum_{i=1}^{m} [1 - (m^4_{(\Phi,A)}, n^4_{\Phi,A)})]$. *To verify that above expression is an entropy of (ffss)s it is enough to verify that $m^4_{(\Phi,A)}, n^4_{\Phi,A)}$ satisfy the conditions of Θ_K. $\Theta_K(a, b) = m^4_{(\Phi,A)}, n^4_{\Phi,A)}$ is a function from $K = \{(m_{(\Phi,A)}, n_{(\Phi,A)}) \in [0,1] \times [0,1] : a^3 + b^3 \le 1\}$ to $[0,1]$. Also $m^4_{(\Phi,A)}, n^4_{\Phi,A)} = 1$ if and only if $m_{(\Phi,A)} = 1$, $n_{\Phi,A)} = 0$ or $m_{(\Phi,A)} = 0$, $n_{\Phi,A)} = 1$ in the domain K.*

Definition 12. *Let $\Phi, \Phi^{'} : [0,1] \to [0,1]$ such that if $a^3 + b^3 \le 1$, then $\Phi(a^3) + \Phi^{'}(b^3) \le 1$ with $a, b \in [0,1]$. Define $D_{\Phi,\Phi'}$ function of the (ffss) $p = (\Phi, D) = [a_{ij}]_{m \times n}$ to $\mathbb{R}^+$ as,*

$$D_{\Phi,\Phi'} = mn - \sum_{j=1}^{n} \sum_{i=1}^{m} \Phi[m_{\Phi(\epsilon_j)}(u_i)] + \Phi^{'}[n_{\Phi(\epsilon_j)}(u_i)] \tag{2}$$

Obviously $0 \leq D_{\Phi,\Phi'}(p) \leq mn$ and $\forall p = [a_{ij}]_{m\times n}$ belonging to $P_{ffss}(\Upsilon)$.

Theorem 10. *If $\Phi : [0,1] \to [0,1]$ satisfies,*

i. Φ is increasing
ii. $\Phi(a) = 0$ if and only if $a = 0$
iii. $\Phi(a) + \Phi(b) = 1$ if and only if $(a,b) = (0,1)$ or $(a,b) = (1,0)$

Then $\Phi(x) + \Phi(y)$ satisfies the conditions (i)-(iv) of the Θ_K function defined previously.

5 Decision Making Application

Let $\Upsilon = \{a_1, a_2, \cdots, a_m\}$ be the universal set under consideration and $\Sigma = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ be the parameter set which consists some qualities of universal set. The algorithm given below explains how to solve the decision making problem using the expression for entropy.

Algortihm:

Step 1: Input each of the (ffss)s $p_1, p_2, \cdots p_k$

Step 2: Compute entropy of each (ffss)s using the expression

$$D(p) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1 - (m^3_{\Phi(\epsilon_j)}(a_i) + n^3_{\Phi(\epsilon_j)}(a_i))].$$

Step 3: Find p_r such that $D(p_r) = \min_{i=1,2,\cdots,k} D(p_i)$.

Step 4: Optimal decision is to select p_r obtained from Step 3.

Step 5: If more than one optimal solution is obtained, any one of them may be chosen.

Example 8. *Consider the infectious diseases set*

$$\Upsilon = \{a_1, a_2, a_3\} = \{HepatitisC, influenzaA(H1N1), norovirus\}.$$

Let's take the $\Sigma = \{p_1, p_2, p_3\} = \{headache, temperature, nausea\}$ set as the parameter set. Then,

***Step 1:** Construct the (ffss)s $(\Phi, A), (\Psi, A), (\Omega, A)$ as follows:*

$$\begin{aligned}
\Phi(p_1) &= \{(a_1,(0.75,0.58)),(a_2,(0.98,0.15)),(a_3,(0.47,0.83))\}\\
\Phi(p_2) &= \{(a_1,(0.82,0.66)),(a_2,(0.59,0.51)),(a_3,(0.26,0.95))\}\\
\Phi(p_3) &= \{(a_1,(0.54,0.79)),(a_2,(0.73,0.55)),(a_3,(0.87,0.51))\}
\end{aligned}$$

$$\begin{aligned}
\Psi(p_1) &= \{(a_1,(0.63,0.87)),(a_2,(0.80,0.72)),(a_3,(0.56,0.68))\}\\
\Psi(p_2) &= \{(a_1,(0.72,0.80)),(a_2,(0.51,0.92)),(a_3,(0.67,0.71))\}\\
\Psi(p_3) &= \{(a_1,(0.82,0.53)),(a_2,(0.88,0.45)),(a_3,(0.73,0.66))\}
\end{aligned}$$

$$\begin{aligned}
\Omega(p_1) &= \{(a_1,(0.42,0.93)),(a_2,(0.56,0.70)),(a_3,(0.88,0.62))\}\\
\Omega(p_2) &= \{(a_1,(0.67,0.79)),(a_2,(0.77,0.64)),(a_3,(0.76,0.39))\}\\
\Omega(p_3) &= \{(a_1,(0.68,0.52)),(a_2,(0.91,0.36)),(a_3,(0.74,0.67))\}
\end{aligned}$$

***Step 2:** Calculate the entropies of $(\Phi, A), (\Psi, A), (\Omega, A)$:*

$$\begin{aligned}
D(\Phi, A) &= \sum_{j=1}^{n}\sum_{i=1}^{m}[1 - (m^4_{\Phi(\epsilon_j)}(a_i) + n^4_{\Phi(\epsilon_j)}(a_i))] = 4.60312528\\
D(\Psi, A) &= 3.96972281\\
D(\Omega, A) &= 4.17700393
\end{aligned}$$

***Step 3:** Find the (ffss) which has the minimum value of entropy, which is (Ψ, A).*

***Step 4:** Optimal decision is to select (Ψ, A).*

Step 5: *Since there is only one optimal decision,*

Comparative Studies:

To compare the proposed entropy measure for (ffss)s, the (Pfs) entrpoy given in [33] is taken.

Example 9. *Take the (ffss)s Φ_1, Φ_2, Φ_3 in the feature space $A = \{a_1, a_2, a_3\}$ as follows:*

$$\begin{aligned}
\Phi_1 &= \{p_1 = (a_1, 0.3, 0.2), (a_2, 0.6, 0.0), (a_3, 0.5, 0.4), p_2 = (a_1, 0.6, 0.3), (a_2, 0.7, 0.2), (a_3, 0.4, 0.3), p_3 = (a_1, 0.8, 0.1), (a_2, 0.8, 0.1), (a_3, 0.6 \\
\Phi_2 &= \{p_1 = (a_1, 0.6, 0.2), (a_2, 0.8, 0.1), (a_3, 0.8, 0.1), p_2 = (a_1, 0.5, 0.5), (a_2, 0.7, 0.2), (a_3, 0.5, 0.4), p_3 = (a_1, 0.7, 0.1), (a_2, 0.6, 0.3), (a_3, 0.6 \\
\Phi_3 &= \{p_1 = (a_1, 0.5, 0.4), (a_2, 0.4, 0.1), (a_3, 0.6, 0.2), p_2 = (a_1, 0.6, 0.2), (a_2, 0.7, 0.1), (a_3, 0.8, 0.1), p_3 = (a_1, 0.9, 0.0), (a_2, 0.5, 0.1), (a_3, 0.6
\end{aligned}$$

The entropy values of (Pfss)s are obtained as $D_{Pfss}(\Phi_1) = 6.63, D_{Pfss}(\Phi_2) = 6.13, D_{Pfss}(\Phi_3) = 6.34$ [33]. The entropy values suggested in this study were measured as $D_{ffss}(\Phi_1) = 7.37, D_{ffss}(\Phi_2) = 7.08, D_{ffss}(\Phi_3) = 7.11$. Then corresponding D_{Pfss} and D_{ffss}, we get Φ_2 has minimum entropy and Φ_1 has maximum entropy. So it can be concluded that proposed equations for entropy are consistent.

Table 12 Comparison of entropies for (Pfss) and (ffss)

	D_{Pfss}	D_{ffss}
Φ_1	6.63	7.37
Φ_2	6.13	7.08
Φ_3	6.34	7.11

6 Conclusion

The aim of this study is to define (ffss)s and to give an entropy measure. Firstly the concept of (ffss) is defined. Later, various operations and properties of (ffss) are discussed. The (ffss)s are a generalization of the concept of fuzzy soft sets. Also, the entropy measure of (ffss) is introduced. We can easily say that ffss is more sensible and more accurate than existing other soft sets models. Then as an application, decision-making problem on (ffss) is proposed. (Pfs) entropy was compared with (ffs) entropy and suggested entropy was found to be consistent.

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Hyperbolic eigenparameter dependent non-selfadjoint discrete Sturm-Liouville equations with a general boundary condition

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Abstract: It is a well-known fact that the representation of the Jost solution and analiticy regions of the Jost function strictly depends on the transformation choosen for eigenparameter. Based on this ideas and using the results of Naimark, in this study, we examine the spectral properties of the non-selfadjoint difference operator L generated in the Hilbert space $l_2(\mathbb{N})$ by the Sturm-Liouville type difference operator and a general boundary condition. In particular, we adopt and generalize the recent results to the hyperbolic eigenparameter dependent case. Hence, we determine the eigenvalues and spectral singularities of the operator.

Keywords: Eigenvalues, Discrete equation, Spectral analysis, Sturm-Liouville equations.
AMS Mathematics Subject Classification [2010]: 47A10, 47A75

1 Introduction

The spectral theory of differential and discrete operators is very vital in the modern analysis, and there has been a lot published about it. It was soon supplemented by close connections to fundamental quantum physics developments. The spectrum of self-adjoint operators already has been carefully examined [1–3]. As new spectral features appeared, further spectral sets were generated. Pre-1980 spectral theory of self-adjoint operators on separable Hilbert spaces, for example, defined spectral sets emerging from natural spectral decompositions on the Hilbert space such as the discrete spectrum, σ_{disc}, essential spectrum, σ_{ess}, singular-continuous spectrum, σ_{sc}, etc. The appearance of spectral singularities in the continuous spectrum for the non-selfadjoint Sturm-Liouville operator was first discovered by Naimark. Spectral singularities are described as points embedded in a continuous spectrum that are the kernel's poles but not the eigenvalues [4, 5]. Physical applications of spectral singularities have been investigated in recent years [6, 7].

The aforementioned and other discoveries piqued people's interest in spectral theory of differential and discrete operators with spectral singularities. The reader is encouraged to the papers [8–21] and respective references for more details.

In this paper, the spectral properties of the operator L generated in $l_2(\mathbb{N})$ by the discrete Sturm-Liouville type equations

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, n \in \mathbb{N} = \{1, 2, ...\}, \tag{1.1}$$

and a general boundary condition

$$\sum_{n=0}^{\infty} k_n y_n = 0, \tag{1.2}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ are complex sequences such that $a_0 = 1,\ k_0 \neq 0$, $\{k_n\}_{n=1}^{\infty} \in l_2(\mathbb{N})$ were taken into consideration. Spectral singularities and the spectrum of discrete Sturm-Liouville type operator under the general boundary condition for the trigonometric eigenparameter case has been considered in [17] . Spectral singularities of the discrete Dirac and discrete Klein-Gordon operators under the general boundary condition has been studied for trigonometric eigenparameter case, too in [18, 19].

Concerning the non-selfadjoint discrete boundary value problems, let us mention some different approaches. For instance, in paper [21] , the eigenparameter of the non-selfadjoint boundary value problem was taken as

$$\lambda = (iz) - (iz)^{-1}, |z| \leq 1.$$

As a result of this transformation, Jost solution obtained the polynomial type representation which is analytic in unit disc. Also, in [20], the spectrum of discrete analogue of Sturm-Liouville equation has been investigated for

$$\lambda = \frac{1}{2}\left(z^{-1} + z\right), |z| \geq 1.$$

A non-standard representation for Jost solution has been obtained under this eigenparameter transformation, too. Therefore, it is clear that there is a gap in the literature investigating the problem of under what transformations of the eigenparameter one can obtain solvable systems for Sturm-Liouville type discrete equations (which is also known as infinite Jacobi matrices).

In contrast to the previous research, recent articles [15, 16] examined discrete Sturm-Liouville equations with hyperbolic eigenparameter. The analycity region of the Jost function is relocated from the upper to the left half-plane by the hyperbolic eigenparameter. As a result of this change, the Jost solution of the operator L has distinct analitic continuity regions.

We examine the general boundary condition of the non-selfadjoint Sturm-Liouville problem with a hyperbolic transformation of the eigenparameter, in addition to a complex valued potential. As a result, the calculations for deriving the Naimark's and Pavlov's criteria for the potential necessitate a new view on trigonometric parameter cases.

In this study, we obtain the Jost function and the resolvent operator of L. The sets of eigenvalues and spectral singularities of L were also introduced. Following the presentation of an example, the problem was subjected to Naimark's and Pavlov's conditions. As a result of generalizing latest discoveries, the foundation for future research in spectral expansion, inverse problems, and scattering theory has been laid.

2 Jost Solution and resolvent of L

The Jost solution and other fundamental properties of the equation (1.1) will be discussed in detail. Following that, using the classical constant coefficients technique for the solutions of differential and discrete equations, the resolvent operator and Jost function of the operator L were derived.

Assume

$$\sum_{n\in\mathbb{N}} n\left(|1-a_n|+|b_n|\right)<\infty. \tag{2.1}$$

Under the assumption (2.1), the following Jost solution has been constructed for $z\in\overline{\mathbb{C}}_{left}:=\{z:z\in\mathbb{C}, Rez\leq 0\}$ and $\lambda=2\cosh z$ with complete analogy with what have been referred in $[9,15,17]$ in previous section that,

$$e_n(z)=\alpha_n e^{nz}\left(1+\sum_{m=1}^{\infty}K_{n,m}e^{mz}\right),\ n\in\mathbb{N}\cup\{0\}. \tag{2.2}$$

Note that $K_{n,m}$ and α_n can be solved uniquely in terms of (a_n) and (b_n) . Moreover, we have the inequality for the kernel $K_{n,m}$

$$|K_{n,m}|\leq C\sum_{r=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty}\left(|1-a_r|+|b_r|\right),\ n\in\mathbb{N}\cup\{0\}, \tag{2.3}$$

Therefore, $e_n(z)$ is analytic with respect to z in $\mathbb{C}_{left}:=\{z:z\in\mathbb{C}, Rez<0\}$ and continuous in $Rez=0$ and it also yields

$$e_n(z)=\alpha_n e^{nz}\left[1+o(1)\right], n\in\mathbb{N}, z=\xi+i\tau, \xi\to-\infty.$$

Similar to $e_n(z)$, assume $\widehat{\varphi}(\lambda)=\{\widehat{\varphi}_n(\lambda)\}=\varphi_n(z)$, $n\in\mathbb{N}\cup\{0\}$ be the solution of (1.1) holding the initial conditions

$$\varphi_0(z)=0,\quad \varphi_1(z)=1.$$

Also, assume

$$\varphi(z)=\widehat{\varphi}(2\cosh z)=\{\widehat{\varphi}_n(2\cosh z)\},\quad n\in\mathbb{N}\cup\{0\}.$$

Clearly, φ is an entire function and

$$\varphi(z)=\varphi(z+2\pi i).$$

Introduce the semi-strips

$$P_0:=\left\{z:z\in\mathbb{C},\ z=\xi+i\tau,\ -\frac{\pi}{2}\leq\tau\leq\frac{3\pi}{2},\xi<0\right\},$$

and

$$P:=P_0\cup\left\{z:z\in\mathbb{C},\ z=\xi+i\tau,\ -\frac{\pi}{2}\leq\tau\leq\frac{3\pi}{2},\xi=0\right\}.$$

The Wronskian of the solutions $y_n(z)$ and $u_n(z)$ of (1.1) is defined as classical

$$W\left[y_n,u_n\right]=a_n\left[y_n u_{n+1}-y_{n+1}u_n\right].$$

Therefore, we get

$$W\left[e_n(z),\varphi_n(z)\right]=e_0(z).$$

Define the functions

$$M(z):=\sum_{n=0}^{\infty}k_n e_n(z), \tag{2.4}$$

and

$$\begin{aligned}\widetilde{M}(z) &:= \sum_{n=0}^{\infty} k_n \varphi_n(z), \\ H_k(z) &:= -\frac{1}{e_0(z)} \Big\{ M(z)\varphi_{k+1}(z) - \widetilde{M}(z) e_{k+1}(z) \\ &\quad -\varphi_{k+1}(z) \sum_{n=k+1}^{\infty} k_n e_n(z) + e_{k+1}(z) \sum_{n=k+1}^{\infty} k_n \varphi_n(z) \Big\}.\end{aligned}$$

Let $G_{nk}(z) := G_{nk}^{(1)}(z) + G_{nk}^{(2)}(z)$ denote the Green's function of the operator L. The classical computations indicate that

$$G_{nk}^{(1)}(z) := \frac{e_n(z) H_k(z)}{M(z)}, \tag{2.5}$$

and

$$G_{nk}^{(2)}(z) := \begin{cases} 0, & k < n, \\ \frac{[\varphi_{k+1}(z)e_n(z) - \varphi_n(z)e_{k+1}(z)]}{e_0(z)}, & k \geq n, \end{cases} \tag{2.6}$$

for all $z \in P$ and $e_0(z) \neq 0$. Hence, the resolvent operator of L is written as

$$(R_\lambda(L)\phi)_n := \sum_{k=0}^{\infty} G_{nk}(z)\phi_{k+1}(z),\ \phi = \{\phi_k\} \in l_2(\mathbb{N}), k \in \mathbb{N} \cup \{0\}. \tag{2.7}$$

3 Main Result

The eigenvalues and spectral singularities of the operator L, as well as their quantitative features, are the focus of this section. To achieve the conditions by which the eigenvalues and spectral singularities are of finite number with finite multiplicities, we shall apply fundamental spectral analysis definitions and boundary uniqueness theorems of analytic functions.

Let's designate the sets of eigenvalues and spectral singularities of the operator L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, correspondingly. (2.5) -(2.7) and the standard spectrum definitions indicate that

$$\sigma_d(L) = \{\lambda : \lambda = 2\cosh z, z \in P_0, M(z) = 0\}, \tag{3.1}$$

$$\sigma_{ss}(L) = \left\{\lambda : \lambda = 2\cosh z,\ z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ M(z) = 0\right\} \setminus \{0\}. \tag{3.2}$$

Now let us proceed with the sets

$$\begin{aligned} R_1 &:= \{z : z \in P_0, M(z) = 0\}, \\ R_2 &:= \left\{z : z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], M(z) = 0\right\}, \end{aligned}$$

and R_3, R_4 as the sets of limit points of the sets R_1 and R_2, correspondingly, and R_5 as the set of zeros in P_0 of the function $M(z)$ with infinite multiplicity. The preceding relationships are obviously valid.

$$R_3 \subset R_2,\ R_4 \subset R_2,\ R_5 \subset R_2, R_1 \cap R_5 = \emptyset,$$

and the linear Lebesgue measures of R_2, R_3, R_4 and R_5 are zero. Because all derivatives of $M(z)$ are continuous up to the real axis, it can be written that

$$R_3 \subset R_5,\ R_4 \subset R_5. \tag{3.3}$$

The sets of eigenvalues and spectral singularities can readily be stated as

$$\begin{aligned} \sigma_d(L) &= \{\lambda : \lambda = 2\cosh z, z \in R_1\}, \\ \sigma_{ss}(L) &= \{\lambda : \lambda = 2\cosh z, z \in R_2\}. \end{aligned}$$

Theorem 1. *Suppose that (2.1) and $\{k_n\}_{n=1}^{\infty} \in l_2(\mathbb{N})$ are true. Consequently,*
i) $\sigma_d(L)$ is bounded, countable and its limit points can lie only in $[-2, 2]$.
ii) The set of spectral singularities of L is subset of $[-2, 2]$, $\mu(\sigma_{ss}(L)) = 0$ where μ denotes the linear Lebesgue measure and $\sigma_{ss}(L) = \overline{\sigma_{ss}(L)}$.

Proof: $M(z)$ is analytic in the upper half-plane and continuous up to the real axis, as is well known. Furthermore, the asymptotic follows

$$M(z) = \alpha_0 [1 + o(1)],\ z \in P_0,\ Rez \to -\infty. \tag{3.4}$$

Using (3.1), (3.2), and (3.4) and analytic function uniqueness theorems, it is simple to conclude i) and ii) [22]. □

Let us build an example for the following simple instance to clarify the above conclusions. Define the operator $\widetilde{L}$ generated in the Hilbert space $l_2(\mathbb{N})$ by

$$\left(\widetilde{l}y\right)_n = y_{n-1}^{(v)} + y_{n+1}^{(v)}, n \in \mathbb{N},$$

and the boundary condition

$$\sum_{n=0}^{\infty} k_n y_n = 0.$$

Let us choose the sequence (k_n) with the terms $k_1 = 1$, $k_0 = -e^{i\frac{\pi}{4}}$ and $k_n = 0$ for $n \geq 2$. Computing the zeros of $M(z)$ for these specific choices and using the classical definitions, one can easily show that

$$\begin{aligned} \sigma_d(\widetilde{L}) &= \emptyset, \\ \sigma_{ss}(\widetilde{L}) &= \left\{\sqrt{2}, 1\right\} \in [-2, 2]. \end{aligned}$$

Clearly, $\sigma_c(\widetilde{L}) = [-2, 2]$ from previous papers [15, 17]. Hence, we verify for this specific example that the set of spectral singularities lies in continuous spectrum and do not belong to the set of discrete spectrum.

Definition 1. The multiplicity of a zero of $M(z)$ in the region P is called as the multiplicity of the corresponding eigenvalue or spectral singularity of the operator L.

Up to this point, the condition has been used to explore the Jost solution, resolvent operator, sets of eigenvalues, and spectral singularities of the operator L. (2.1). Now we'll look into the effects of stricter conditions on potential, such as Naimark's and Pavlov's conditions.
We shall assume

$$\sum_{n=1}^{\infty} e^{\varepsilon n^{\beta}} \left(|1 - a_n| + |b_n| + |k_n|\right) < \infty, \ \varepsilon > 0, \ \frac{1}{2} \leq \beta \leq 1. \tag{3.5}$$

For $\beta = 1$, (3.5) reduces to Naimark's condition:

$$\sum_{n=1}^{\infty} e^{\varepsilon n} \left(|1 - a_n| + |b_n| + |k_n|\right) < \infty, \ \varepsilon > 0. \tag{3.6}$$

Theorem 2. *If condition (3.6) is satisfied, the operator L has a finite number of eigenvalues and spectral singularities, each of which has a finite multiplicity.*

Proof: The following inequality can be calculated using (2.3) and (3.6).

$$|K_{n,m}| \leq C \exp\left(\frac{-\varepsilon}{2}(n + m)\right), \tag{3.7}$$

is satisfied for all $C > 0$ constant, $n = 0, 1, 2, ...$ and $m = 1, 2, ...$ Using (2.2), (2.8), (3.6) and (3.7) and after some mathematics, one derives,

$$|M(z)| \leq \sum_{m=1}^{\infty} e^{-m\left(\frac{\varepsilon}{4} - Rez\right)}. \tag{3.8}$$

(3.8) suggests that $M(z)$ continues analytically from real axis to the left half-plane $Rez < \frac{\varepsilon}{4}$. Furthermore, $M(z)$ is a $2\pi i$ periodic function, the limit points of its zeros in the region P can not be in the interval

$$\left\{z \in \mathbb{C} : z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}.$$

Theorem 1 confirms the finiteness of eigenvalues and spectral singularities of L as a result of these findings. □

Apparently, (3.6) assures that $M(z)$ continues analytically from the real axis to the left half-plane. Take the condition (3.5) for $\frac{1}{2} \leq \beta < 1$,

$$\sum_{n=1}^{\infty} e^{\varepsilon n^{\beta}} \left(|1 - a_n| + |b_n|\right) < \infty, \ \varepsilon > 0. \tag{3.9}$$

It is evident that $M(z)$ has analicity in the left half-plane and infinite differentiability on the imaginary axis. Analytic continuation of $M(z)$ from the real axis to the lower half-plane is not accomplished under condition (3.9). As a conclusion, a novel approach for analyzing the finiteness of eigenvalues and spectral singularities of L is necessitated. The following theorem [17] will find a way to deal with this challenge.

Theorem 3. *[17] Suppose the 2π periodic function ξ is anaytic in the open half-plane, all of its derivatives are continuous in the closed upper half-plane and*

$$\sup_{z\in P}\left|\xi^{(k)}(z)\right| \leq \eta_k,\ k\in\mathbb{N}\cup\{0\}. \tag{3.10}$$

If the set G with linear Lebesgue measure zero is the set of all zeros of the function ξ with infinite multiplicity in P, and

$$\int_0^\omega \ln t(s)d\mu(G_s) > -\infty,$$

where $\mu(G_s)$ is the Lebesgue measure of the s-neighborhood of G, $t(s)=\inf_k \frac{\eta_k s^k}{k!}$, $k\in\mathbb{N}\cup\{0\}$, and $\omega\in(0,2\pi)$ is an arbitrary constant, then $\xi\equiv 0$.

Theorem 4. *If (3.9) holds, then $R_5=\emptyset$.*

Proof: The following inequality for the k. th derivative of $M(z)$ can be obtained from (3.9), (2.2) and (2.3) and after some algebra

$$\left|M^{(k)}(z)\right| \leq \eta_k, k\in\mathbb{N}\cup\{0\},$$

where

$$\eta_k = 2^k C\sum_{m=1}^\infty m^k \exp\left(-\varepsilon m^\beta\right), \tag{3.11}$$

and $C>0$ is a constant. As a next step, one obtains the inequality for η_k using the classical inequalities in the literature

$$\eta_k \leq 2^k C\int_0^\infty x^k e^{-\varepsilon x^\beta}dx \leq Dd^k k! k^{k\frac{1-\beta}{\beta}},$$

where D and d are constants depending C, ε and β.

Now, the previous theorem can be adopted to our case. Taking into account $t(s)=\inf_k \frac{\eta_k s^k}{k!}$, $k\in\mathbb{N}\cup\{0\}$, $\mu(R_{5,s})$ is the Lebesgue measure of the s-neighborhood of R_5 and η_k is defined by (3.11), the following inequality is clear

$$\int_0^\omega \ln t(s)d\mu(R_{5,s}) > -\infty. \tag{3.12}$$

We get the inequality

$$t(s) \leq D\exp\left\{-\frac{1-\beta}{\beta}e^{-1}d^{-\frac{\beta}{1-\beta}}s^{-\frac{\beta}{1-\beta}}\right\}, \tag{3.13}$$

using (3.11). (3.12) and (3.13) yield,

$$\int_0^\omega s^{-\frac{\beta}{1-\beta}}d\mu(R_{5,s}) < \infty. \tag{3.14}$$

Because of $\frac{\beta}{1-\beta}\geq 1$, the inequality (3.14) is true for arbitrary s if and only if $\mu(R_{5,s})=0$ or $R_5=\emptyset$. □

Theorem 5. *If (3.9) holds to be true, then the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof: We are supposed to prove that $M(z)$ has a finite number of zeros with finite multiplicities in the region P. From Theorem 4 and (3.3), we have $R_3=R_4=\emptyset$. Therefore, the accummulation points of the bounded sets R_1 and R_2 do not exist. Due to the these reasons, $H(z)$ must have only finite number of zeros in the region P. Because of $R_5=\emptyset$, these zeros must be of finite multiplicity. □

4 Conclusion

In this paper, the non-selfadjoint singular boundary value problem generated in $l_2(\mathbb{N})$ including the Sturm-Liouville type difference equation and a general boundary condition for hyperbolic eigenparameter has been investigated. After presenting the basic definitions and Jost solution properties, we constructed the Naimark's and Pavlov's conditions for the problem and proved that the eigenvalues and spectral singularities are of finite number with finite multiplicities.

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Two Applications of the Duhamel product

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Abstract: Let $p \in [1, +\infty)$ be an integer. We study some properties of integration operator $V, Vf(z) = \int_0^z f(t)\,dt$, on the space of analytic functions

$$\ell_A^p := \ell_A^p(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : f(z) \sum_{n=0}^{\infty} a_n z^n \text{ and } (a_n)_{n\geq 0} \in \ell^p \right\},$$

where $a_n = \frac{f^{(n)}(0)}{n!}$, $n \geq 0$, is the n th Taylor coefficients of f. Namely, we characterize the sets of so-called extended eigenvalues and extended eigenvectors of operator V. We also calculate the spectral multiplicity of operator of the form $V \oplus A$, where A is a appropriate bounded linear operator on a Banach space X.

1 Introduction

Let p, $1 < p < +\infty$, be an integer, and let $\ell_A^p = \ell_A^p(\mathbb{D})$ be a space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $\sum_{n=0}^{\infty} |a_n|^p < +\infty$. With the norm $\|f\|_p := \left(\sum_{n=0}^{\infty} |a_n|^p\right)^{1/p} < +\infty$ the space ℓ_A^p is a Banach space. It is well known that $\left(\ell_A^p\right)^* = \ell_A^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. In this article, we consider the classical Volterra integration operator V, $Vf(z) = \int_0^z f(t)\,dt$, on the space ℓ_A^p and study its some new properties. Namely, by using the Duhamel product method, we investigate the set of extended eigenvalues and characterize the corresponding set of extended eigenvectors (Section 2.1); we prove an addition formula $\mu(V \oplus A) = \mu(V) + \mu(A)$ for the spectral multiplicity of operator $V \oplus A$ with a suitable bounded linear operator A acting in the Banach space (Section 2.2).
Before stating our results, we need some necessary background. For a Banach space X we denote by $\mathcal{B}(X)$ the set of all bounded linear operators on X. Recall that if X is a separable Banach space and $A \in \mathcal{B}(X)$, if $\text{span}\left\{A^k E : k = 0, 1, 2, \ldots\right\} = X$, then $E \subset X$ is a cyclic subspace; vector $x \in X$ is known as the cyclic vector ($x \in \text{Cyc}(A)$) of operator A, is $\text{span}\left\{A^k x : k = 0, 1, 2, \ldots\right\} = X$, where span denotes the closed linear hull of the set in X. Spectral multiplicity $\mu(A)$ of operator A is the following number (or the symbol ∞):

$$\mu(A) := \inf\left\{\dim E : \text{span}\left\{A^k E : k \geq 0\right\} = X\right\}.$$

If $\mu(A) = 1$, A is a cyclic operator . For example, the Volterra integration operator V and the shift operator S, $Sf(z) = zf$, are cyclic operators in many function spaces.
For a given operator $A \in \mathcal{B}(X)$ the number $\lambda \in \mathbb{C}$ is called the extended eigenvalue of operator A if there exists a nonzero operator $B \in \mathcal{B}(X)$ such that $AB = \lambda BA$. Such operator B is called the corresponding extended eigenvector for operator A (see [1]). The set of all extended eigenvalues of operator A is denoted by ext(A). Note that these topics were more popular after the celebrated result of Lomonosov [7] concerning to the existence of hyperinwariant subspace for operators commuting with compact operator on the Banach space. Recall that a subspace $E \subset X$ is hyperinwariant with respect to A if $BE \subset E$ for all operators B commuting with operator A.
In the present paper, we investigate these two problems mentioned above, that is, we study the structure of extended eigenvalues and extended eigenvectors of an integration operator V on the space ℓ_A^p; for this, we use the Duhamel product (see [11, 12]), which is defined for any two functions f, g in ℓ_A^p by the formula

$$(f \circledast g)(z) := \frac{d}{dz}\int_0^z f(z-t)\,g(t)\,dt = \int_0^z f'(z-t)\,g(t)\,d(t) + f(0)\,g(z). \tag{1}$$

The Duhamel operator is defined by $\mathcal{D}_f g := f \circledast g$, $g \in \ell_A^p$. This product is also utilized in the study of spectral multiplicity $\mu(V \oplus A)$.

2 Main Results

2.1 The extended eigenvalues and extended eigenvectors for integration operator

In this section, we describe the sets of extended eigenvalues and extended eigenvectors of operator V on the space ℓ_A^p. The main tool is the use of Duhamel product (1), which firstly used to Karaev in [5], see also Gürdal [2] and Tapdigoglu [9, 10].
Note that it is easy to see that $\ker(V) = \{0\}$, so $\lambda = 0$ is not an extended eigenvalue of V, i.e., $0 \notin \text{ext}(V)$.

Here we will prove that, in fact, $\text{ext}(V) = \mathbb{C}\setminus\{0\}$.

Theorem 1. *Let $\lambda \in \mathbb{C}\setminus\{0\}$ and $1 < p < \infty$, $X \in \mathcal{B}(\ell_A^p)$ be a nonzero operator and V be an integration operator on ℓ_A^p.*

(i) If $\lambda \in \overline{\mathbb{D}}$, then $VX = \lambda XV$ if and only if $XC_\lambda = \mathcal{D}_{X\mathbf{1}}$, where $\mathcal{D}_{X\mathbf{1}}$ is the Duhamel operator on ℓ_A^p and $(C_\lambda f) = f(\lambda z)$ is a composition operator on ℓ_A^p.

(ii) If $\lambda \in \mathbb{C}\setminus\overline{\mathbb{D}}$, then $VX = \lambda XV$ if and only if $X = \mathcal{D}_{X\mathbf{1}} C_{1/\lambda}$, that is

$$Xf(z) = \frac{d}{dz}\int_0^z (X\mathbf{1})(z-t)\, f\left(\frac{t}{\lambda}\right) dt, f \in \ell_A^p.$$

Proof: (i) It is easy to see from the formula (1) that

$$V^n f = \frac{z^n}{n!} \circledast f,\ f \in \ell_A^p, \tag{2}$$

for all $n \geq 0$. Let $VX = \lambda XV$. Then we have that $V^n X = \lambda^n XV^n$ for each $n \geq 0$, and therefore $\lambda^n XV^n f = V^n Xf$ for all $f \in \ell_A^p$. In particular, $\lambda^n XV^n \mathbf{1} = V^n X\mathbf{1}$, and according to (2), we get that

$$X\left(\frac{(\lambda z)^n}{n!} \circledast \mathbf{1}\right) = \left(\frac{z^n}{n!} \circledast X\mathbf{1}\right),$$

or $X(\lambda z)^n = z^n \circledast X\mathbf{1} = X\mathbf{1} \circledast z^n, n \geq 0$. Since the polynomials are dense in ℓ_A^p, from this we decude that $(Xf)(\lambda z) = X\mathbf{1} \circledast f = \mathcal{D}_{X\mathbf{1}} f$ for all $f \in \ell_A^p$, and hence $XC_\lambda f = \mathcal{D}_{X\mathbf{1}} f$, thus $XC_\lambda = \mathcal{D}_{X\mathbf{1}}$, as desired.
Moreover if $XC_\lambda = \mathcal{D}_{X\mathbf{1}}$, then for any polynomial p we achieve that

$$\begin{aligned}
VXp(z) &= VXCp\left(\lambda^{-1} z\right) = V\mathcal{D}_{X\mathbf{1}} p\left(\lambda^{-1} z\right) \\
&= \mathcal{D}_{X\mathbf{1}} Vp\left(\lambda^{-1} z\right) = XC_\lambda Vp\left(\lambda^{-1} z\right) = XC_\lambda\left(z \circledast p\left(\lambda^{-1} z\right)\right) \\
&= \lambda XC_{1/\lambda}\left(\lambda^{-1} z \circledast p\left(\lambda^{-1} z\right)\right) = \lambda XC_\lambda (Vp)\left(\lambda^{-1} z\right) \\
&= \lambda XVp(z),
\end{aligned}$$

this brings the proof of (i), on account of density of polynomials in ℓ_A^p.
(ii) Let $\lambda XV = VX$. Then $\frac{1}{\lambda} VX = XV$, and therefore $\frac{1}{\lambda^n} V^n X = XV^n, n \geq 0$. Now the same arguments, which were used in the proof of (i), yield that $Xf(z) = X\mathbf{1} \circledast \mathbf{f}\left(\frac{z}{\lambda}\right)$, $f \in \ell_A^p$, which means that $X = \mathcal{D}_{X\mathbf{1}} C_{1/\lambda}$, i.e.,

$$(Xf)(z) = \frac{d}{dz}\int_0^z (X\mathbf{1})(z-t)\, f\left(\frac{t}{\lambda}\right) dt.$$

It remains only to demonstrate every operator X of the form $X = \mathcal{D}_{X\mathbf{1}} C_{1/\lambda}$ satisfies the equation $\lambda XV = VX$. In fact, for any $f \in \ell_A^p$ we have

$$\begin{aligned}
(XVf)(z) &= \left(\mathcal{D}_{X\mathbf{1}} C_{1/\lambda} Vf\right)(z) = \mathcal{D}_{X\mathbf{1}} (Vf)\left(\frac{z}{\lambda}\right) = X\mathbf{1} \circledast (Vf)\left(\frac{z}{\lambda}\right) \\
&= X\mathbf{1} \circledast \left(\frac{z}{\lambda} \circledast f\left(\frac{z}{\lambda}\right)\right) = \frac{z}{\lambda} \circledast \left(X\mathbf{1} \circledast f\left(\frac{z}{\lambda}\right)\right) = \frac{z}{\lambda} \circledast \mathcal{D}_{X\mathbf{1}} C_{1/\lambda} f(z) \\
&= \frac{1}{\lambda} V\mathcal{D}_{X\mathbf{1}} C_{1/\lambda} f(z) = \frac{1}{\lambda} VXf(z).
\end{aligned}$$

The theorem is proven. □

2.2 On the spectral multiplicity of a direct sum of operators V and A

Recall that the direct sum $A \oplus B$ of operators $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ is defined by $(A \oplus B)(x \oplus y) = Ax \oplus By$, $x \oplus y \in X \oplus Y$. It is well-known that (see, for instance, Nikolskii [6])

$$\max\{\mu(A), \mu(B)\} \leq \mu(A \oplus B) \leq \mu(A) + \mu(B). \tag{3}$$

In the following theorem the spectral multiplicity $\mu(V \oplus A)$ of operator $V \oplus A$ with some appropriate summand $A \in \mathcal{B}(\ell_A^p)$ is calculated.

Theorem 2. *Let V be an integration operator on ℓ_A^p, where $1 < p < \infty$, and let $A \in \mathcal{B}\left(\ell_A^p\right)$ be an operator such that $\left(\sum_{n=0}^{\infty}\left(n!\left\|A^n x\right\|\right)^q\right)^{1/q} < \infty$ for every $x \in X$, where $\frac{1}{p}+\frac{1}{q}=1$. Then $\mu(V \oplus A)=\mu(V)+\mu(A)=1+\mu(A)$.*

Proof: It follows from inequalities in (3) that if $\mu(A)=+\infty$, then the proof is trivial, and therefore we assume that $\mu(A)=n<+\infty$. We prove that $\mu(V \oplus A)=n+1$ (the fact that $\mu(V)=1$ is clear from the density of polynomials in ℓ_A^p, $1<p<+\infty$). Suppose in contrary that $\mu(V \oplus A)<n+1$, that is $\mu(V \oplus A)=n$. Let $\{g_i \oplus x_i\}_{i=1}^n$ be a cyclic set for operator $V \oplus A$. Then it is easy to see that $\{g_i\}_{i=1}^n$ is a cyclic set of V, i.e., $\{g_i\}_{i=1}^n \in \operatorname{Cyc}(V)$ (see, for instance, [6]). Then there exists a number $i_0 \in \{1,2,\ldots,n\}$ such that $g_{i_0}(0) \neq 0$; we suppose without losing generality that $i_0=1$, so $g_1(0)\neq 0$, otherwise we obtained $\operatorname{span}\left\{g_1, Vg_1, V^2 g_2, \ldots\right\} \subset \left\{g \in \ell_A^p(\mathbb{D}) : g(0)=0\right\} \neq \ell_A^p$, which contradicts to the cyclicity property of g_1.

It is known that $\left(\ell_A^p, \circledast\right)$ is a Banach algebra (see [4, 8]) . Since g_1 is invertible in the this algebra, there exists a function $G_1 \in \ell_A^p$ such that $(G_1 \circledast g_1)(z) \equiv 1$, since $(G_1 \circledast g_1)(0)=G_1(0)\,g_1(0)=1$, (see formula (1)), we see that $G_1(0)\neq 0$. Now we consider the following $n\times n$ matrix-function

$$N(z)=\begin{pmatrix} G_1 & 0 & 0 & \cdots & 0\\ -g_2\circledast G_1 & 1 & 0 & \cdots & 0\\ -g_3\circledast G_1 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ -g_n\circledast G_1 & 0 & 0 & \cdots & 1\end{pmatrix}.$$

Let $\mathcal{B}$ be a classical Borel transformation from the space $Hol(\mathbb{D})$ of all analytic functions to the space of formally power series $\mathbb{C}[[\mathbb{Z}]]$ over the field of complex numbers $\mathbb{C}$ defined by

$$\mathcal{B}\left(\sum_{n=0}^{\infty}\widehat{g}(n)\,z^n\right) := \sum_{n=0}^{\infty} n!\,\widehat{g}(n)\,\mathbb{Z}^n.$$

The formula $\mathcal{B}^{-1}\left(\sum_{n=0}^{\infty} a_n\mathbb{Z}^n\right) := \sum_{n=0}^{\infty}\frac{a_n}{n!}z^n$ defines the inverse Borel transform $\mathcal{B}^{-1}$. It is known (and easy to verify) that if $g,h\in Hol(\mathbb{D})$ and V is an integration operator on $Hol(\mathbb{D})$, then

$$g\circledast h=(\mathcal{B}g)(V)\,h=(\mathcal{B}h)(V)\,g, \tag{4}$$

where $(\mathcal{B}g)(V)\,h := \sum_{n=0}^{\infty} n!\,\widehat{g}(n)\,(V^n h)(z)$. Denoting $\overrightarrow{g}=\begin{pmatrix} g_1\\ g_2\\ \vdots\\ g_n\end{pmatrix}$ and considering (4), we get :

$$(\mathcal{B}N)(V)\,\overrightarrow{g} := \begin{pmatrix} (\mathcal{B}G_1)(V) & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0}\\ (\mathcal{B}(-g_2\circledast G_1))(V) & \mathrm{I} & \mathbf{0} & \ldots & \mathbf{0}\\ (\mathcal{B}(-g_3\circledast G_1))(V) & \mathbf{0} & \mathrm{I} & \ldots & \mathbf{0}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ (\mathcal{B}(-g_n\circledast G_1))(V) & \mathbf{0} & \mathbf{0} & \ldots & \mathrm{I}\end{pmatrix}\begin{pmatrix} g_1\\ g_2\\ g_3\\ \vdots\\ g_n\end{pmatrix}$$

$$=\begin{pmatrix} (\mathcal{B}G_1)(V)\,g_1\\ (\mathcal{B}(-g_2\circledast G_1))(V)\,g_1+g_2\\ (\mathcal{B}(-g_3\circledast G_1))(V)\,g_1+g_3\\ \vdots\\ (\mathcal{B}(-g_n\circledast G_1))(V)\,g_1+g_n\end{pmatrix}$$

$$=\begin{pmatrix} G_1\circledast g_1\\ (-g_2\circledast G_1)\circledast g_1+g_2\\ (-g_3\circledast G_1)\circledast g_1+g_3\\ \vdots\\ (-g_n\circledast G_1)\circledast g_1+g_n\end{pmatrix}$$

$$=\begin{pmatrix} 1\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

Since $\circledast$-det $(\mathcal{B}N)(0)=(\mathcal{B}G_1)(0)\neq 0$, it is easy to show that $(\mathcal{B}G)(V)$ and $(\mathcal{B}G)(A)$ are invertible operators on $\left(\ell_A^p\right)^n := \ell_A^p\times\ell_A^p\times\ldots\times\ell_A^p$ and $X^n := X\times X\times\ldots\times X$, respectively. It is also not difficult to see that (see Karaev [3])

$$\left\{\left((\mathcal{B}N)(V)\,\overrightarrow{g}\right)_i \oplus \left((\mathcal{B}N)(A)\,x\right)_i : i=1,2,\ldots,n\right\}$$

is the cyclic set of operator $V+A$. Therefore, we get a now cyclic set $\{\mathbf{1}\oplus y_1, \mathbf{0}\oplus y_2, \ldots, \mathbf{0}\oplus y_n\}$, and hence for any $x\in X$ there exists a family of polynomials $\left\{P_{m,i}\right\}_{i=1}^n$ such that $\lim_{m\longrightarrow\infty} p_{m,1}(V)\,\mathbf{1}=0$ in ℓ_A^p and $\lim_{m\longrightarrow\infty}\sum_{i=1}^n p_{m,i}(A)\,y_i=x$ in X. From this by using (4), we obtain that $\lim_{m \longrightarrow \infty} q_{m,1}(z) = 0$ in ℓ_A^p, where

$$q_{m,1}(z) := \left(\mathcal{B}^{-1} p_{m,1}\right)(z) = \sum_{k \geq 0} \frac{1}{k!} \widehat{p}_{m,1}(k) z^k,$$

where $\mathcal{B}^{-1}$ is the inverse Borel transform. Then, by using the condition of the theorem we have

$$\begin{aligned}
\left\|p_{m,1}(A) y_1\right\|_X &= \left\|\sum_{k \geq 0} \widehat{p}_{m,1}(k) A^k y_1\right\|_X \leq \sum_{k \geq 0} \left|p_{m,1}(k)\right| \left\|A^k y_1\right\|_X \\
&= \sum_{k \geq 0} \frac{1}{k!} \left|\widehat{p}_{m,1}(k)\right| k! \left\|A^k y_1\right\| \\
&\leq \left(\sum_{k \geq 0} \left(\frac{1}{k!} \left|\widehat{p}_{m,1}(k)\right|^p\right)^{1/p}\right) \left(\sum_{k \geq 0} \left(k! \left\|A^k y_1\right\|\right)^q\right)^{1/q} \\
&= C_{y_1}^{1/2} \left(\sum_{k \geq 0} \left|\widehat{q}_{m,1}(k)\right|^p\right)^{1/p} = C_{y_1}^{1/2} \left\|q_{m,1}\right\|_{\ell_A^p} \longrightarrow 0, m \longrightarrow \infty.
\end{aligned}$$

Thus,

$$\lim_{m \longrightarrow \infty} \sum_{i=2}^{n} p_{m,i}(A) y_i = x.$$

Since the vector x is arbitrary, the latter assertion means that $\{y_i\}_{i=2}^n$ is a cyclic set for A and card $\left\{y_2, y_{3,\ldots,} y_n\right\} = n-1$, hence $\mu(A) \leq n-1$. But, this contradicts to $\mu(A) = n$. The theorem is proven. □

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Some further properties of predictors in correctly-reduced linear models

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Abstract: In this study, we consider the best linear unbiased predictors (BLUPs) in the context of partitioned linear models and their correctly-reduced models. Some properties of the BLUPs and their analytical expressions are given under these models. We derive some results about the comparison of covariance matrices of BLUPs with other types of predictors by using formulas in matrix algebra, especially ranks of block matrices and elementary matrix operations. Also, results for special cases are given.

Keywords: BLUP, correctly-reduced models, covariance matrix, rank.

1 Introduction and preliminary results

This paper is concerned with the comparison of covariance matrices of predictors in linear regression models. We consider a partitioned linear model and its correctly-reduced models. Before proceeding, we introduce the notations used in this paper. Let $\mathbb{R}^{m\times n}$ stand for the set of all $m \times n$ real matrices. $\mathbf{A}'$, $\boldsymbol{r}(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$, and $\mathbf{A}^+$ denote the transpose, the rank, the column space, and the Moore–Penrose generalized inverse of $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ denotes the identity matrix of order m. $\mathbf{E}_{\mathbf{A}} = \mathbf{A}^{\perp} = \mathbf{I}_m - \mathbf{A}\mathbf{A}^+$ stands for the orthogonal projectors. The inequality $\mathbf{A} \succcurlyeq \mathbf{0}$ means that symmetric matrix $\mathbf{A}$ is a positive semi-definite matrix in the Löwner partial ordering.

Consider a linear model with partitioned form

$$\mathcal{M} : \mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \boldsymbol{\varepsilon} = \begin{bmatrix}\mathbf{X}_1, & \mathbf{X}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}_1', & \boldsymbol{\alpha}_2'\end{bmatrix}' + \boldsymbol{\varepsilon} = \mathbf{X}_1\boldsymbol{\alpha}_1 + \mathbf{X}_2\boldsymbol{\alpha}_2 + \boldsymbol{\varepsilon} \tag{1}$$

$$\text{with} \quad \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0} \quad \text{and} \quad \mathrm{cov}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) = \mathrm{D}(\boldsymbol{\varepsilon}) = \sigma^2\boldsymbol{\Sigma},$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1, & \mathbf{X}_2\end{bmatrix} \in \mathbb{R}^{n\times k}$ is a known matrix of arbitrary rank with $\mathbf{X}_i \in \mathbb{R}^{n\times k_i}$, $\boldsymbol{\alpha} = \begin{bmatrix}\boldsymbol{\alpha}_1', & \boldsymbol{\alpha}_2'\end{bmatrix}' \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters with $\boldsymbol{\alpha}_i \in \mathbb{R}^{k_i\times 1}$, $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times 1}$ is an unobservable vector of random errors, σ^2 is a positive unknown parameter, and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a known positive semi-definite matrix of arbitrary rank, $i = 1, 2$, $k_1 + k_2 = k$.

Reduced linear models are obtained by using linear transformations on linear models. They are one of the different forms of partitioned linear models to meet the analysis' requirements. Especially, they can be considered when estimation/prediction problems on general parametric functions of partial parameters are considered. To examine the relations between the linear model $\mathcal{M}$ in (1) and its reduced models, we can consider the following reduced model of $\mathcal{M}$:

$$\mathcal{M}_1 : \mathbf{X}_2^{\perp}\mathbf{y} = \mathbf{X}_2^{\perp}\mathbf{X}_1\boldsymbol{\alpha}_1 + \mathbf{X}_2^{\perp}\boldsymbol{\varepsilon}. \tag{2}$$

The reduced model $\mathcal{M}_1$ in (2) is obtained by pre-multiplying $\mathbf{X}_2^{\perp}$ on the both sides of partitioned model in (1) and this model is also known as a correctly-reduced linear model of $\mathcal{M}$. In this study, we assume that $\mathcal{M}$ is consistent, i.e., $\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{X}, & \boldsymbol{\Sigma}\end{bmatrix}$ holds with probability 1, see, [13]. We note that $\mathcal{M}_1$ is consistent, i.e., $\mathbf{X}_2^{\perp}\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{X}_2^{\perp}\mathbf{X}_1, & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}\end{bmatrix}$ holds with probability 1, under the assumption of consistency of $\mathcal{M}$.

To establish the results on predictors of all unknown vectors with partial parameters , we can consider the following vector

$$\boldsymbol{\phi}_1 = \mathbf{K}_1\boldsymbol{\alpha}_1 + \mathbf{H}\boldsymbol{\varepsilon} \tag{3}$$

for given matrices $\mathbf{K}_1 \in \mathbb{R}^{s\times k_1}$ and $\mathbf{H} \in \mathbb{R}^{s\times n}$. Then, according to the assumptions on expectation vector and covariance matrix in (1), we obtain

$$\mathrm{cov}(\boldsymbol{\phi}_1, \mathbf{y}) = \sigma^2\mathbf{H}\boldsymbol{\Sigma}, \;\; \mathrm{cov}(\boldsymbol{\phi}_1, \mathbf{X}_2^{\perp}\mathbf{y}) = \sigma^2\mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}, \;\; \mathrm{D}(\boldsymbol{\phi}_1) = \sigma^2\mathbf{H}\boldsymbol{\Sigma}\mathbf{H}'.$$

In this study, we consider the best linear unbiased predictors (BLUPs) as predictors. For the comparison process, we use covariance matrices of BLUPs. BLUPs are defined according to the unbiasedness criteria of predictors and the minimum covariance matrix requirement in the Löwner partial ordering. Under our considerations, we review the predictability/estimability requirement of $\boldsymbol{\phi}_1$ in (3) and its special cases under $\mathcal{M}_1$ before giving the definition of the BLUP.

(a) $\boldsymbol{\phi}_1$ is predictable by $\mathbf{X}_2^{\perp}\mathbf{y}$ in $\mathcal{M}_1 \Longleftrightarrow \mathscr{C}(\mathbf{K}_1') \subseteq \mathscr{C}[(\mathbf{X}_2^{\perp}\mathbf{X}_1)'] \Longleftrightarrow \mathbf{K}_1\boldsymbol{\alpha}_1$ is estimable by $\mathbf{X}_2^{\perp}\mathbf{y}$ in $\mathcal{M}_1$,
(b) $\mathbf{X}_1\boldsymbol{\alpha}_1$ is estimable by $\mathbf{X}_2^{\perp}\mathbf{y}$ in $\mathcal{M}_1 \Longleftrightarrow \mathscr{C}(\mathbf{X}_1') \subseteq \mathscr{C}[(\mathbf{X}_2^{\perp}\mathbf{X}_1)']$,
(c) $\mathbf{X}_2^{\perp}\mathbf{X}_1\boldsymbol{\alpha}_1$ is always estimable under $\mathcal{M}_1$ and $\boldsymbol{\varepsilon}$ is always predictable under $\mathcal{M}_1$,

see, e.g., [1]. Further, if $\boldsymbol{\phi}_1$ is predictable under $\mathcal{M}_1$ then it is predictable under $\mathcal{M}$. Let $\boldsymbol{\phi}_1$ be predictable under $\mathcal{M}_1$. If there exists $\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y}$ such that

$$\mathrm{D}(\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y} - \boldsymbol{\phi}_1) = \text{ min subject to } \mathrm{E}(\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y} - \boldsymbol{\phi}_1) = \mathbf{0}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y}$ is defined to be the BLUP of $\boldsymbol{\phi}_1$ under $\mathcal{M}_1$ and is denoted by $\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y} = \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1) = \mathrm{BLUP}_{\mathcal{M}_1}(\mathbf{K}_1\boldsymbol{\alpha}_1 + \mathbf{H}\boldsymbol{\varepsilon})$. If $\mathbf{H} = \mathbf{0}$ in $\boldsymbol{\phi}_1$, $\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y}$ corresponds the best linear unbiased estimator (BLUE) of $\mathbf{K}_1\boldsymbol{\alpha}_1$, denoted by $\mathrm{BLUE}_{\mathcal{M}_1}(\mathbf{K}_1\boldsymbol{\alpha}_1)$, under $\mathcal{M}_1$; see, e.g., [3] and [12].

The results, in the present paper, are established by making use of formulas of ranks of block matrices and elementary matrix operations which are effective algebraic tools in matrix theory. Three types of elementary row and column operations for block matrices are reviewed as follows; see, e.g., [15].

(a) Interchange two block rows or two block columns in a partitioned matrix.
(b) Multiply a block row by a nonsingular matrix from the left or a block column by a nonsingular matrix from the right in a partitioned matrix.
(c) Multiply a block row by a matrix from the left and add it to another block row or multiply a block column by a matrix from the right and add it to another block column in a partitioned matrix.

The related subject can also be found in [2], [4]-[9], [17] and [18]. A group of well-known formulas for ranks of block matrices are collected in the following lemma; see [10] and [14].

Lemma 1.1. *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, $\mathbf{D} \in \mathbb{R}^{l\times k}$, $\mathbf{F} \in \mathbb{R}^{m\times t}$, and let $\mathbf{X} \in \mathbb{R}^{m\times m}$ be a symmetric matrix. Then,*

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A}, & \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{E_A}\mathbf{B}) = \boldsymbol{r}(\mathbf{B}) + \boldsymbol{r}(\mathbf{E_B}\mathbf{A}), \tag{4}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{C}\mathbf{E}_{\mathbf{A}'}) = \boldsymbol{r}(\mathbf{C}) + \boldsymbol{r}(\mathbf{A}\mathbf{E}_{\mathbf{C}'}), \tag{5}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{bmatrix} = \boldsymbol{r}(\mathbf{B}) + \boldsymbol{r}(\mathbf{C}) + \boldsymbol{r}(\mathbf{E_B}\mathbf{A}\mathbf{E}_{\mathbf{C}'}), \tag{6}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{D}), \tag{7}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{D} - \mathbf{C}\mathbf{A}^{+}\mathbf{B}) \;\; \textit{if} \;\; \mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{A}) \;\; \textit{and} \;\; \mathscr{C}(\mathbf{C}') \subseteq \mathscr{C}(\mathbf{A}'), \tag{8}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B}\\ \mathbf{B}' & \mathbf{0}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\end{bmatrix} + \boldsymbol{r}(\mathbf{B}), \;\; \textit{and} \;\; \boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B} & \mathbf{F}\\ \mathbf{B}' & \mathbf{0} & \mathbf{0}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B} & \mathbf{F}\end{bmatrix} + \boldsymbol{r}(\mathbf{B}) \;\; \textit{if} \;\; \mathbf{X} \succcurlyeq \mathbf{0}. \tag{9}$$

The following well-known result was given by [11].

Lemma 1.2. *The linear matrix equation $\mathbf{AX} = \mathbf{B}$ is consistent if and only if $\boldsymbol{r}\begin{bmatrix}\mathbf{A}, & \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A})$, or equivalently, $\mathbf{A}\mathbf{A}^{+}\mathbf{B} = \mathbf{B}$. In this case, the general solution of $\mathbf{AX} = \mathbf{B}$ can be written in the following form $\mathbf{X} = \mathbf{A}^{+}\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{U}$, where $\mathbf{U}$ is an arbitrary matrix.*

2 Main results

In this section, some results on the comparison of covariance matrices of BLUPs with other types of predictors under the correctly-reduced linear model $\mathcal{M}_1$ are derived by using block matrices' rank formulas. Firstly, we review the fundamental equations of BLUPs of all unknown vectors and then we give some properties of BLUPs.

The fundamental BLUP equation and some of the properties of the BLUPs under $\mathcal{M}_1$ are given as follows; see, e.g., [16]. Let $\boldsymbol{\phi}_1$ be predictable under $\mathcal{M}_1$. Then

$$\begin{aligned} &\mathrm{E}(\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y} - \boldsymbol{\phi}_1) = \mathbf{0} \text{ and } \mathrm{D}(\mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y} - \boldsymbol{\phi}_1) = \min \\ &\Longleftrightarrow \mathbf{L}_1\left[\mathbf{X}_2^{\perp}\mathbf{X}_1, \quad \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right] = \left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]. \end{aligned} \tag{10}$$

The matrix equation in (10) is consistent and the $\mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)$ is given by

$$\mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1) = \mathbf{L}_1\mathbf{X}_2^{\perp}\mathbf{y} = \left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp} + \mathbf{U}_1\mathbf{W}_1^{\perp}\mathbf{X}_2^{\perp}\right)\mathbf{y}, \tag{11}$$

by using the general solution $\mathbf{L}_1$ of (10), where $\mathbf{U}_1 \in \mathbb{R}^{s\times n}$ is an arbitrary matrix and $\mathbf{W}_1 = \left[\mathbf{X}_2^{\perp}\mathbf{X}_1, \quad \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]$. In particular,

$$\mathbf{L}_1 \text{ is unique } \Longleftrightarrow \boldsymbol{r}(\mathbf{W}_1) = n,$$

$$\mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1) \text{ is unique with probability 1 } \Longleftrightarrow \mathcal{M}_1 \text{ is consistent.}$$

Further, the rank of $\mathbf{W}_1$ satisfies the equalities

$$\boldsymbol{r}(\mathbf{W}_1) = \boldsymbol{r}\left[\mathbf{X}_2^{\perp}\mathbf{X}_1, \quad \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}\right] = \boldsymbol{r}\left[\mathbf{X}_2^{\perp}\mathbf{X}_1, \quad (\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}\right].$$

The covariance matrices of $\mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)$ and $\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)$ are unique and satisfy the equalities

$$\mathrm{D}[\mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)] = \sigma^2\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}\left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\right)',$$

$$\begin{aligned} \mathrm{D}[\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)] = \sigma^2 &\left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp} - \mathbf{H}\right)\boldsymbol{\Sigma} \\ &\times\left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp} - \mathbf{H}\right)'. \end{aligned}$$

Theorem 2.1. *Let consider the correctly-reduced model $\mathcal{M}_1$ and assume that $\boldsymbol{\phi}_1$ is predictable under $\mathcal{M}_1$. Let $\mathbf{G} = \mathbf{G}' \in \mathbb{R}^{s\times s}$ and $\mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)$ be as given in* (11)*. Denote*

$$\mathbf{M} = \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Sigma}\mathbf{H}' & \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{H}\boldsymbol{\Sigma} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' - \mathbf{G} & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{X}_1' & \mathbf{K}_1' & \mathbf{0} & \mathbf{0} \\ \mathbf{X}_2' & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then

$$\boldsymbol{r}(\mathbf{G} - \mathrm{D}[\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)]) = \boldsymbol{r}(\mathbf{M}) - \boldsymbol{r}(\mathbf{X}_2) - \boldsymbol{r}(\mathbf{X}).$$

In consequence, $\mathbf{G} = \mathrm{D}[\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}_1)] \Longleftrightarrow \boldsymbol{r}(\mathbf{M}) = \boldsymbol{r}(\mathbf{X}_2) + \boldsymbol{r}(\mathbf{X})$.

Proof: The rank of the difference between $\mathbf{G}$ and $\mathrm{D}[\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)]$ is

$$\begin{aligned} \boldsymbol{r}(\mathbf{G} - \mathrm{D}[\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)]) = \boldsymbol{r}&\left(\mathbf{G} - \left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp} - \mathbf{H}\right)\right. \\ &\left.\times\boldsymbol{\Sigma}\left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp} - \mathbf{H}\right)'\right). \end{aligned} \tag{12}$$

We can apply (8) to (12) by using the equality $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{+}\boldsymbol{\Sigma}$. Then

$$\begin{aligned} &\boldsymbol{r}(\mathbf{G} - \mathrm{D}[\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)]) \\ &= \boldsymbol{r}\begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Sigma}\left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp} - \mathbf{H}\right)' \\ \left(\left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]\mathbf{W}_1^{+}\mathbf{X}_2^{\perp} - \mathbf{H}\right)\boldsymbol{\Sigma} & \mathbf{G} \end{bmatrix} \\ &\quad - \boldsymbol{r}(\boldsymbol{\Sigma}) \\ &= \boldsymbol{r}\left(\begin{bmatrix} \boldsymbol{\Sigma} & -\boldsymbol{\Sigma}\mathbf{H}' \\ -\mathbf{H}\boldsymbol{\Sigma} & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right] \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{W}_1 \\ \mathbf{W}_1' & \mathbf{0} \end{bmatrix}^{+}\right. \\ &\quad\left.\times\begin{bmatrix} \mathbf{X}_2^{\perp}\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K}_1, \quad \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\right]' \end{bmatrix}\right) - \boldsymbol{r}(\boldsymbol{\Sigma}) \end{aligned} \tag{13}$$

is obtained. We can reapply (8) to (13) since $\mathscr{C}(\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}) = \mathscr{C}(\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}) \subseteq \mathscr{C}(\mathbf{W}_1)$ and $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}_1, & \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\end{bmatrix}'\right) \subseteq \mathscr{C}(\mathbf{W}_1')$, where $\mathbf{W}_1 = \begin{bmatrix}\mathbf{X}_2^{\perp}\mathbf{X}_1, & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\end{bmatrix}$. Then, by using elementary matrix operations and (4)-(9),

$$
\begin{aligned}
&\boldsymbol{r}(\mathbf{G} - \mathrm{D}[\boldsymbol{\phi}_1 - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\phi}_1)])\\
&= \boldsymbol{r}\begin{bmatrix}
\mathbf{0} & -\mathbf{X}_2^{\perp}\mathbf{X}_1 & -\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp} & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma} & \mathbf{0}\\
-\mathbf{X}_1'\mathbf{X}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1'\\
-(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{H}'\\
\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma} & -\boldsymbol{\Sigma}\mathbf{H}'\\
\mathbf{0} & \mathbf{K}_1 & \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp} & -\mathbf{H}\boldsymbol{\Sigma} & \mathbf{G}
\end{bmatrix}\\
&\quad - \boldsymbol{r}\begin{bmatrix}\mathbf{X}_2^{\perp}\mathbf{X}_1, & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\end{bmatrix} - \boldsymbol{r}(\boldsymbol{\Sigma})\\
&= \boldsymbol{r}\begin{bmatrix}
-\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & -\mathbf{X}_2^{\perp}\mathbf{X}_1 & -\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp} & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{H}'\\
-\mathbf{X}_1'\mathbf{X}_2^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{K}_1'\\
-(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{0} & \mathbf{0} & (\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{H}'\\
\mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{K}_1 & \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp} & \mathbf{G} - \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}'
\end{bmatrix}\\
&\quad - \boldsymbol{r}\begin{bmatrix}\mathbf{X}_2^{\perp}\mathbf{X}_1, & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}\end{bmatrix}\\
&= \boldsymbol{r}\begin{bmatrix}
-\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & -\mathbf{X}_2^{\perp}\mathbf{X}_1 & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{H}'\\
-\mathbf{X}_1'\mathbf{X}_2^{\perp} & \mathbf{0} & \mathbf{K}_1'\\
\mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{K}_1 & \mathbf{G} - \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}'
\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\mathbf{X}_2^{\perp}\mathbf{X}_1, & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}\end{bmatrix}\\
&\quad + \boldsymbol{r}((\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp}\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1)^{\perp})\\
&= \boldsymbol{r}\begin{bmatrix}
\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{H}' & \mathbf{X}_2^{\perp}\mathbf{X}_1\\
\mathbf{H}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' - \mathbf{G} & \mathbf{K}_1\\
\mathbf{X}_1'\mathbf{X}_2^{\perp} & \mathbf{K}_1' & \mathbf{0}
\end{bmatrix} + \boldsymbol{r}\begin{bmatrix}
\mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp} & \mathbf{X}_2^{\perp}\mathbf{X}_1\\
\mathbf{X}_1'\mathbf{X}_2^{\perp} & \mathbf{0}
\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\mathbf{X}_2^{\perp}\mathbf{X}_1, & \mathbf{X}_2^{\perp}\boldsymbol{\Sigma}\mathbf{X}_2^{\perp}\end{bmatrix}\\
&\quad - 2\boldsymbol{r}(\mathbf{X}_2^{\perp}\mathbf{X}_1)\\
&= \boldsymbol{r}\begin{bmatrix}
\boldsymbol{\Sigma} & \boldsymbol{\Sigma}\mathbf{H}' & \mathbf{X}_1 & \mathbf{X}_2\\
\mathbf{H}\boldsymbol{\Sigma} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' - \mathbf{G} & \mathbf{K}_1 & \mathbf{0}\\
\mathbf{X}_1' & \mathbf{K}_1' & \mathbf{0} & \mathbf{0}\\
\mathbf{X}_2' & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix} + \boldsymbol{r}\begin{bmatrix}
\boldsymbol{\Sigma} & \mathbf{X}_1 & \mathbf{X}_2\\
\mathbf{X}_1' & \mathbf{0} & \mathbf{0}\\
\mathbf{X}_2' & \mathbf{0} & \mathbf{0}
\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}
\mathbf{X}_1 & \boldsymbol{\Sigma} & \mathbf{X}_2\\
\mathbf{0} & \mathbf{X}_2' & \mathbf{0}
\end{bmatrix} - 2\boldsymbol{r}(\mathbf{X}_2)\\
&\quad - 2\boldsymbol{r}(\mathbf{X}_2^{\perp}\mathbf{X}_1)\\
&= \boldsymbol{r}\begin{bmatrix}
\boldsymbol{\Sigma} & \boldsymbol{\Sigma}\mathbf{H}' & \mathbf{X}_1 & \mathbf{X}_2\\
\mathbf{H}\boldsymbol{\Sigma} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' - \mathbf{G} & \mathbf{K}_1 & \mathbf{0}\\
\mathbf{X}_1' & \mathbf{K}_1' & \mathbf{0} & \mathbf{0}\\
\mathbf{X}_2' & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix} + \boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma}, & \mathbf{X}\end{bmatrix} + \boldsymbol{r}(\mathbf{X}) - \boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma}, & \mathbf{X}\end{bmatrix} - 3\boldsymbol{r}(\mathbf{X}_2) - 2\boldsymbol{r}(\mathbf{X}_2^{\perp}\mathbf{X}_1)\\
&= \boldsymbol{r}\begin{bmatrix}
\boldsymbol{\Sigma} & \boldsymbol{\Sigma}\mathbf{H}' & \mathbf{X}_1 & \mathbf{X}_2\\
\mathbf{H}\boldsymbol{\Sigma} & \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' - \mathbf{G} & \mathbf{K}_1 & \mathbf{0}\\
\mathbf{X}_1' & \mathbf{K}_1' & \mathbf{0} & \mathbf{0}\\
\mathbf{X}_2' & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix} - \boldsymbol{r}(\mathbf{X}_2) - \boldsymbol{r}(\mathbf{X}). \qquad (14)
\end{aligned}
$$

(14) establishes the required results. □

Many consequences can be derived from Theorem 2.1 for different choices of the matrices $\mathbf{K}_1$ and $\mathbf{H}$. Some of these are given in the following.

Corollary 2.1. *Let consider the correctly-reduced model $\mathcal{M}_1$ and assume that $\mathbf{X}_1\boldsymbol{\alpha}_1$ is estimable under $\mathcal{M}_1$. Let $\mathbf{G} = \mathbf{G}' \in \mathbb{R}^{s\times s}$. Denote*

$$
\mathbf{M}_1 = \begin{bmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_1 & \mathbf{X}_2\\
\mathbf{0} & -\mathbf{G} & \mathbf{X}_1 & \mathbf{0}\\
\mathbf{X}_1' & \mathbf{X}_1' & \mathbf{0} & \mathbf{0}\\
\mathbf{X}_2' & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix} \quad \textit{and} \quad \mathbf{M}_2 = \begin{bmatrix}
\boldsymbol{\Sigma} & \boldsymbol{\Sigma} & \mathbf{X}_1 & \mathbf{X}_2\\
\boldsymbol{\Sigma} & \boldsymbol{\Sigma} - \mathbf{G} & \mathbf{0} & \mathbf{0}\\
\mathbf{X}_1' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{X}_2' & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}.
$$

Then

$$\boldsymbol{r}(\mathbf{G} - \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_1}(\mathbf{X}_1\boldsymbol{\alpha}_1)]) = \boldsymbol{r}(\mathbf{M}_1) - \boldsymbol{r}(\mathbf{X}_2) - \boldsymbol{r}(\mathbf{X}).$$

$$\boldsymbol{r}(\mathbf{G} - \mathrm{D}[\boldsymbol{\varepsilon} - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\varepsilon})]) = \boldsymbol{r}(\mathbf{M}_2) - \boldsymbol{r}(\mathbf{X}_2) - \boldsymbol{r}(\mathbf{X}).$$

In consequence,

$$\mathbf{G} = \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_1}(\mathbf{X}_1\boldsymbol{\alpha}_1)] \Longleftrightarrow \boldsymbol{r}(\mathbf{M}_1) = \boldsymbol{r}(\mathbf{X}_2) + \boldsymbol{r}(\mathbf{X}).$$

$$\mathbf{G} = \mathrm{D}[\boldsymbol{\varepsilon} - \mathrm{BLUP}_{\mathcal{M}_1}(\boldsymbol{\varepsilon})] \Longleftrightarrow \boldsymbol{r}(\mathbf{M}_2) = \boldsymbol{r}(\mathbf{X}_2) + \boldsymbol{r}(\mathbf{X}).$$

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Generalized Berezin number inequalities

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Abstract: Let A be a positive bounded linear operator acting on a complex Hilbert space $\mathcal{H}$. Let $\operatorname{ber}_A(X)$ denote the A-Berezin number of an operator X. In this paper, we give new inequalities of A-Berezin number of operators on the reproducing kernel Hilbert space. Some more related results are also obtained. In particular, we show that

$$\operatorname{ber}_A^n(X) \leq \frac{1}{2^{n-1}} \operatorname{ber}_A(X^n) + \|X\|_{A-\mathrm{Ber}} \sum_{s=1}^{n-1} \frac{1}{2^s} \operatorname{ber}_A^{n-s-1}(X) \|X^s\|_{A-\mathrm{Ber}},$$

for all $n = 2, 3, ...$

Keywords: A-Berezin symbol, reproducing kernel Hilbert space, positive operator, semi-inner product.

1 Introduction

The purpose of this section is to gather all of the technical components needed to read the text. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a Hilbert functional space (H.f.s.) on some set Ω with the reproducing kernel $k_\xi \in \mathcal{H}$, i.e., $f(\xi) = \langle f, k_\xi \rangle$ for all $f \in \mathcal{H}$ and all $\xi \in \Omega$. It is supposed that for every $\xi \in \Omega$ there exists a function $f_\xi \in \mathcal{H}$ such that $f_\xi(\xi) \neq 0$, or equivalently there is no $\xi_0 \in \Omega$ such that $f(\xi_0) = 0$ for all $f \in \mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$ with an inner product $\langle .,. \rangle$ and the corresponding norm $\|.\|$. For an operator $X \in \mathcal{B}(\mathcal{H})$, its Berezin symbol (B.s.) $\widetilde{X}$ is defined by

$$\widetilde{X}(\xi) = \left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle, \ \xi \in \Omega,$$

where $\widehat{k}_\xi := \frac{k_\xi}{\|k_\xi\|_{\mathcal{H}}}$ is the normalized reproducing kernel of $\mathcal{H}$. For more facts about H.f.s. and B.s., see, [4, 5, 21].

$R(X)$, $N(X)$, $\overline{R(X)}$ and X^* stand for the range, null space, closure of the range and adjoint of X, respectively, for each operator $X \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for any $x \in H$ and we write a positive operator as $A \geq 0$. It is clear that a positive operator A induces a positive semi-definite sesquilinear form $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$ for all $x, y \in \mathcal{H}$. The semi-norm induced by $\langle \cdot, \cdot \rangle_A$ is given by $\|x\|_A = \langle Ax, x \rangle^{1/2} = \left\| A^{1/2}x \right\|$. Then $\|.\|_A$ is a norm on $\mathcal{H}$ iff A is injective operator and the semi-normed space $(\mathcal{B}(\mathcal{H}), \|.\|_A)$ is complete iff $R(A)$ is closed. An operator $Y \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of an operator X if $\langle Xx, y \rangle_A = \langle x, Yy \rangle_A$ holds for all $x, y \in \mathcal{H}$. The set of all operators in $\mathcal{B}(\mathcal{H})$ admitting A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas theorem [8], we get

$$\mathcal{B}_A(\mathcal{H}) = \left\{ X \in \mathcal{B}(\mathcal{H}) : R(X^*A) \subset R(A) \right\}.$$

If $X \in \mathcal{B}_A(\mathcal{H})$ then X admits an A-adjoint operators. Moreover, there exists a distinguished A-adjoint operator of X, namely the reduced solution of the equation $AX = X^*A$, i.e., $X^\# = A^\dagger X^* A$, where $A^\dagger$ is the Moore-Penrose inverse of A. Also, by applying Douglas theorem, we can see that

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \left\{ X \in \mathcal{B}(\mathcal{H}) : \exists c > 0;\ \|Xx\|_A \leq c\|x\|_A,\ \forall x \in \mathcal{H} \right\}.$$

It is well-known that the semi-inner product $\langle .,. \rangle$ induces an inner product on the quotient space $\mathcal{H}/N(A)$ which is not complete unless $R(A)$ is closed. However, a canonical construction due to de Branges and Rovnyak [7] shows that the completion of $\mathcal{H}/N(A)$ is isometrically isomorphic to the Hilbert space $R\left(A^{1/2}\right)$ with the inner product

$$\left\langle A^{1/2}x, A^{1/2}y \right\rangle_{R(A^{1/2})} := \left\langle P_{\overline{R(A^{1/2})}}x,\ P_{\overline{R(A^{1/2})}}y \right\rangle \tag{1}$$

for all $x, y \in \mathcal{H}$. For the sequel, the Hilbert space $\left(R\left(A^{1/2}\right), \langle .,. \rangle_{R(A^{1/2})}\right)$ will be denoted by $R\left(A^{1/2}\right)$. Moreover, we have $\langle Ax, Ay \rangle_{R(A^{1/2})} = \langle x, y \rangle_A$ for all $x, y \in \mathcal{H}$, whence $\|Ax\|_{R(A^{1/2})} = \|x\|_A$ for all $x \in \mathcal{H}$. The definitions and properties needed in this paper are shown in [2, 6, 13, 22, 26, 27].

In 2022, Huban [19] and Gürdal and Başaran [17] obtained the notions of A-Berezin number, A-Berezin set and A-Berezin norm of operators, and discussed the further generalizations and refinements of A-Berezin number.

The A-Berezin set, A-Berezin number and A-Berezin norm of operators $X \in \mathcal{B}_A(\mathcal{H})$ are defined, respectively, by

$$\mathrm{Ber}_A(X) := \left\{ \left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A : \xi \in \Omega \right\}, \, \mathrm{ber}_A(X) := \sup_{\xi \in \Omega} \left| \left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A \right|,$$

and

$$\|X\|_{A-\mathrm{Ber}} := \sup_{\xi \in \Omega} \left\| AX\widehat{k}_{\mathcal{H},\xi} \right\|_{\mathcal{H}}.$$

We get the Berezin number if $A = I$. As a result, the Berezin number of Hilbert functional space operators and the Berezin norm of operators are both generalized by this new idea.

In this work, we give several inequalities involving A-Berezin radius. In particular, we show and generalize recent some numerical radius and Berezin number inequalities of bounded linear operators due to Dragomir [9–11], Guesba [16], Kittaneh et al. [23] and Huban et al. [20].

2 A-Berezin number inequalities

2.1 Prerequisites

We need several requirements before we can express and prove our findings. To begin, consider the following facts: [3] demonstrated that for each $X \in \mathcal{B}(\mathcal{H})$, $X \in \mathcal{B}_{A1/2}(\mathcal{H})$ occurs iff there exists a unique operator $\widetilde{X} \in \mathcal{B}\left(R\left(A^{1/2}\right)\right)$ such that $V_A X = \widetilde{X} V_A$ with $V_A : H \to R\left(A^{1/2}\right)$ is defined by $V_A x = Ax$. Moreover, we have $\|X\|_A = \|X\|_{\mathcal{B}\left(R\left(A^{1/2}\right)\right)}$ for each $X \in \mathcal{B}_{A1/2}(\mathcal{H})$

The following four lemmas are stated in order to prove our primary conclusions in this study. In the case of operators in Hilbert spaces, the first lemma is known as the Hölder-McCarthy inequality (see [25]).

Lemma 1. *If $X \in \mathcal{B}(\mathcal{H})$, $X \geq 0$, and $x \in \mathcal{H}$ is an any unit vector, then*

$$\left\langle X^r x, x \right\rangle \leq \langle Xx, x \rangle^r, \text{ for all } r \in [0, 1], \tag{2}$$

$$\langle Xx, x \rangle^r \leq \left\langle X^r x, x \right\rangle, \text{ for all } r \geq 1. \tag{3}$$

Lemma 2. *([1]) If a_i, $i = \overline{1,k}$, is a positive real number, then we have*

$$\left(\sum_{i=1}^{k} a_i \right)^n \leq k^{n-1} \sum_{i=1}^{k} a_i^n, \tag{4}$$

for every $n = 1, 2, ...$

Lemma 3. *([12]) If $X_j \in \mathcal{B}_A(\mathcal{H})$, $i = \overline{1,k}$, then we have*

$$\widetilde{\sum_{j=1}^{k} X_j} = \sum_{j=1}^{k} \widetilde{X_j} \tag{5}$$

and

$$X \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \Rightarrow \widetilde{X^n} = \left(\widetilde{X}\right)^n. \tag{6}$$

Lemma 4. *([24]) If $X \in \mathcal{B}_A(\mathcal{H})$, then we have $\widetilde{X^\#} = \left(\widetilde{X}\right)^*$.*

The following lemmas are important in our subsequent proofs, which may be found in [10, 26].

Lemma 5. *If x, y, e are vectors in $\mathcal{H}$ and $\|e\| = 1$, then*

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} \left(\|x\| \, \|y\| + |\langle x, y \rangle| \right). \tag{7}$$

$$\left| \langle x, e \rangle_A \langle e, y \rangle_A \right| \leq \frac{1}{2} \left(\|x\|_A \, \|y\|_A + \left| \langle x, y \rangle_A \right| \right). \tag{8}$$

Lemma 6. *If $x, y \geq 0$ and $\alpha \in (0, 1)$, then we have*

$$\alpha x + (1 - \alpha) y \leq \left(\alpha x^p + (1 - \alpha) y^p \right)^{1/p} \tag{9}$$

for all $p \geq 1$.

2.2 Main Results

We can first give the following result required for the next theorem.

Remark 1. *The inequality in [20, Theorem 3.11] states that*

$$\operatorname{ber}^n\left(Y^*X\right) \leq \frac{1}{2}\left\|\left(Y^*Y\right)^n + \left(X^*X\right)^n\right\|_{\text{ber}} \tag{10}$$

for $n \geq 1$.

We now provide one of the section's primary outcomes.

Theorem 1. *If* $Y, X \in \mathcal{B}_A(\mathcal{H})$*, then*

$$\operatorname{ber}_A^n\left(Y^{\#}X\right) \leq \frac{1}{2}\left\|\left(Y^{\#}Y\right)^n + \left(X^{\#}X\right)^n\right\|_A$$

for all $n = 1, 2, \ldots$

Proof: $Y, X \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ because $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$. As a result, there exists unique $\widetilde{Y}$ and $\widetilde{X}$ in $\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)$ such that $V_AY = \widetilde{Y}V_A$, $V_AX = \widetilde{X}V_A$. Since $R\left(A^{\frac{1}{2}}\right)$ is a complex Hilbert space and the inequality (10),

$$\operatorname{ber}^n\left(\widetilde{Y^*X}\right) \leq \frac{1}{2}\left\|\left(\widetilde{Y^*Y}\right)^n + \left(\widetilde{X^*X}\right)^n\right\|_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)}$$

is obtained. From (5) and (6), we get

$$\operatorname{ber}^n\left(\widetilde{Y^*X}\right) \leq \frac{1}{2}\left\|\left(\left(\widetilde{Y}\right)^*\widetilde{Y}\right)^n + \left(\left(\widetilde{X}\right)^*\widetilde{X}\right)^n\right\|_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)}.$$

Applying Lemma 4 to the operators $\left(\widetilde{Y}\right)^*$ and $\left(\widetilde{X}\right)^*$, we have

$$\begin{aligned}\operatorname{ber}^n\left(\widetilde{Y^{\#}X}\right) &\leq \frac{1}{2}\left\|\left(\widetilde{Y^{\#}Y}\right)^n + \left(\widetilde{X^{\#}X}\right)^n\right\|_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)} \\ &= \frac{1}{2}\left\|\widetilde{\left(Y^{\#}Y\right)^n + \left(X^{\#}X\right)^n}\right\|_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)},\end{aligned}$$

and so,

$$\operatorname{ber}_A^n\left(Y^{\#}X\right) \leq \frac{1}{2}\left\|\left(Y^{\#}Y\right)^n + \left(X^{\#}X\right)^n\right\|_{A-\text{Ber}}.$$

The evidence is now complete. □

The following conclusion applies Theorem 1 directly to the situation $Y = X^{\#}$.

Corollary 1. *If* $X \in \mathcal{B}_{A,r}(\mathcal{H})$*, then we have*

$$\operatorname{ber}_A^n\left(X^2\right) \leq \frac{1}{2}\left\|\left(XX^{\#}\right)^n + \left(X^{\#}X\right)^n\right\|_{A-\text{Ber}}.$$

The following result will be needed for further investigation.

Theorem 2. *Let* $X \in \mathcal{B}(\mathcal{H})$*. Then we have*

$$\operatorname{ber}^2(X) \leq \frac{1}{2}\left(\operatorname{ber}\left(X^2\right) + \|X\|_{\text{Ber}}^2\right). \tag{11}$$

Proof: Now, by choosing in (7), $e = \widehat{k}_\xi$, $x = X\widehat{k}_\xi$ and $y = X^*\widehat{k}_\xi$, we get

$$\frac{1}{2}\left(\left\|X\widehat{k}_\xi\right\|\left\|X^*\widehat{k}_\xi\right\| + \left|\left\langle X\widehat{k}_\xi, X^*\widehat{k}_\xi\right\rangle\right|\right) \geq \left|\left\langle X\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^2 \tag{12}$$

for any $\xi \in \Omega$. Taking the supremum in (12) over $\xi \in \Omega$, we deduce the desired inequality

$$\operatorname{ber}^2(X) \leq \frac{1}{2}\left(\operatorname{ber}\left(X^2\right) + \|X\|_{\text{Ber}}^2\right).$$

□

The following is a generalization of the inequality in (11).

Theorem 3. *If $X \in \mathcal{B}_{A,r}(\mathcal{H})$, then, we have*

$$\operatorname{ber}_A^n(X) \leq \frac{1}{2^{n-1}} \operatorname{ber}_A(X^n) + \|X\|_{A-\mathrm{Ber}} \sum_{s=1}^{n-1} \frac{1}{2^s} \operatorname{ber}_A^{n-s-1}(X) \|X^s\|_{A-\mathrm{Ber}},$$

for any $n = 2, 3, \ldots$

Proof: Let $\xi \in \Omega$ be arbitrary. First we need to show that

$$\left|\left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^n \leq \frac{1}{2^{n-1}} \left|\left\langle X^n\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right| + \sum_{s=1}^{n-1} \frac{1}{2^s} \left|\left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^{n-s-1} \left\|X^s\widehat{k}_\xi\right\|_A \left\|X^{\#}\widehat{k}_\xi\right\|_A, \tag{13}$$

for any $n = 2, 3, \ldots$ To establish the needed inequality, we shall utilize induction on n. Simply substituting $e = \widehat{k}_\xi$, $x = X\widehat{k}_\xi$ and $y = X^{\#}\widehat{k}_\xi$ in (8), established that that the inequality (13) is valid for $n = 2$. Presume, on the other hand, that (13) holds true for n. We get

$$\left\|X^n\widehat{k}_\xi\right\|_A \left\|X^{\#}\widehat{k}_\xi\right\|_A + \left|\left\langle X^{n+1}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right| \geq 2\left|\left\langle X^n\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right| \left|\left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|$$

by using the inequality (8) with $x = X^n\widehat{k}_\xi$ and $y = X^{\#}\widehat{k}_\xi$. We get

$$\begin{aligned} \left\|X^n\widehat{k}_\xi\right\|_A \left\|X^{\#}\widehat{k}_\xi\right\|_A + \left|\left\langle X^{n+1}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right| &\geq 2^n \left|\left\langle X^n\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^{n+1} \\ &\quad - \sum_{s=1}^{n-1} 2^{n-s} \left|\left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^{n-s} \left\|X^s\widehat{k}_\xi\right\|_A \left\|X^{\#}\widehat{k}_\xi\right\|_A \end{aligned}$$

under the premise. Then, we have

$$\left|\left\langle X^{n+1}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right| \geq 2^n \left|\left\langle X^n\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^{n+1} - \sum_{s=1}^{n} 2^{n-s} \left|\left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^{n-s} \left\|X^s\widehat{k}_\xi\right\|_A \left\|X^{\#}\widehat{k}_\xi\right\|_A.$$

Thus,

$$\left|\left\langle X^n\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^{n+1} \leq \frac{1}{2^n} \left|\left\langle X^{n+1}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right| + \sum_{s=1}^{n} \frac{1}{2^n} \left|\left\langle X\widehat{k}_\xi, \widehat{k}_\xi \right\rangle_A\right|^{n-s} \left\|X^s\widehat{k}_\xi\right\|_A \left\|X^{\#}\widehat{k}_\xi\right\|_A.$$

and so by taking the supermum over $\xi \in \Omega$ in (13) and using the fact $\left\|X^{\#}\right\|_A = \|X\|_A$ for every $X \in \mathcal{B}_A(\mathcal{H})$, we get

$$\operatorname{ber}_A^n(X) \leq \frac{1}{2^{n-1}} \operatorname{ber}_A(X^n) + \|X\|_{A-\mathrm{Ber}} \sum_{s=1}^{n-1} \frac{1}{2^s} \operatorname{ber}_A^{n-s-1}(X) \|X^s\|_{A-\mathrm{Ber}}$$

as required. □

The following is a consequence of Theorem 3.

Corollary 2. *If $X \in \mathcal{B}_{A,r}(\mathcal{H})$, then we have*

$$\operatorname{ber}_A^2(X) \leq \frac{1}{2}\left(\operatorname{ber}_A\left(X^2\right) + \|X\|_{A-\mathrm{Ber}}^2\right).$$

Theorem 4. *([18, Theorem 2.12]) Let $Y, X \in \mathcal{B}(\mathcal{H})$. Then*

$$\operatorname{ber}^2(Y+X) \leq \operatorname{ber}^2(Y) + \operatorname{ber}^2(X) + \|Y\|_{\mathrm{Ber}} \|X\|_{\mathrm{Ber}} + \operatorname{ber}(Y^*X).$$

The following result is an extension of Theorem 4.

Theorem 5. *If $Y, X \in \mathcal{B}_{A,r}(\mathcal{H})$, then we have*

$$\operatorname{ber}_A^2(Y+X) \leq \operatorname{ber}_A^2(Y) + \operatorname{ber}_A^2(X) + \|Y\|_{A-\mathrm{Ber}} \|X\|_{A-\mathrm{Ber}} + \operatorname{ber}_A\left(Y^{\#}X\right).$$

Proof: We have

$$\begin{aligned}\left|\left\langle (Y+X)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2 &= \left|\left\langle \left(Y_{\widehat{k}_\xi}+X_{\widehat{k}_\xi}\right),\widehat{k}_\xi\right\rangle_A\right|^2\\ &= \left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A+\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2\\ &\le \left(\left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A+\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|\right)^2\\ &= \left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\left|\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+2\left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|\left|\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|.\end{aligned}$$

By taking $x=X\widehat{k}_\xi$ and $y=Y\widehat{k}_\xi$ in the inequality (8), we have

$$\begin{aligned}&\left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\left|\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+2\left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|\left|\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|\\ &\le \left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\left|\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\|Y\|_{A-\mathrm{Ber}}\|X\|_{A-\mathrm{Ber}}+\left|\left\langle Y^{\#}X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|\end{aligned}$$

and

$$\left|\left\langle (Y+X)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2 \le \left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\left|\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\|Y\|_{A-\mathrm{Ber}}\|X\|_{A-\mathrm{Ber}}+\left|\left\langle Y^{\#}X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|. \tag{14}$$

By taking the supremum over $\xi\in\Omega$ in the above inequality,

$$\sup_{\xi\in\Omega}\left|\left\langle (Y+X)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2 \le \sup_{\xi\in\Omega}\left\{\left|\left\langle Y\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\left|\left\langle X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|^2+\|Y\|_{A-\mathrm{Ber}}\|X\|_{A-\mathrm{Ber}}+\left|\left\langle Y^{\#}X\widehat{k}_\xi,\widehat{k}_\xi\right\rangle_A\right|\right\}$$

which is equivalent to

$$\mathrm{ber}_A^2(Y+X)\le \mathrm{ber}_A^2(Y)+\mathrm{ber}_A^2(X)+\|Y\|_{A-\mathrm{Ber}}\|X\|_{A-\mathrm{Ber}}+\mathrm{ber}_A\left(Y^{\#}X\right)$$

as desired inequality. □

In the following, we define

$$B=\mathrm{Re}_A X:=\frac{1}{2}\left(X+X^{\#}\right) \text{ and } C=\mathrm{Im}_A X:=\frac{1}{2i}\left(X-X^{\#}\right)$$

for any arbitrary operator $X=B+iC\in\mathcal{B}_{A,r}(\mathcal{H})$

We are now ready to express the following theorem.

Theorem 6. *If $X_j\in\mathcal{B}_{A,r}(\mathcal{H})$ with $X_j=B_j+iC_j$ for $j=1,...,k$, then we have*

$$\mathrm{ber}_A^n\left(\sum_{j=1}^k X_j\right)\le\left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^k\left\|B_j^{2n}+C_j^{2n}\right\|_{A-\mathrm{Ber}}^{1/2}$$

for all $n\ge 1$. In particular, $\mathrm{ber}(X)\le\left(\sqrt{2}\right)^{n-1}\left\|B^{2n}+C^{2n}\right\|_{A-\mathrm{Ber}}^{1/2}$.

Proof: Let $\xi\in\Omega$ be arbitrary. It is not difficult to demonstrate that

$$\left|\sum_{j=1}^k\left\langle X_j\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^n\le\left(\sum_{j=1}^k\left(\left|\left\langle B_j\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2+\left|\left\langle C_j\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2\right)^{\frac{1}{2}}\right)^n.$$

By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|\sum_{j=1}^{k}\left\langle X_j\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^n &\le \left(\sum_{j=1}^{k}\left(\left\|B_j\widehat{k}_\xi\right\|^2\left\|\widehat{k}_\xi\right\|^2+\left\|C_j\widehat{k}_\xi\right\|^2\left\|\widehat{k}_\xi\right\|^2\right)^{\frac{1}{2}}\right)^n\\
&= \left(\sum_{j=1}^{k}\left(\left\langle\left(B_j^*B_j\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle\left(C_j^*C_j\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)^{\frac{1}{2}}\right)^n\\
&= \left(\sum_{j=1}^{k}\left(\left\langle B_j^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle C_j^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)^{\frac{1}{2}}\right)^n\\
&\le k^{n-1}\sum_{j=1}^{k}\left(\left\langle B_j^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle C_j^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)^{\frac{n}{2}}\\
&\text{(by the inequality (4))}\\
&\le \left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left(\left\langle B_j^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^n+\left\langle C_j^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^n\right)^{\frac{1}{2}}\\
&\text{(by the inequality (9))}\\
&\le \left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left(\left\langle\left(B_j^2\right)^n\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle\left(C_j^2\right)^n\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)^{\frac{1}{2}}\\
&\text{(by the inequality (3))}\\
&= \left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left\langle\left(B_j^{2n}+C_j^{2n}\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\frac{1}{2}}\\
&\le \left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left\|\left(B_j^{2n}+C_j^{2n}\right)\widehat{k}_\xi\right\|^{\frac{1}{2}}.
\end{aligned}$$

and so,

$$\left|\sum_{j=1}^{k}\left\langle X_j\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^n \le \left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left\|\left(B_j^{2n}+C_j^{2n}\right)\widehat{k}_\xi\right\|^{1/2}.$$

Taking the supremum over $\xi\in\Omega$ in above inequality, we deduce

$$\operatorname{ber}^n\left(\sum_{j=1}^{k}X_j\right)\le\left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left\|B_j^{2n}+C_j^{2n}\right\|^{1/2}.$$

Since $X_j,B_j,C_j\in\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, so there exists unique $\widetilde{X_j}$, $\widetilde{B_j}$, and $\widetilde{C_j}$ in $\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)$ such that $V_AB_j=\widetilde{B_j}V_A$, $V_AC_j=\widetilde{C_j}V_A$, $V_AX_j=\widetilde{X_j}V_A$. This implies that

$$\operatorname{ber}^n\left(\sum_{j=1}^{k}\widetilde{X}_j\right)\le\left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left\|\left(\widetilde{B_j}\right)^{2n}+\left(\widetilde{C_j}\right)^{2n}\right\|^{1/2}_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)}.$$

By using (5) and (6), we deduce

$$\operatorname{ber}^n\left(\widetilde{\sum_{j=1}^{k}X_j}\right)\le\left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left\|\widetilde{B_j^{2n}+C_j^{2n}}\right\|^{\frac{1}{2}}_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)}.$$

which is equivalent to

$$\operatorname{ber}_A^n\left(\sum_{j=1}^{k}X_j\right)\le\left(\sqrt{2k}\right)^{n-1}\sum_{j=1}^{k}\left\|B_j^{2n}+C_j^{2n}\right\|^{1/2}_{A-\mathrm{Ber}}.$$

This completes the proof. ☐

Finally, we will provide the following outcome.

Theorem 7. *If $X_j \in \mathcal{B}_{A,r}(\mathcal{H})$ with $X_j = B_j + iC_j$ for $j = 1, ..., k$, then, we have*

$$\text{ber}_A^n\left(\sum_{j=1}^{k} X_j\right) \leq 2^{\frac{n}{2}-1} k^{n-1} \sum_{j=1}^{n} \left\| (B_j + C_j)^{2n} + (B_j - C_j)^{2n} \right\|_{A-\text{Ber}}$$

for all $n \geq 1$. In particular, $\text{ber}_A^n(X) \leq 2^{\frac{n}{2}-1} \left\| (B+C)^{2n} + (B-C)^{2n} \right\|_{A-\text{Ber}}^{1/2}$.

Proof: Let $\xi \in \Omega$ be arbitrary. By using the inequalities in (2), (3) and (4), we get

$$\begin{aligned}
\left|\sum_{j=1}^{k} \left\langle X_j \widehat{k}_\xi, \widehat{k}_\xi \right\rangle\right|^n &\leq \left(\sum_{j=1}^{k} \left(\left\langle B_j \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^2 + \left\langle C_j \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^2\right)^{\frac{1}{2}}\right)^n \\
&= \left(\sum_{j=1}^{k} \left(\frac{1}{2}\left(\left\langle (B_j + C_j) \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^2 + \left\langle (B_j - C_j) \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^2\right)\right)^{\frac{1}{2}}\right)^n \\
&\leq k^{n-1} 2^{-\frac{n}{2}} \sum_{j=1}^{k} \left(\left\langle (B_j + C_j) \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^2 + \left\langle (B_j - C_j) \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^2\right)^{\frac{n}{2}} \\
&\leq k^{n-1} 2^{-\frac{n}{2}} \sum_{j=1}^{k} \left(\left\langle (B_j + C_j)^2 \widehat{k}_\xi, \widehat{k}_\xi \right\rangle + \left\langle (B_j - C_j)^2 \widehat{k}_\xi, \widehat{k}_\xi \right\rangle\right)^{\frac{n}{2}} \\
&\leq k^{n-1} 2^{\frac{n}{2}-1} \sum_{j=1}^{k} \left(\left\langle (B_j + C_j)^2 \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^n + \left\langle (B_j - C_j)^2 \widehat{k}_\xi, \widehat{k}_\xi \right\rangle^n\right)^{\frac{1}{2}} \\
&\leq k^{n-1} 2^{\frac{n}{2}-1} \sum_{j=1}^{k} \left(\left\langle (B_j + C_j)^{2n} \widehat{k}_\xi, \widehat{k}_\xi \right\rangle + \left\langle (B_j - C_j)^{2n} \widehat{k}_\xi, \widehat{k}_\xi \right\rangle\right)^{\frac{1}{2}} \\
&= k^{n-1} 2^{\frac{n}{2}-1} \sum_{j=1}^{k} \left(\left\langle \left((B_j + C_j)^{2n} + (B_j - C_j)^{2n}\right) \widehat{k}_\xi, \widehat{k}_\xi \right\rangle\right)^{\frac{1}{2}}.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we see that

$$\left|\sum_{j=1}^{k} \left\langle X_j \widehat{k}_\xi, \widehat{k}_\xi \right\rangle\right|^n \leq k^{n-1} 2^{\frac{n}{2}-1} \sum_{j=1}^{k} \left(\left\| (B_j + C_j)^{2n} + (B_j - C_j)^{2n} \right\|\right).$$

Taking the supremum over $\xi \in \Omega$ in above inequality, we have

$$\text{ber}^n\left(\sum_{j=1}^{k} X_j\right) \leq k^{n-1} 2^{\frac{n}{2}-1} \sum_{j=1}^{n} \left\| (B_j + C_j)^{2n} + (B_j - C_j)^{2n} \right\|.$$

Now, for all $j = 1, 2, ..., k$, we have $X_j,\ B_j,\ C_j \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, so there exists unique $\widetilde{X_j},\ \widetilde{B_j}$, and $\widetilde{C_j}$ in $\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)$ such that $V_A B_j = \widetilde{B_j} V_A$, $V_A C_j = \widetilde{C_j} V_A$, $V_A X_j = \widetilde{X_j} V_A$. This implies that

$$\text{ber}^n\left(\sum_{j=1}^{k} \widetilde{X_j}\right) \leq k^{n-1} 2^{\frac{n}{2}-1} \sum_{j=1}^{n} \left\| \left(\widetilde{B_j} + \widetilde{C_j}\right)^{2n} + \left(\widetilde{B_j} - \widetilde{C_j}\right)^{2n} \right\|_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)}.$$

By using (5), (6) and $\|X\|_A = \|X\|_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)}$, $X \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, we deduce

$$\text{ber}^n\left(\widetilde{\sum_{j=1}^{k} X_j}\right) \leq 2^{\frac{n}{2}-1} k^{n-1} \sum_{j=1}^{n} \left\| (B_j + C_j)\ \widetilde{2n +}\ (B_j - C_j)^{2n} \right\|_{\mathcal{B}\left(R\left(A^{\frac{1}{2}}\right)\right)}$$

and

$$\text{ber}_A^n\left(\sum_{j=1}^{k} X_j\right) \leq 2^{\frac{n}{2}-1} k^{n-1} \sum_{j=1}^{n} \left\| (B_j + C_j)^{2n} + (B_j - C_j)^{2n} \right\|_{A-\text{Ber}}.$$

This concludes the theorem's proof. □

We recommend [14, 15] for more current studies on Berezin radius inequalities for operators and other relevant results.

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Traffic Sign Recognition via Convolutional Neural Network

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Abstract: In this paper, Convolutional Neural Network(CNN) based traffic sign recognition has been introduced for various conditions that an autonomous vehicle may encounter. The recognition performance of the CNN based traffic sign recognizer has been examined for nominal, snowy, obstacle and damaged cases. For this purpose, CNN has been trained primarily by using nominal traffic signs. Then, for validation phase, the approximation accuracy of CNN based traffic sign recognizer has been evaluated for traffic signs covered with snow, obstacle and damaged conditions. Even in the toughest conditions, CNN correctly recognizes traffic signs with approximately 70% accuracy. The acquired results indicate that the CNN can be deployed successfully in traffic sign recognition.

Keywords: Convolutional Neural Network, Deep Learning, Optimization Theory, Traffic Sign Recognition.

1 Introduction

Convolutional Neural Networks(CNN) have been frequently used in pattern recognition-based engineering problems in recent years due to their superior non-linear estimation capabilities. The most significant advantage of the CNN structure compared to Multi Layer Perceptron(MLP) is that the dimension of the data is reduced by using various filters in the convolution layers and thus the low-dimensional data is prepared for the MLP structure. Thus, the computational load of the MLP is reduced.

The pattern recognition problem in which the CNN structure is deployed most effectively is the recognition of traffic signs. In recent years, the control and driving performance of autonomous vehicles is directly related to the correct recognition of traffic signs pattern. For this reason, artificial intelligence-based pattern recognition algorithms are frequently used in autonomous vehicles.

In technical literature, there are various traffic sign recognition architectures based on machine learning. Castellano et al[1] proposed an automatic traffic sign detection method based on k-Nearest Neighbors and Support Vector Machines(SVM) to enhance performance limitations of current automatic sign detection systems, specially for achromatic signs and variable lighting conditions. Fourier descriptors with SVM based classification algorithm is deployed for the description of sign shapes[1]. Lorsakul and Suthakorn[2] introduced neural network-based traffic sign recognizer. Image processing techniques, such as, threshold technique, Gaussian filter, Canny edge detection, Contour and Fit Ellipse deployed in preprocessing phase[2]. The supervised backpropogation algorithm is deployed to train MLP structure with sigmoid activation functions[2]. Zhang et al [3] utilized multiscale cascaded regional convolutional neural network(RCNN) structure in traffic sign recognizer to overcome some drawbacks of traffic sign detection problem such as undetected smal signs and false signs owing to interferences caused by illumination variation, bad weather and some signs similiar to the traffic signs. Cao et al[4] deployed faster region-based CNN(Faster R-CNN) for small object detection in traffic to deal with the positioning deviation caused by traditional methods and recognizing small traffic signs lost in the complexity of the background.For this purpose, Convolutional feature fusion and soft-non-maximum supression(NMS) algorithm are employed to enhance recognition accuracy[4]. Tabernik and Skocaj[5] proposed to use Mask RCNN(extension of faster RCNN) based architecture for the problem of detecting and recognizing a large number of traffic-sign categories for the main purpose of automating the traffic-sign inventory management. Stochastic gradient descent is deployed as learning algorithm[5].

In this paper, CNN based traffic sign recognition has been proposed for self-driving cars. More complex patterns with snowy, obstacle and damaged features have been applied to assess the recognition performance of the CNN structure.

This paper is organized as follows: the fundamentals of CNN are overviewed in Section section 2. Traffic sign recognition via CNN is given in Section section 3. The performance of the CNN traffic sign recognizer is evaluated in Section section 4. The paper ends with a brief conclusion part in Section section 5.

2 Convolutional Neural Network

In this section, the fundamentals of ANN and CNN are overviewed. Since the CNN structure is exactly the same as MLP, except for the filters that provide the preprocessing process, and CNN is based on MLP, the basis of ANN topology is given in Section subsection 2.1. In section subsection 2.2, the CNN architecture is explained.

2.1 An Overview of Artificial Neural Network

Artificial neural networks are learning systems inspired by human biological neural networks. ANNs aim is to mimic the learning mechanism of neuron by utilizing optimization theory. The MIMO(multi input-multi output) MLP architecture is illustrated in Figure 1 which is composed of input, hidden and output layers, respectively where $w^h_{S,N}$, is hidden layer weights,b^h_S denotes the bias of hidden neurons, $\Phi(,)$ stands for activation function and $w^o_{m,S}$ represents the weights of output layer[6, 7]. In order to provide nonlinearity, various nonlinear activation functions can be used in hidden layer. Depending on the complexity of the identification problem, it is possible to increase the number of the hidden layers. ANN aims to find the optimal weights which map input feature vector to the given output by optimizing the network weights. For this purpose, first order and second order learning algorithms such as gradient descent and Newton-Raphson can be employed to minimize learning error. In addition to the weights that provide the connection between neurons, the parameters of the activation functions can also be optimized. Since MLPs are accepted as universal approximators, it is possible to employ ANN to solve various engineering problem by obtaining the optimal set of weights for MLP.

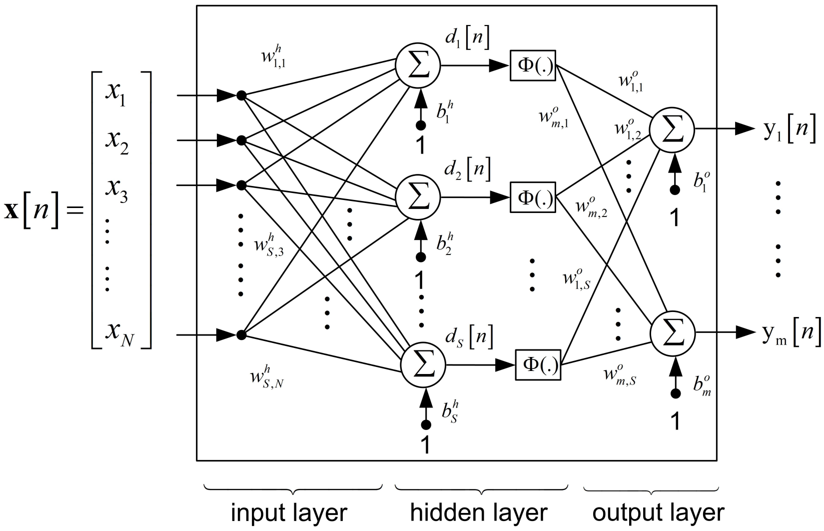

Fig. 1: MIMO MLP Structure[8, 9]

2.2 Convolutional Neural Network

Convolutional neural network is one of the most popular deep learning algorithms. CNN is generally used for image classification, object detection, recommender systems, medical image analyses, natural language processing(NLP) etc.[4, 9, 11]. CNN illustrated in Figure 2 is composed of convolutional, pooling layers and fundamental MLP architecture. In convolutional and pooling layer, it is aimed to obtain the

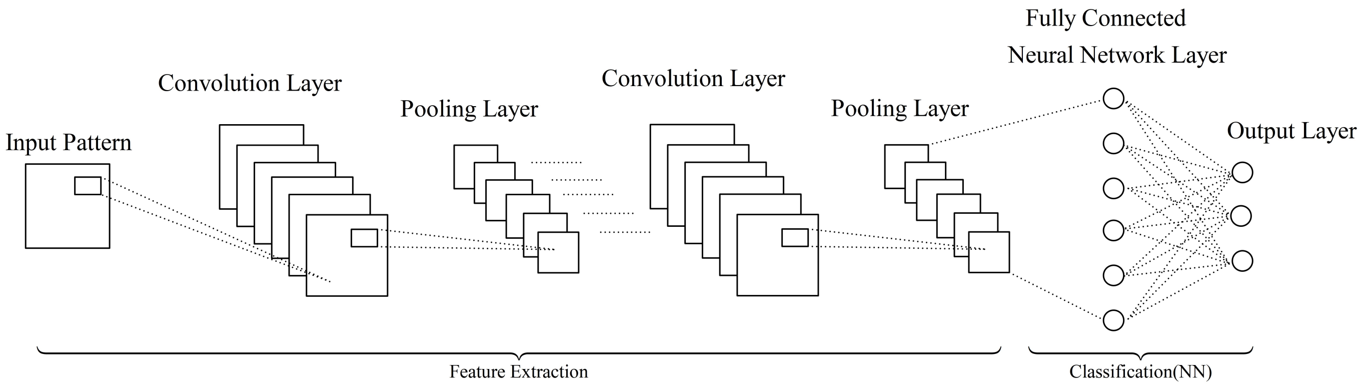

Fig. 2: CNN Architecture[12, 13]

significant features for MLP structure. This process can be considered as preprocessing operation to reduce the dimension of the input pattern. Therefore, one of the main purposes of convolutional neural networks is to reduce the number of the parameters in ANN[11]. In the convolution and pooling layers, filters are applied to the input pattern in order to extract the features of the input image. If the input pattern is considered as a traffic sign, the convolution and pooling layer allows to find the features of the plate shape such as rectangular, circle etc. and the characteristic of the sign such as stop, zigzag, speed limit etc. The pooling process is applied to reduce the complexity for further layers[11]. Max-pooling method is commonly deployed as pooling method[11]. In the max pooling process, the highest value pixels among some pixels in the image are taken and the input is made simpler. Depending on the complexity of the input pattern, convolution and pooling operations can be increased as desired. Thus, the complex input pattern in input layer is made suitable for MLP by reducing its dimension to increase the performance of MLP. Using the input-output data set, the network parameters of the MLP can be optimized to learn the correlation among input-output data via derivative or evolutionary based optimization algorithms.

3 Traffic Sign Recognition via CNN

CNN is frequently deployed in traffic sign detection. The data set composed of various traffic signs is divided into train and test data sets before training phase[14]. By using convolutional and pooling layers, the features of the input training pattern are extricated to simplfy the recognition problem for MLP structure. These features can be the outer shape of the sign, the color of the sign, the characteristic of the sign such as stop, zigzag, speed limit etc[14, 15]. Then, the outputs of convolutional and pooling layer are flattened as an input vector for MLP. Then, the outputs

Fig. 3: CNN Approximation accuracy for traffic sign detection for nominal case (%).

of convolutional and pooling layer are flattened as an input vector for MLP. Then, using the processed input data for MLP and output data, the weights of the MLP can be optimized by using learning algorithms. After each training phase, test data is applied to the CNN. When the testing error starts to increase, training phase is terminated in order to prevent overfitting. Owing to the non-convex structure of the objective function in CNN, the algorithm may get stuck at local minima. In order to overcome this situation, some weights in MLP can be randomly changed to jump from this stuck local minima. In addition to this, it is possible to deploy evolutionary or swarm based learning algorithm to train weights of the CNN. In validation phase, the recognition accuracy of the model which belows a certain threshold value can be ignored. In addition to the recognition accuracy of CNN, the computational load is significant in applications. Therefore, CNN model with low computational load is more preferable than those with the same recognition accuracy.

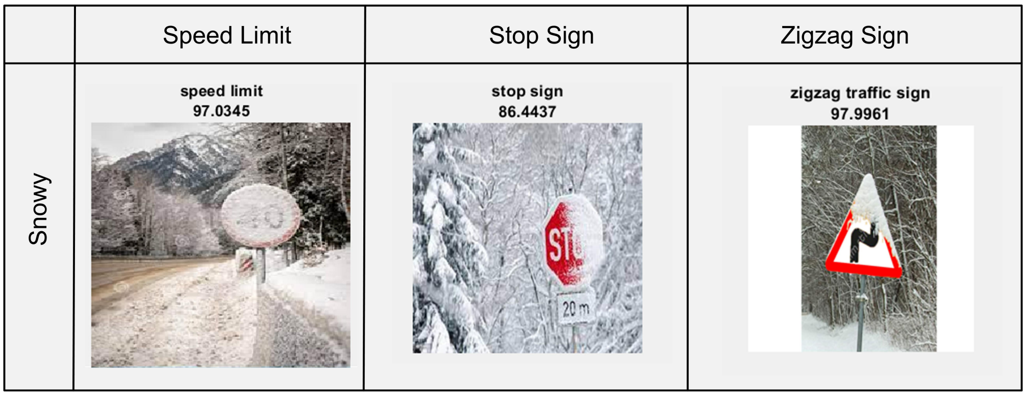

Fig. 4: CNN Approximation accuracy for traffic sign detection for snowy case (%).

4 Simulation Results and Discussion

The performance of the CNN has been examined for four different cases: Nominal, snowy, obstacle, damaged. The CNN architecture is trained by using the nominal case data. CNN is trained by using 58, 92 and 136 training data for speed limit, stop sign and zigzag traffic signs, respectively. In order to evaluate the robustness and performance of the CNN, the CNN is tested on obstacle, damaged and snowy cases in addition to nominal case. The performance of the CNN for nominal case is shown in Figure 3. As it is expected, the CNN model has the best approximation accuracy in nominal case. As illustrated in Figure 4 for snowy case, the traffic signs are covered with snow. In spite of this condition, the minimum prediction accuracy of the CNN is about 87%. Since the trees grow in front of the traffic signs and they get in front of the sign until the authorities correct this situation, so that it becomes impossible to recognize the sign or it becomes difficult to recognize. Therefore, the CNN performance is assessed with respect to obstacle case as illustrated in Figure 5. As shown in Figure 5 for obstacle case, CNN has about 91% recognition performance. Owing to the weather conditions, human factor, traffic accidents, the traffic signs may wear out and get damaged. For this reason, it becomes difficult to recognize these damaged signs. Therefore, the performance of the CNN has been tested by considering damaged cases as depicted in Figure 6. As illustrated for damaged case, CNN has 70% accurate recognition performance. In a nutshell, even in the worst condition given in Figure 6, CNN has almost 70% accuracy. By using traffic signs under snowy, obstacle and damaged cases in training data set, the approximation performance can be enhanced for snowy, obstacle and damaged cases.

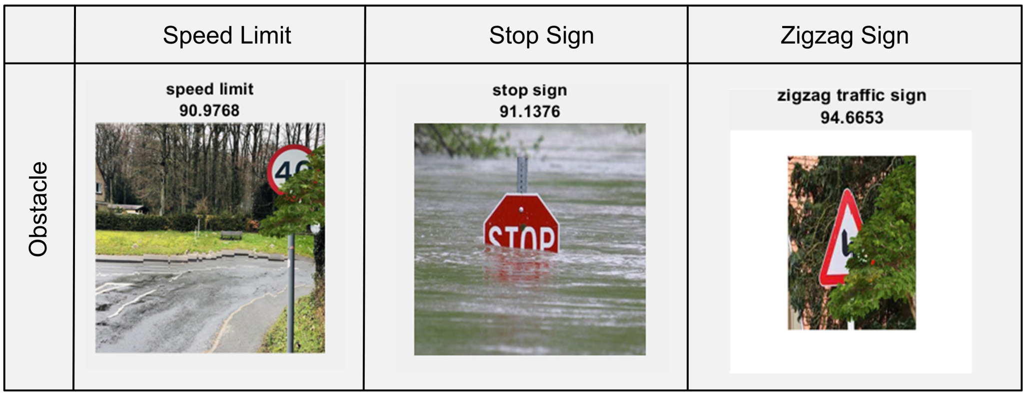

Fig. 5: CNN Approximation accuracy for traffic sign detection for obstacle case (%).

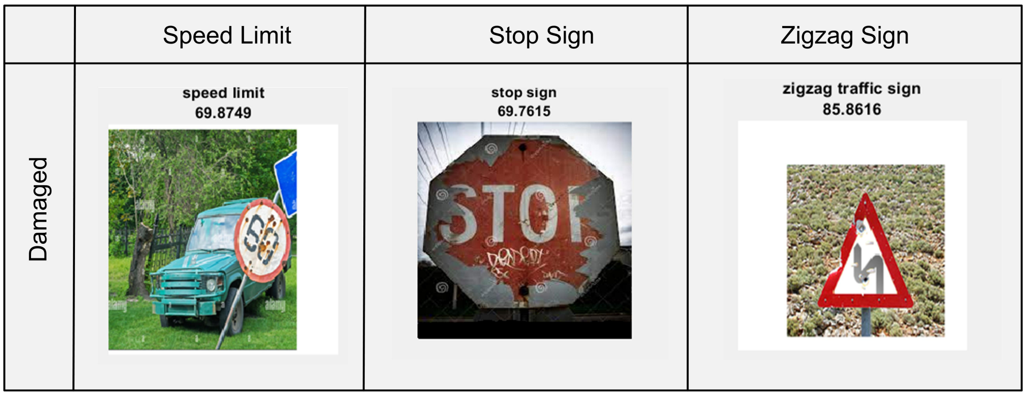

Fig. 6: CNN Approximation accuracy for traffic sign detection for damaged case (%).

5 Conclusion and Future Works

In this paper, traffic sign recognizer based on CNN has been introduced for various forceful conditions in pattern recognition. The validation performance of the CNN based traffic sign recognizer has been assessed for snowy, obstacle and damaged cases. Even in damaged conditions, which is one of the most difficult conditions for pattern recognition, CNN has approximately 70% approximation accuracy. Only nominal data has been utilized in training phase of the CNN. By using snowy, obstacle and damaged patterns, the recognition performance of CNN can be enhanced. As future work, in addition to traffic sign recognition, it is intended to be used in recognition of pedestrians and obstacles on road such as stones, potholes etc.

Acknowledgement

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An Improved Adaptive Fuzzy PID Controller based on Peak Observer for Nonlinear Dynamical Systems

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Abstract: In this paper, an adjustment mechanism based on peak observer for fuzzy PID controller has been proposed for nonlinear dynamical systems. The peak observer based adjustment mechanism proposed by Qiao and Mizumoto [1] has been improved for nonlinear control systems by taking into account adaptation of all controller parameters and peak observer parameter. The peak observer mechanism is constituted by considering mth order term derived over tracking error. Thus, in addition to scaling coefficients of fuzzy PID controller, the degree of the observer is also considered as a design parameter. Therefore, in order to enhance closed-loop system performance, the degree of the term used in observer is optimized by using grid search algorithm. The performance of introduced adaptive fuzzy controller has been examined on nonlinear continously stirred tank reactor (CSTR) system. The obtained results indicate that the proposed adjustment mechanism is quite successful for the adaptive control of nonlinear dynamical systems.

Keywords: Adaptive Control, Fuzzy PID Controller, Grid Search, Peak Observer, Peak Observer based Optimization.

1 Introduction

Fuzzy control is one of the most popular and convenient field where fuzzy set theory is most frequently deployed. Fuzzy controllers provide very successful performance in the control of systems whose mathematical model is difficult to obtain, especially those with non-linear and uncertain dynamics. Fuzzy PID has an effective control structure because it combines the information representation feature of fuzzy logic and the robustness of the PID controller.

Adaptive control architectures can be utilized so as to enhance the adaptability of the fuzzy PID controller parameters against the uncertainties in the system dynamics and the disturbances affecting the system behaviour.

In technical literature, there are various adadaptation architecture for fuzzy controllers. Chou and Lu [2] presented a self-tuning fuzzy controller based on adjustment of scaling factors. In this structure, the adaptation rules are transformed into numerical tuning tables by applying the appropriate membership functions, so that the adaptation of the scaling factors is just matrix mapping. Jung et al (1995) [3] proposed a real-time adaptation mechanism based on instantaneous system fuzzy performance (ISFP). This adaptation method uses a variable reference setting directory. An instantaneous system fuzzy performance uses two reference setting indices for overshoot and non-overshoot, respectively [3]. Maeda and Murakami [4] proposed a mechanism that adapts the controller parameters according to a fuzzy rule base created depending on the exceedance value, the time to reach the reference and the amplitude value. Mudi and Pal [5] proposed a structure that adapts the output scaling coefficients online according to the trend of the controlled system. The rule base required for the output scaling coefficients is defined depending on the derivative of the tracking error and the tracking error [5]. Zheng [6] proposed a mechanism by which the peak values and scaling coefficients of the membership functions and rules are adapted[6]. Chung et al [7] presented a mechanism for adapting the tracking error coefficient, tracking error derivative coefficient and integrator scaling coefficients for a PI type fuzzy controller, depending on the tracking error and the derivative of the error[7]. Chao and Teng [8] presented an adaptation mechanism based on the gradient descent method for the adaptation of fuzzy PD scaling coefficients.

In this paper, an incremental fuzzy PID controller in which all scaling coefficients are adjusted via peak observer has been proposed for nonlinear dynamical systems. The peak observer based adjustment mechanism introduced by Qiao and Mizumoto [1] has been enhanced to tune all controller parameters. In addition to scaling coefficients, the degree of the term produced by peak observer is considered as a design parameter, and optimized via grid search algorithm. Thus, the flexibility of the fuzzy PID controller and peak observer is deepened so as to improve the performance of the nonlinear control system. This organization of the paper is given as follows: the fundamentals of peak observer based adaptation mechnanism introduced by Qiao and Mizumoto [1] has been overviewed in Section 2. The contribution introduced in this paper has been detailed in Section 3. The adaptation performance of the proposed mechanism has been evaluated on a nonlinear CSTR system in Section 4. The paper ends with a brief conclusion part in Section 5

2 An Overview of Adaptive Fuzzy PID Controller based on Peak Observer

The block diagram of incremental fuzzy PID controller is shown in Figure 1 where K and K_d input parameters to scale tracking error and derivative of tracking error, and α and β are output parameters to weight the PD and PI parts of the fuzzy PID controller, respectively[9–11]. The incremental fuzzy PID control law is composed of PD and PI control laws as follows:

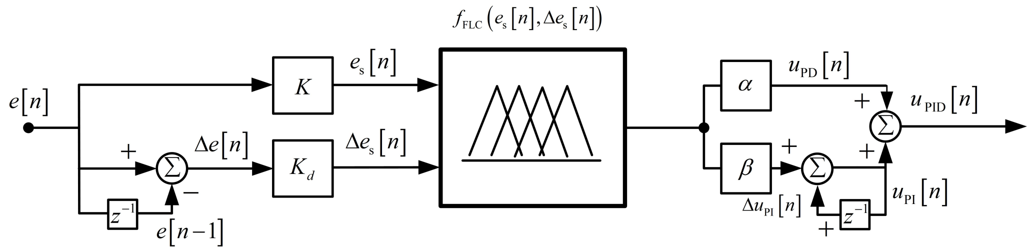

Fig. 1: Incremental Fuzzy PID Controller[1, 9, 11].

$$u_{PID}\left[n\right] = \overbrace{\alpha f_{FLC}\left(e_s\left[n\right], \Delta e_s\left[n\right]\right)}^{u_{PD}[n]} + \overbrace{\beta f_{FLC}\left(e_s\left[n\right], \Delta e_s\left[n\right]\right) + u_{PI}\left[n-1\right]}^{u_{PI[n]}} \tag{1}$$

where $e_s\left[n\right]$ and $\Delta e_s\left[n\right]$ are scaled error and derivative of error, α and β are output scaling coefficients. The input membership functions in FLC are given in Figure 2. As illustrated in Figure 2, four(4) rules are fired from the rule base depending on current $e_s\left[n\right]$ and $\Delta e_s\left[n\right]$ values[10, 11]. Product-sum inference method and center of gravity defuzzification method [1] are utilized in FLC. Thus, the output of the FLC can be derived as follows [1, 10, 11]:

$$\begin{aligned} f_{\text{FLC}}\left(e_s\left[n\right], \Delta e_s\left[n\right]\right) = & \overbrace{A_i\left(e_s\left[n\right]\right)B_j\left(\Delta e_s\left[n\right]\right)}^{w_{i\ j}} u_{i\ j} + \overbrace{A_{i+1}\left(e_s\left[n\right]\right)B_j\left(\Delta e_s\left[n\right]\right)}^{w_{i+1\ j}} u_{i+1\ j} \\ & + \underbrace{A_i\left(e_s\left[n\right]\right)B_{j+1}\left(\Delta e_s\left[n\right]\right)}_{w_{i\ j+1}} u_{i\ j+1} + \underbrace{A_{i+1}\left(e_s\left[n\right]\right)B_{j+1}\left(\Delta e_s\left[n\right]\right)}_{w_{i+1\ j+1}} u_{i+1\ j+1} \end{aligned} \tag{2}$$

where w_{ij}'s denote the firing strength of the corresponding fired rule, and corresponding membership values are given as follows[1, 10, 11]:

$$\begin{aligned} A_i\left(e_s\left[n\right]\right) &= \frac{e_{i+1} - e_s\left[n\right]}{e_{i+1} - e_i}\ ,\ A_{i+1}\left(e_s\left[n\right]\right) = \frac{e_s\left[n\right] - e_i}{e_{i+1} - e_i} \\ B_j\left(\Delta e_s\left[n\right]\right) &= \frac{\dot{e}_{j+1} - \Delta e_s\left[n\right]}{\dot{e}_{j+1} - \dot{e}_j}\ ,\ B_{j+1}\left(\Delta e_s\left[n\right]\right) = \frac{\Delta e_s\left[n\right] - \dot{e}_j}{\dot{e}_{j+1} - \dot{e}_j} \end{aligned} \tag{3}$$

Fuzzy rule base proposed in [1] to constitute FLC is given in Table 1, and rule base is depictured in Figure 3.

Table 1: Fuzzy Control Rule Base[1].

MFs	$\dot{e}_{-2}$	$\dot{e}_{-1}$	$\dot{e}_0$	$\dot{e}_1$	$\dot{e}_2$
e_{-2}	-1.0	-0.7	-0.5	-0.3	0.0
e_{-1}	-0.7	-0.4	-0.2	0	0.3
e_0	-0.5	-0.2	0.0	0.2	0.5
e_1	-0.3	0.0	0.2	0.4	0.7
e_2	0.0	0.3	0.5	0.7	1.0

In order to analyse the dynamic behavior of the fuzzy PID controller over standart PID parameters, the control law can be linearized in the neighborhood of fired rules as detailed in [1]. Thus, the control law can be reexpressed as[1]:

$$\begin{aligned} u &= A + Pe_s\left[n\right] + D\Delta e_s\left[n\right] \\ A &= u_{ij} - Pe_i - D\dot{e}_j \\ P &= \frac{u_{i+1\ j} - u_{ij}}{e_{i+1} - e_i} \\ D &= \frac{u_{i\ j+1} - u_{ij}}{\dot{e}_{j+1} - \dot{e}_j} \end{aligned} \tag{4}$$

If the input-output scaling coefficients are substituted in (4) and the terms are matched with conventional PID controller, the equivalent standart PID components can be acquired as âĂIJ$\alpha KP + \beta K_d D$âĂİ proportional term, âĂIJβKPâĂİ integral term and âĂIJ$\alpha K_d D$âĂİ derivative term. In order to improve the performance of fuzzy PID, Qiao and Mizumoto [1] proposed peak observer based adjustment mechanism which tunes the controller parameters depending on peak values of controlled system output. Peak observer based adjustment mechanism is illustrated in Figure 4. The adjustment mechanism aims to decrease the integral term(βKP) while increasing the derivative term($\alpha K_d D$) so as

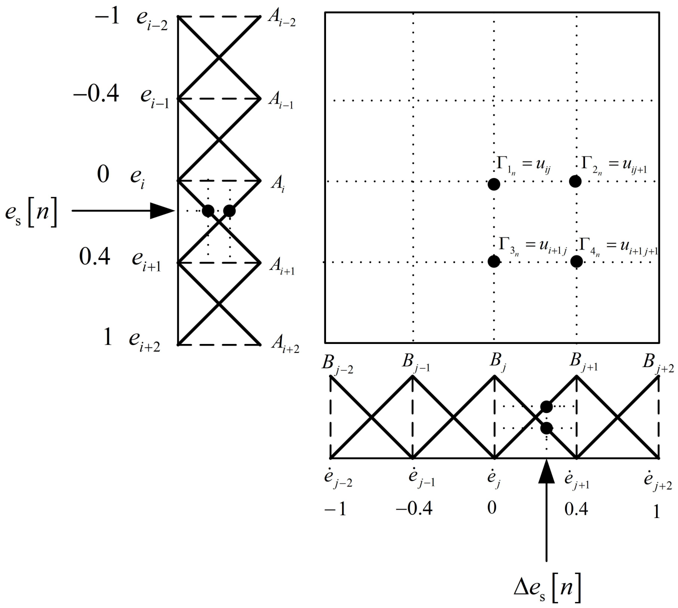

Fig. 2: Input Membership Functions and Fuzzy Rule Base[1, 10, 11].

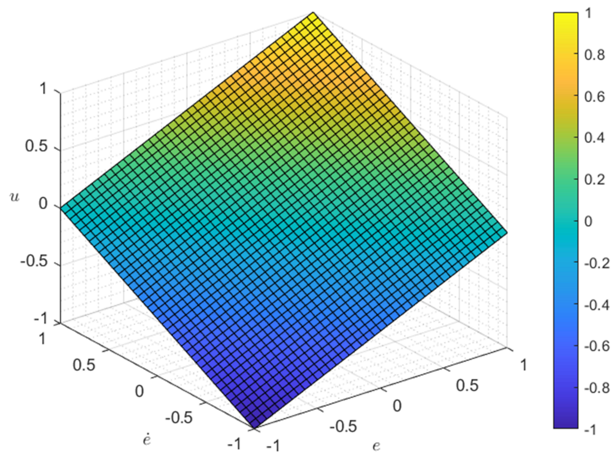

Fig. 3: Linear Fuzzy Rule Base constituted via Table 1[1].

to strengthen the resistance against the overshoots and oscillations in controlled system[1]. For this purpose, K_d input and β output scaling coefficients can be adjusted as in (5) depending on the absolute error value($\delta_k = |e_k|$) observed at peak times:

$$K_d = \frac{K_{d0}}{\delta_k},\ \beta = \beta_0 \delta_k \tag{5}$$

where $t_k,\ k \in \{1, 2, 3, \cdots\}$ are the peak times[1].

3 Adaptive Fuzzy PID Controller based on Peak Observer with mth Order Term

The aim in this study is to adapt all scaling coefficients of the fuzzy PID controller inspired by the peak observer based adaptation mechanism proposed by Qiao and Mizumoto in [1]. For this purpose, the K parameter, which scales the error signal, and the α parameter, which scales the derivative term, are also adapted. The derivative coefficient is increased while the integrator is decreased by keeping proportional term fixed. Thus, all scaling coefficients can be updated as follows:

$$\begin{bmatrix} K_{new} \\ K_{d_{new}} \\ \alpha_{new} \\ \beta_{new} \end{bmatrix} = \begin{bmatrix} K\delta_m \\ \frac{K_d}{\delta_m} \\ \frac{\alpha}{\delta_m} \\ \beta\delta_m \end{bmatrix} \tag{6}$$

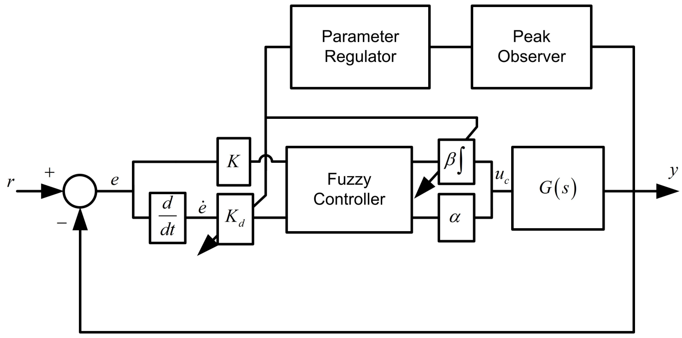

Fig. 4: Peak Observer based Adjustment Mechanism[1].

where δ_m indicates the corresponding peak observer value and âĂIJmâĂİ is the power of this term. The introduced adaptation mechanism is illustrated in Figure 5. The internal structure of fuzzy PID controller with adaptive parameters is depicted in the Figure 6 where triangular

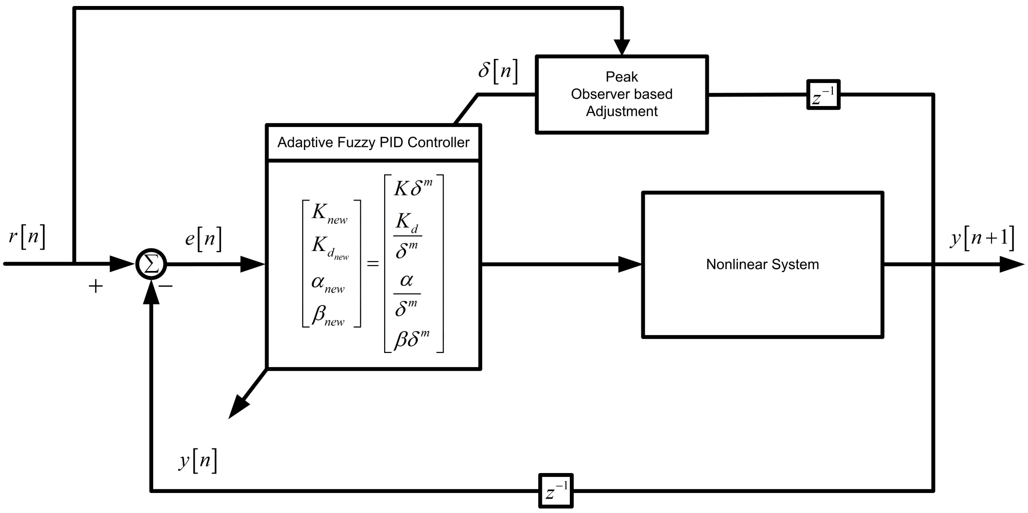

Fig. 5: Adaptive Fuzzy MIMO PID controller based on Peak Observer.

type membership functions given in Figure 2 and the fuzzy rule base in Table 1 and Figure 3 are deployed to constitute fuzzy rules. Product

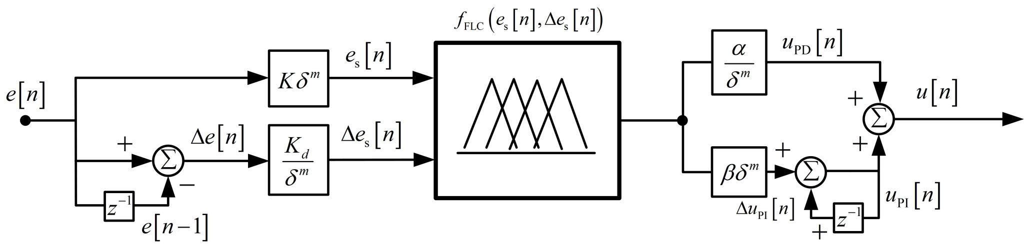

Fig. 6: Adaptive Fuzzy PID Controller.

operation and center of gravity method are utilized as inference mechanism and defuzzification method respectively. When the proposed adaptive mechanism is compared to an equivalent standard PID controller, the proportional term is attained as âĂIJ$\alpha KP + \beta K_d D$âĂİ, the integral term is obtained as âĂIJ$\beta_0 K_0 \delta^{2m} P$âĂİ, and derivative term is derived as âĂIJ $\frac{\alpha K_d D}{\delta^{2m}}$ âĂIJ. It is very significant to emphasize that the equivalent PID parameters obtained for $m = 0.5$ and the equivalent parameters of the controller structure proposed by Qiao and Mizumoto [1] are different since the P, A and D terms are a function of the scaled error and derivative of error as given in (4).

4 Simulation Results

The adaptation mechanism has been examined on a nonlinear CSTR system. The schematic diagram of CSTR system is depicted in Figure 7 [10, 12, 13]. The dynamics of the CSTR can be expressed by using the following differential equations:

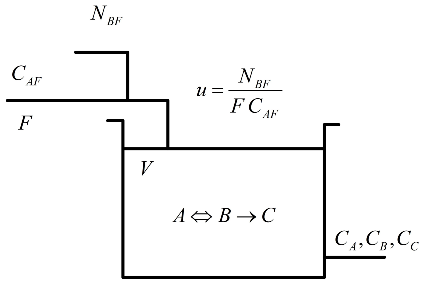

Fig. 7: CSTR system[10, 12, 13].

$$
\begin{aligned}
\dot{x}_1(t) &= 1 - x_1(t) - Da_1 x_1(t) + Da_2 x_2^2(t) \\
\dot{x}_2(t) &= -x_2(t) + Da_1 x_1(t) - Da_2 x_2^2(t) - Da_3 d_2(t) x_2^2(t) + u(t) \\
\dot{x}_3(t) &= -x_3(t) + Da_3 d_2(t) x_2^2(t) + u(t)
\end{aligned}
\tag{7}
$$

where $x_1(t)$ and $x_2(t)$ denote the inlet reactants(A,B), and $x_3(t)$ represents the product C, $Da_1 = 3$, $Da_2 = 0.5$, $Da_3 = 1$, $u(t)$ is the control signal, $x_3(t)$ is the system output, $d_2(t)$ is time varying system parameter[13–15]. Grid search algorithm has been employed to

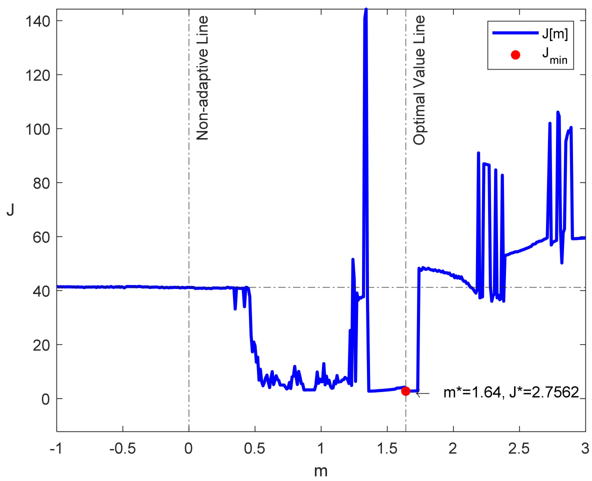

Fig. 8: Optimization of n term via Grid Search Algorithm.

determine the optimal m value(degree of the δ term) that minimize the following objective function:

$$
J(m) = \int_{t=0}^{t=t_f} \left[|e(t)| + |\frac{du(t)}{dt}| \right] dt \tag{8}
$$

It has been aimed to minimize the tracking error and alternation on control signals as given in (8) for $m \in [-1 \;\; 3]$ with 0.01 intervals. The results of the grid search algorithm is illustrated in Figure 8. The initial parameters are chosen as $K_0 = 2.5$, $K_{d,0} = 0.5$, $\alpha_0 = 0.25$ and $\beta_0 = 7.5$. The tracking performances of the CSTR system for various m values are illustrated in Figure 9 in response to step function. The performances of the non-adaptive fuzzy PID controller($m = 0$) and adjustment mechanism proposed by Qiao and Mizumoto [1] are illustrated in Figure 9 (a). While the non-adaptive controller includes oscillation, the controller structure proposed by Qiao and Mizumoto [1] improves the tracking performance of the closed-loop system. As mentioned in Section 3, the equivalent standard PID parameters for controller proposed by Qiao and Mizmoto [1] and the proposed controller structure in this paper can be assumed to be equivalent. However, the difference of the P A and D terms in (4) ensures that the equivalent standard PID parameters are different. As given in Figure 9 (b), the introduced controller for $m = 0.5$ has better tracking performance than fuzzy PID controller proposed by Qiao and Mizmoto [1]. The performace of the fuzzy PID controller for different values of m are shown in Figure 9 (c,d). The tracking performance for optimal m value is depicted in Figure 9 (d). The performace index in (8) is deployed to compare the controller performances numerically as given in Figure 10. As can be explicitly seen from Figure 9(d) and Figure 10, the adaptive fuzzy PID controller with optimal m value has the best tracking performance. The tracking performance of the introduced adaptive fuzzy PID controller with optimal m value is assessed for staircase input signal as given in Figure 11. As illustrated in Figure 11, the closed-loop system tracks the desired reference signal as close as possible. The evaluations of the controller parameters for optimal m value($m* = 1.64$) are depicted in Figure 12. Depending on the alternation of the desired reference signal, the controller parameters are adjusted to minimize the tracking error. As given in Figure 12, the resistance against the overshoot and oscillation of the system can be increased by increasing the equivalent derivative term via α and K_d parameters[1]. Similarly, K and β parameters are diminished to decrease the equivalent integral term. The main drawback of peak observer based adjustment is that the controller parameters are not updated and are fixed by the next peak time. This situation is an open problem to enhance the performance of the adjustment mechanism.

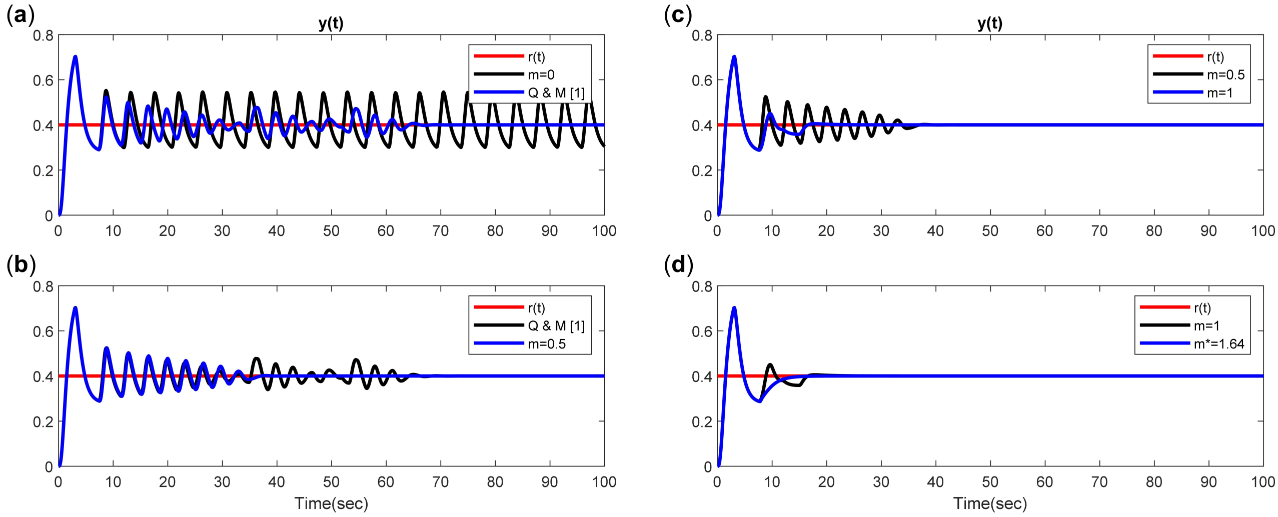

Fig. 9: System Outputs for various m values.

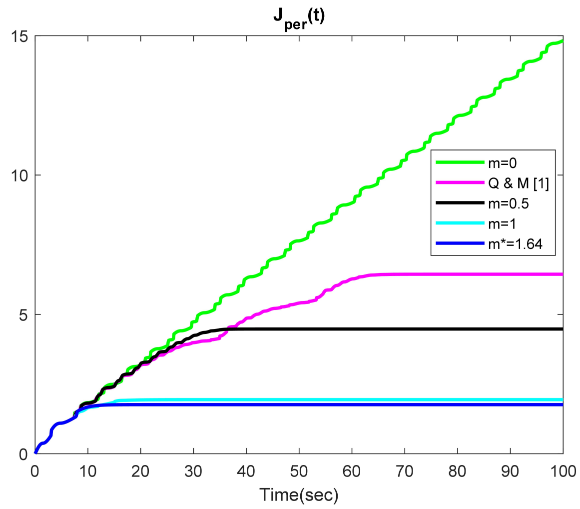

Fig. 10: Performace indeces for various m values.

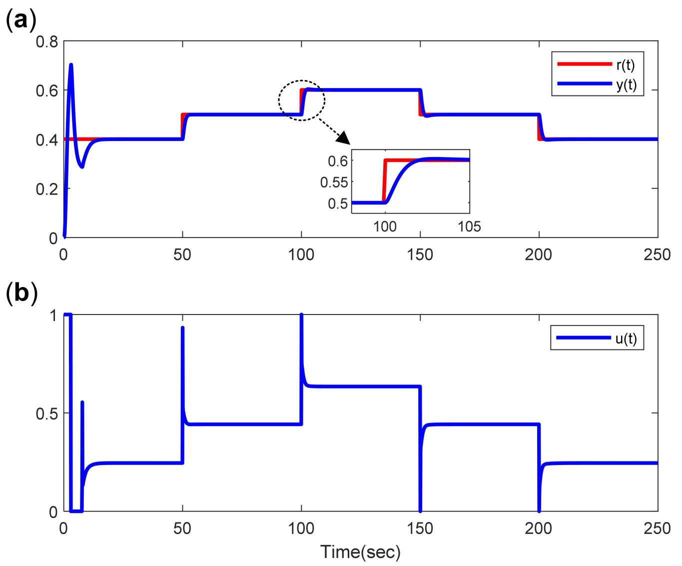

Fig. 11: System Output(a) and Control Signal(b) for Adaptive Fuzzy PID($m* = 1.64$).

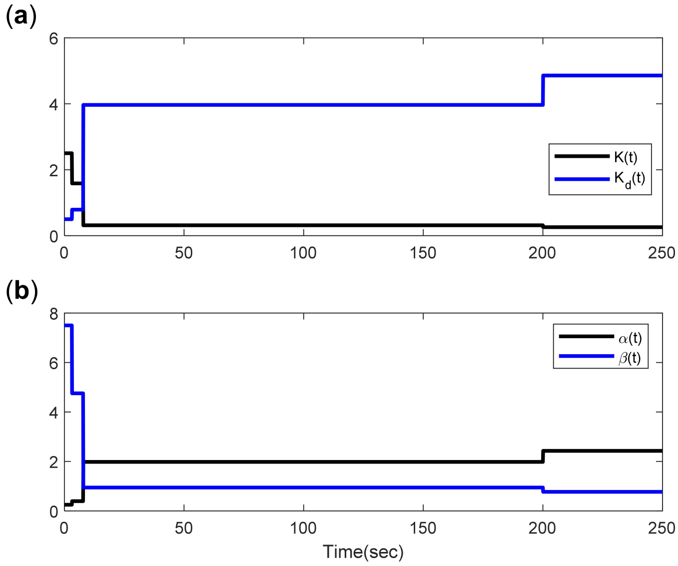

Fig. 12: Input Scaling Coefficients(a), and Output Scaling Coefficients(b) ($m* = 1.64$).

5 Conclusion

In this paper, the adjustment mechnanism proposed by Qiao and Mizumoto [1] for K_d and β parameters has been enhanced for all controller parameters of fuzzy PID controller. In addition to this, the degree of the peak observer is considered as a design parameter and optimal value for peak observer is optimized via grid search algorithm. The performance of the introduced mechanism is examined on nonlinear CSTR system by comparing with various degrees of peak observer and Qiao and Mizumoto adaptation method. It has been observed that adaptation of all controller parameters and degree of peak observer improves the tracking performance of the fuzzy PID controller for nonlinear dynamical system. As future works, it is aimed to introduce novel adjustment mechanism for fuzzy PID controllers to overcome the drawbacks of the peak observer based adaptation mechanism.

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A Study On The Fermi Model

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Abstract: Fermi system is a relativistic field theory for the interaction of fermions. It is two-dimensional massless model. Since the fermi scale interaction model is the simplest nonlinear spinor model, it has an important place in particle physics as a toy model. Fermi model exhibits soliton type solutions. The finite-energy, steady-wave solutions of the classical equations of motion of Lagrangian field theories are called solitons. In this study, the dynamics of soliton type solutions of the Fermi system are investigated analyzed by constructing spatial-temporal evolution and phase space diagrams.

Keywords: spinor, lagrangian, nonlinear dynamics, phase space, Fermi

1 Introduction

In 1926, SchrÃűdinger turned the strange behavior of electrons in an atom discovered by Bohr into precise mathematical equations, that is, it was understood that the behavior of small bodies could be determined by nonlinear quantum wave equations, and the success of the spinor-field nonlinear wave equation solutions written by Dirac in electron and anti-electron interpretation encouraged theoretical physicists to write new nonlinear field equations and seek physical wave solutions of these equations. With the discovery of new particles, these theoretical studies and searches became even more attractive. Especially since the 1950s, a great increase has been observed in these efforts in the world of theoretical physics. A number of theoretical models with broad symmetries that are hoped to cover all particles have been developed and proposed. For this purpose, the 2-dimensional massless Fermi model was proposed by Walter Thirring [1] in 1958. It is a relativistic field theory for the interaction of fermions. The interest in Fermi interactions has increased again due to the extraordinary weight of the top quark compared to other quarks and leptons [2–4].Vector-vector interaction is closely related to Quantum electrodynamics and is also studied in a variety of other contexts. The model plays an important role in particle physics as it represents a fully solvable coherent field theory with the n-point correlation known as analytical. In higher dimensions, the model cannot be resolved and renormalized in perturbation theory. Fermi model which is among the simplest interacting field theories, has been one of the remarkable study areas in the literature due to its thermodynamic properties [5–8]. Also the model has soliton-type solutions.

It is known that solitons are very important because the soliton approach is universal in different fields of modern nonlinear science. Solitons are defined as localized waves that can propagate without changing their shape and velocity properties and are stable against mutual interactions. Solitons were found in the solution of nonlinear wave equations by applied mathematicians in the 19^{th} century. The first observation of solitons was made by a British marine engineer, J. Scoot Russel, in 1938, by examining wave motion in narrow water channels. Russell found that there are stable, unchanging waves between long and shallow water waves, and that there may be a relationship between their velocity and amplitude. Solitons, which are the solutions of nonlinear wave equations discovered by applied mathematicians in the 19th century, were later used to explain some problems in solid state physics, particle physics and plasma physics. The fact that they have a vacuum state has been interpreted as corresponding to meaningful solutions in particle physics. In this study, Fermi model is analyzed as a toy model by constructing the spatial-temporal evolution and phase spaces plots to better understanding the dynamics of soliton solutions.

2 Model

The Lagrangian for extended Fermi model is

$$L(\phi) = \bar{\Psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\Psi + \lambda\bar{\Psi}\Psi U(\phi), \tag{1}$$

where U, is a function of $\phi = \bar{\Psi}\Psi$. The solitonic solution for Fermi model was firstly discovered by M. Soler[9]. The Equation of motion for Fermi model is given below,

$$i\gamma^{\mu}\partial_{\mu}\Psi - m\Psi + \lambda V\left(\bar{\Psi}\Psi\right) = 0. \tag{2}$$

Here λ coupling constant and fermion field Ψ has scale dimension $\frac{1}{2}$. In order to solve the equation we will use soler ansatzs given in Eq. 3.

$$\Psi = e^{-i\omega t}\begin{bmatrix} g(r)\begin{pmatrix}1\\0\end{pmatrix} \\ if(r)\begin{pmatrix}cos\theta\\ e^{i\phi}sin\theta\end{pmatrix}\end{bmatrix}, \tag{3}$$

here f and g are real functions of r and $r = x^2 + y^2 + z^2 + t^2$. Fermi model has 4 component time and 3 dimensional spatial dimensions. If we insert Eq. 3 in to Eq. 2 then we found,

$$f' + \frac{2}{r} f + (m - \omega)\, g + 2\lambda g \left(g^2 + f^2\right) = 0 \tag{4a}$$

$$g' + (m + \omega)\, f - 2\lambda f \left(g^2 + f^2\right) = 0. \tag{4b}$$

Numerical calculations we transform the Eq. 4a and 4b dimensionless for by using,

$$f(r) = \left(\frac{(m+\omega)}{2\lambda}\right)^{\frac{1}{2}} F(\rho)$$

$$g(r) = \left(\frac{(m+\omega)}{2\lambda}\right)^{\frac{1}{2}} G(\rho)$$

$$r = \frac{\rho}{m+\omega}.$$

Finally we obtain dimesionless form of the fermi model given below,

$$F' + \frac{2}{\rho} F + \nu G - G\left(G^2 + F^2\right) = 0 \tag{5a}$$

$$G' + F - F\left(G^2 + F^2\right) = 0. \tag{5b}$$

3 Numerical Results

System fix points are $(F, G) = (0, 0.176068); (0, 0); (0, -0.176068)$ for $\nu = 0.031$ and $(F, G) = (0, 0.774597); (0, 0); (0, -0.774597)$ for $\nu = 0.6$, respectively. The Jacobian matrix for the system is

$$J = \begin{bmatrix} -\frac{2}{r} + 2FG & F^2 + 3G^2 - \nu \\ -1 + 3F^2 + G^2 & 2FG \end{bmatrix}. \tag{6}$$

The Eigenvalues of the system are given below for both $\nu 0.031$ and $\nu 0.6$,

	fix points	λ_1	λ_2
$\nu = 0.031$	$(0, 0.176068)$	$\frac{0.03(-33.29-\sqrt{1108.23-66.58r^2})}{r}$	$\frac{0.03(-33.29+\sqrt{1108.23-66.58r^2})}{r}$
	$(0, 0)$	$\frac{0.015(-64.51-\sqrt{4162.33-129.032r^2})}{r}$	$\frac{0.015(-64.51-\sqrt{4162.33+129.032r^2})}{r}$
	$(0, -0.176068)$	$\frac{0.03(-33.29-\sqrt{1108.23-66.58r^2})}{r}$	$\frac{0.03(-33.29+\sqrt{1108.23-66.58r^2})}{r}$
$\nu = 0.6$	$(0, 0.774597)$	$\frac{0.24(-4.16-\sqrt{17.36-8.33r^2})}{r}$	$\frac{0.24(-4.16-\sqrt{17.36+8.33r^2})}{r}$
	$(0, 0)$	$\frac{0.3(-3.33-\sqrt{11.11-6.66r^2})}{r}$	$\frac{0.3(-3.33-\sqrt{11.11+6.66r^2})}{r}$
	$(0, -0.774597)$	$\frac{0.24(-4.16-\sqrt{17.36-8.33r^2})}{r}$	$\frac{0.24(-4.16-\sqrt{17.36+8.33r^2})}{r}$

Table 1 Eigenvalues for $\nu = 0.031$.

According to Table 1, the system has singularity for $r = 0$. All eigenvalues are positive and negative real number for all $r > 0$. We solve Eq. 5a and 5b Runge-Kutta method by using matlab. We fixed the initial conditions $(F(0); G(0)) = (-0.1; 0.5)$. We evaluate the system 0.1 to 400 with step size 0.001 both $\nu = 0.031$, $\nu = 0.1$ and $\nu = 0.6$. The system exhibit regular dynamics along the flow. According to the phase space displays and spatial-temporal evolution graphics system has damped and the attractor for the system is a sink.

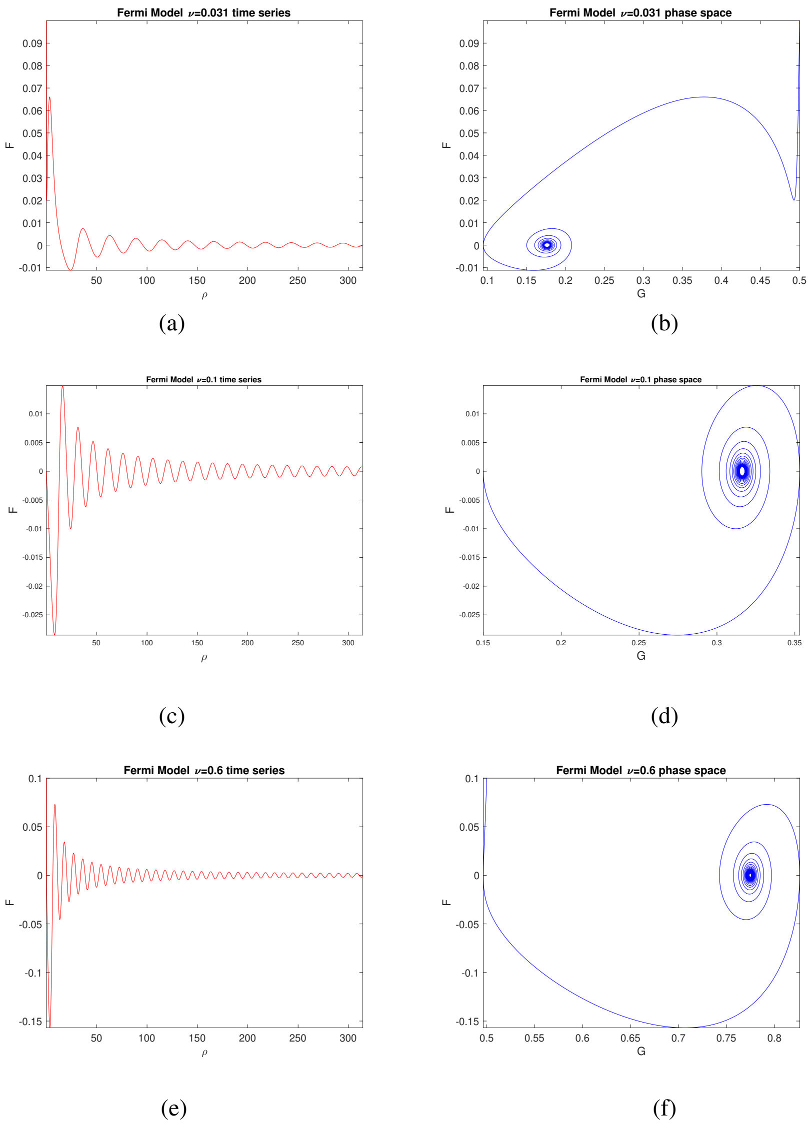

Fig. 1: (a), (b) Fermi Model spatial-temporal evolution and phase space for $\nu = 0.031$, respectively; (c) and (d) spatial-temporal evolution and phase space for $\nu = 0.1$, respectively; (e) and (f) spatial-temporal evolution and phase space for $\nu = 0.6$, respectively

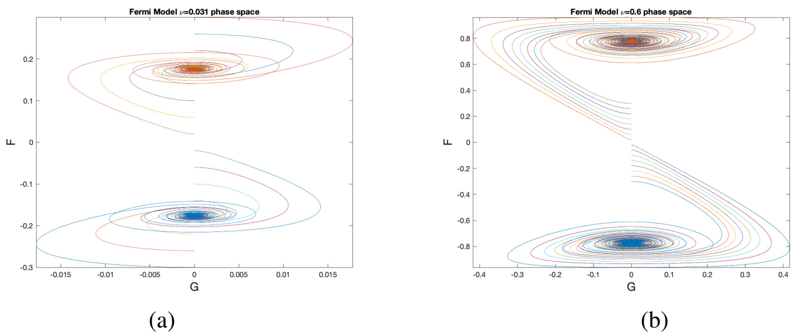

Fig. 2: Fermi Model phase spaces displays for possible ten different initial conditions for a) $\nu = 0.031$ and b) $\nu = 0.6$.

4 Conclusion

Studies on field theories are continuing. The set of formations, which we can call the standard model of particle physics, has led to important developments. Important developments in particle physics have made it important to find large symmetrical physical solutions of nonlinear field equations emerging with models by developing new techniques and to discuss the physical properties and location of these solutions. In this paper, we investigate the nonlinear dynamics of Fermi Model. Firstly, we obtain the dimensionless form of the model from the equation of motion. After that we solve the equation numerically. There are three fix points for the model. There is singularity for the fix points $(F, G) = (0, 0)$ both for $\nu = 0.0031$ and $\nu = 0.6$. The other two fix points are asymptotically stable saddle points in phase space (Fig. 2). According to the phase space displays and spatial-temporal evolution graphics system has damped and the attractor for the system is a sink.

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The Dynamic of Generalized Gursey Model

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Abstract: The spinor field equation proposed by Feza Gursey in 1956 is the first nonlinear spinor wave equation with conformal invariance. Soliton type solutions are found by adding the mass term to the equation for certain values of the coupling constant. Also the system is open to the spin particle structuring outside of fermions. In this study, the generalized version of the Gursey model is given and spatial-temporal evolution and phase spaces plots are constructed to examine the dynamics of the system. Considering the importance of the model in particle physics, chaos analysis techniques were applied to make the dynamics of the nonlinear structure of the system more understandable.

Keywords: spinor, lagrangian, nonlinear dynamics, phase space, Gursey

1 Introduction

After the success in the electron and anti-electron (positron) interpretation of the nonlinear spinor field wave equation was created by Dirac, Heisenberg spent years formulating a "theory of everything" using fermions alone [1, 2]. Later, another attempt in this direction came with the work of Gursey [3]. As a possible basis for a unitary definition of elementary particles, Gursey proposed a new spinor wave equation similar to the nonlinear generalization of Heisenberg's Dirac equation, which exhibits additional invariance with respect to conformal transformations [3]. Gursey model was proposed in 1956 to realize Heisenberg's dream [3]. This system is the first nonlinear spinor wave equation to have conformal symmetry. Due to these properties, Gursey nonlinear spinor wave equation has a wider dynamic symmetry than Dirac equation and the equations proposed by Heisenberg et al. It is also open to other spin particle structuring outside of fermions. For these reasons, the model is very important in particle physics. Due to the importance of the gursey model in particle physics, the generalized version of the system with soliton solutions is discussed in this study. For certain values of coupling constants, soliton-type solutions are found by adding the mass term to the generalized gursey system [4]. It is known that the finite energy, stable wave solutions of the classical equations of motion of Lagrangian field theories are called solitons. Solitons were found in the solution of nonlinear wave equations by applied mathematicians in the 19^{th} century [5, 6]. Solitary waves and solitons arise in a wide range of areas such as particle physics, optics, Bose-Einstein condensates and biological models. Topological classical solutions, including solitons, are classified as fixed, static and dependent on both space and time. Soliton solutions of the expanded form of Gursey spinor equation [7] and Wu-Yang type monopole solutions were found [8].

Since nonlinear equations do not have exact solutions, phase diagrams and strange attractors give information about nonlinear dynamic structure. Numerical methods and the phase spaces of the solutions obtained from these methods are applied to have a view on the dynamics and evolution of nonlinear equation solutions. Therefore, in this study, spatial-temporal evolution and phase spaces structures are applied to understand the dynamics of the nonlinear generalized Gursey equation system.

2 Model

Internal symmetry form for Gursey model was suggested in 1982 and integral of orbit with quantization was also shown for the model [7]. It was realized that the version of applied the Solar ansatz Gursey model with mass term and internal symmetry and the studies before the Gursey model had not separate from the each other. The problem was solved adding the term including the axial symmetry [9]. The Lagrangian for generalized Gursey model is

$$L = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - m\bar{\Psi}\Psi + \lambda_1\left[\left(\bar{\Psi}\gamma^{\mu}\Psi\right)\left(\bar{\Psi}\gamma_{\mu}\Psi\right)\right]^{\frac{2}{3}} + \lambda_2\left[\left(\bar{\Psi}\gamma^{5}\Psi\right)\left(\bar{\Psi}\gamma_{5}\Psi\right)\right]^{\frac{2}{3}}. \quad (1)$$

The Equation of motion for generalized Gursey model is given below,

$$i\gamma^{\mu}\partial_{\mu}\Psi - m\Psi + \frac{4}{3}\lambda_1\frac{\left(\bar{\Psi}\gamma^{\mu}\Psi\right)\gamma_{\mu}\Psi}{\left[\left(\bar{\Psi}\gamma^{\mu}\Psi\right)\left(\bar{\Psi}\gamma_{\mu}\Psi\right)\right]^{\frac{1}{3}}} + \frac{4}{3}\lambda_2\frac{\left(\bar{\Psi}\gamma^{5}\Psi\right)\gamma_{5}\Psi}{\left[\left(\bar{\Psi}\gamma^{5}\Psi\right)\left(\bar{\Psi}\gamma_{5}\Psi\right)\right]^{\frac{1}{3}}} = 0, \quad (2)$$

here λ_1 and λ_2 are coupling constant and fermion field Ψ has scale dimension $\frac{3}{2}$. In order to solve the equation we will use soler ansatzs given in Eq. 3.

$$\Psi = e^{-i\omega t} \begin{bmatrix} g(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ if(r) \begin{pmatrix} cos\theta \\ e^{i\phi} sin\theta \end{pmatrix} \end{bmatrix}, \tag{3}$$

here f and g are real functions of r and $r = x^2 + y^2 + z^2 + t^2$. Fermi model has 4 component time and 3 dimensional spatial dimensions. If we insert Eq. 3 in to Eq. 2 then we found,

$$f' + \frac{2}{r} f + (m - \omega) g - \frac{4}{3} \lambda_2 g \left(g^2 - f^2 \right)^{\frac{1}{3}} \left(1 - \left| \frac{\lambda_1}{\lambda_2} \right|^3 \right)^{\frac{1}{3}} = 0, \tag{4a}$$

$$g' + (m + \omega) f - \frac{4}{3} \lambda_2 f \left(g^2 - f^2 \right)^{\frac{1}{3}} \left(1 - \left| \frac{\lambda_1}{\lambda_2} \right|^3 \right)^{\frac{1}{3}} = 0. \tag{4b}$$

Where $|\lambda_2| = \frac{1}{(2)^{\frac{1}{3}}} |\lambda_1|$, $\lambda = \frac{4}{3}\lambda_2$. The equations become,

$$f' + \frac{2}{r} f + (m - \omega) g + \lambda g \left(g^2 + f^2 \right)^{\frac{1}{3}} = 0 \tag{5a}$$

$$g' + (m + \omega) f + \lambda f \left(g^2 + f^2 \right)^{\frac{1}{3}} = 0. \tag{5b}$$

Numerical calculations we transform the Eq. 5a and 5b dimensionless for by using,

$$\begin{aligned} f(r) &= (m + \omega)^{\frac{3}{2}} F(\rho) \\ g(r) &= (m + \omega)^{\frac{3}{2}} G(\rho) \\ r &= \frac{\rho}{m + \omega}. \end{aligned}$$

Finally we obtain dimesionless form of the fermi model given below,

$$F' + \frac{2}{\rho} F + \nu G + \lambda_1 G \left(G^2 + F^2 \right)^{\frac{1}{3}} = 0 \tag{6a}$$

$$G' + F + \lambda_2 F \left(G^2 + F^2 \right)^{\frac{1}{3}} = 0. \tag{6b}$$

3 Numerical Results

We fixed the $\lambda_1 = 0.8$ and $\lambda_2 = -0.1$. System fix points are $(F, G) = (0, 0.0441942); (0, 0); (0, -0.0441942)$ for $\nu = 0.1$ and $(F, G) = (0, 0.649519); (0, 0); (0, -0.649519)$ for $\nu = 0.6$, respectively. The Jacobian matrix for the system is

$$J = \begin{bmatrix} -\frac{2}{r} + \frac{2FG\lambda_1}{3(F^2+G^2)^{\frac{3}{2}}} & \frac{2G^2\lambda_1}{3(F^2+G^2)^{\frac{3}{2}}} + (F^2 + G^2)^{\frac{1}{3}} \lambda_1 - \nu \\ -1 + \frac{2F^2\lambda_2}{3(F^2+G^2)^{\frac{3}{2}}} + (F^2 + G^2)^{\frac{1}{3}} \lambda_2 & \frac{2FG\lambda_2}{3(F^2+G^2)^{\frac{3}{2}}} \end{bmatrix}. \tag{7}$$

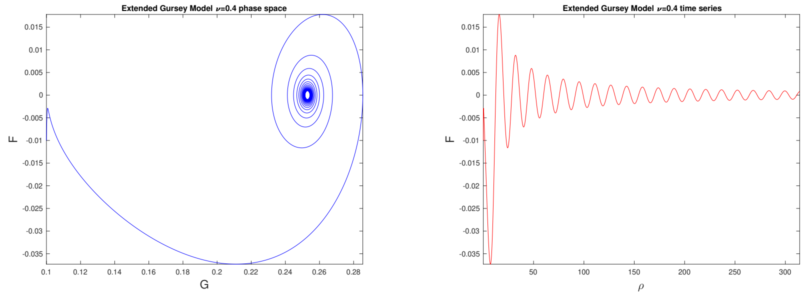
The Eigenvalues of the system are given below for both $\nu 0.031$ and $\nu 0.6$,

	fix points	λ_1	λ_2
$\nu = 0.1$	$(0, 0.0441942)$	$\frac{0.03(-29.62-\sqrt{877.91-59.25r^2})}{r}$	$\frac{0.03(-29.62+\sqrt{877.91-59.25r^2})}{r}$
	$(0, 0)$	Indeterminate	Indeterminate
	$(0, -0.0441942)$	$\frac{0.03(-29.62-\sqrt{877.91-59.25r^2})}{r}$	$\frac{0.03(-29.62+\sqrt{877.91-59.25r^2})}{r}$
$\nu = 0.6$	$(0, 0.774597)$	$\frac{0.28(-3.51-\sqrt{12.36-7.03r^2})}{r}$	$\frac{0.28(-3.51+\sqrt{12.36-7.03r^2})}{r}$
	$(0, 0)$	Indeterminate	Indeterminate
	$(0, -0.774597)$	$\frac{0.28(-3.51-\sqrt{12.36-7.03r^2})}{r}$	$\frac{0.28(-3.51+\sqrt{12.36-7.03r^2})}{r}$

Table 1 Eigenvalues for $\nu = 0.031$.

According to Table 1, the system has singularity for $r = 0$. In addition, there is no eigenvalue for $(F, G) = (0, 0)$ due to singularity for both $\nu = 0.1$ and 0.6. All eigenvalues are positive and negative real number for all $r > 0$. We solve Eq. 6a and 6b Runge-Kutta method by using matlab. We fixed the $\lambda_1 = 0.8$ and $\lambda_2 = -0.1$ and initial conditions $(F(0); G(0)) = (-0.01; 0.1)$. We evaluate the system 0.1 to 400 with step size 0.001 both $\nu = 0.1$, $\nu = 0.4$ and $\nu = 0.6$. According to the phase space displays and spatial-temporal evolution graphics system has damped and the attractor for the system is a sink.

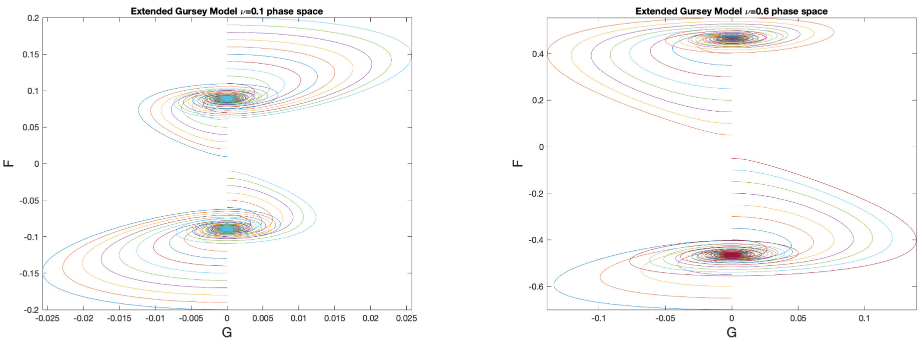
(a) (b)



(c) (d)

(e) (f)

Fig. 1: (a), (b) Generalized Gursey Model spatial-temporal evolution and phase space for $\nu = 0.1$, (c) and (d) spatial-temporal evolution and phase space for $\nu = 0.4$ respectively; (e) and (f) spatial-temporal evolution and phase space for $\nu = 0.6$, respectively



(a) (b)

Fig. 2: Generalized Gursey Model phase spaces displays for possible ten different initial conditions for a) $\nu = 0.1$ and b) $\nu = 0.6$..

4 Conclusion

In this study, the dynamical structuring and spatial evolution of Soler soliton solutions of the generalized version of the four-dimensional pure spinor conformal invariant nonlinear Gursey wave were examined according to the system parameters. Firstly, we obtain the dimensionless form of the model from the equation of motion. After that we solve the equation numerically and phase diagrams expressing the spatial evolution were drawn. There are three fix points for the model. There is singularity for the fix points $(F, G) = (0, 0)$ both for $\nu = 0.1$ and $\nu = 0.6$. The other two fix points are asymptotically stable saddle points in phase space (Fig. 2). . Considering the importance of the model in particle physics, we think that this study is important to understand the dynamics of the nonlinear structure of the system. According to the phase space displays and spatial-temporal evolution graphics system has damped and the attractor for the system is a sink.

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An approximate analytical approach for immiscible process in the porous medium with inclination impact

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Abstract: Examining the impact of slope on immiscible phase flow in a porous media is the main objective of the paper. By analyzing the governing equation using the variational iteration method, the saturation profile in the fingering occurrence corresponds chances of getting a solution. The effect on concentration rate, which has been explored at a different scale, is shown by parametric values. Maple software has been used to create tabular and graphical representations.

Keywords: Capillary pressure, Equation of continuity, Fractured porous media, Variational Iteration Method (VIM).

1 Introduction

Imbibition is a phenomenon in which a soaking phase spontaneously infuses into a porous medium, swamping the non-wetting phase and causing a counterflow of the resident fluid from the medium to the wetting phase. This procedure emerged as a significant recovery mechanism while water is flooded into damaged reservoirs containing heterogeneous porous medium. Tavassoli, Zimmerman and Blunt [1] invested an approximate solution for the flood hydrocarbon reservoirs with water or to remove non-aqueous phase liquid (NAPL) with water. For one-dimensional flow, they determined the recovery of the non-wetting phase as a function of time. Zimmerman and Bodvarsson [2] approached an approximative closed-form solution for the one-dimensional hydration in an unsaturated porous medium with van Genuchten type model parameters which is obtained using the "boundary layer" or "integral" technique. Mirzaei-Paiaman [3] investigates numerical simulation tests which demonstrate that there are considerable variations between Counter-current spontaneous imbibition with and without the gravitational forces in terms of the final recovery and imbibition velocity.

Many researchers have a huge interest in fluid flow problem which is based on real-world problems. These can be investigated with the help of different numerical or analytical methods. In many oil recovery process problems, the adomain decomposition method was used, for these problems, an analytical solution for the saturation rate as well as the recovery rate for various models is described [4], for the Fingero-Imbibition occurrence [5], the counter-current imbibition occurrence in heterogeneous porous media with gravitational and inclination effect [6], or with various porous materials [7]. The homotopy analysis method has been used for problems like a motion of immiscible fluids with some inclination effect [8], fluid flow across fractured porous media of different porous materials [9], the restrained invertible issue predicated on the nonlinear convection-diffusion equation in the multiphase porous media ïňĆows [10], counter-current imbibition phenomenon in a heterogeneous [11] and homogeneous [12] porous medium which is further investigated by optimal homotopy analysis method [13,14], one-dimensional groundwater recharge phenomenon [15], finite difference method [16] is studied to get a numerical solution for imbibition phenomenon in porous media for double phase flow.

The variational iteration method (VIM) is an approximate analytical method that is used many times to solve a nonlinear ordinary and partial differential equation. The VIM differs from all the other perturbation and non-traditional perturbation methods in that it provides a solution in a few iterations, and hence it is being used to solve nonlinear problems. The VIM has been successfully used to solve a wide variety of nonlinear problems, as well as with nonlinear Newell-Whitehead-Segel equation[17], nonlinear equation in one-dimensional destabilization occurrence in homogeneous porous media in horizontal direction [20], solution for the Glioblastoma Tumor Cells Growth in Homogeneous Medium[18], space and time fraction KdV equation [19] and for solving Burger's equations [21,22].

Figure (1) depicts the structure of the fingering occurrence. Throughout this analysis, we implemented some slope and gravitational effects to identify an analytical solution to the nonlinear partial differential equation that emerges during the fingering occurrence in a porous medium. We also discussed how the inclined surface, capillary pressure, and relative permeability including both phases in the fluid reservoir affect the saturation rate for the fingering occurrence.

2 Formulation of the model

Consequently, this was described as a cylindrical frame with a length L_0 that has all surfaces encased in a contamination surface apart from one open end. The cylindrical slab is slanted at an angle with the ground surface as indicated in figures 1 and the imbibition surface is represented by face $\alpha = 0$. In addition, it is assumed that the presence of magnetic fluid components in the injected fluids results in a low capillary

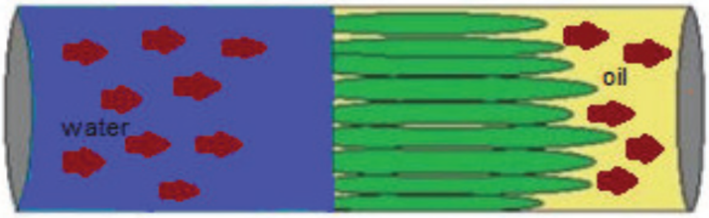

Fig. 1: Schematic representation of the fingering occurance.

pressure during another wetting phase and a thin layer of magnetic field on the surface. The counter-current imbibition exists also in the line of the interface because of the low capillary pressure of the wetting phase. The conservation of mass equation is given below for this model [23].

$$\phi \frac{\partial M_w}{\partial t} + \frac{\partial n_w}{\partial x} = 0 \tag{1}$$

$$\phi \frac{\partial M_0}{\partial t} + \frac{\partial n_0}{\partial x} = 0 \tag{2}$$

According to Darcy's law, the speeds of oil and water are

$$n_w = -\frac{AA_w}{\mu_w}\left(\frac{\partial p_w}{\partial x} - \rho_w g \sin\alpha\right) \tag{3}$$

$$n_0 = -\frac{AA_0}{\mu_0}\left(\frac{\partial p_0}{\partial x} - \rho_0 g \sin\alpha\right) \tag{4}$$

The difference in pressure at the interface of two immiscible fluids, such as oil and water in the reservoir, is known as capillary pressure p_c. The value of p_c is only because capillary pressure seems to have an effect on the distribution of fluids.

$$p_c = p_0 - p_w \tag{5}$$

Due to phase saturation, we get

$$M_w + M_0 = 1 \tag{6}$$

As we use Equations (3) and (4) to modify the value of nw in Equations (1) and (2), we obtain

$$\phi \frac{\partial M_w}{\partial t} = \frac{\partial}{\partial x}\left|\frac{AA_w}{\mu_w}\left(\frac{\partial p_w}{\partial x} - \rho_w g \sin\alpha\right)\right| \tag{7}$$

$$\phi \frac{\partial M_0}{\partial t} = \frac{\partial}{\partial x}\left[\frac{AA_0}{\mu_0}\left(\frac{\partial p_0}{\partial x} - \rho_0 g \sin\alpha\right)\right] \tag{8}$$

Integrating Equation (7) and (5)

$$\phi \frac{\partial M_w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{AA_w}{\mu_w}\left(\frac{\partial p_0}{\partial x} - \frac{\partial p_c}{\partial x} - \rho_w g \sin\alpha\right)\right] \tag{9}$$

now, Equation (9), (8) and (6), we get as:

$$\frac{\partial}{\partial x}\left[\frac{AA_w}{\mu_w}\left(\frac{\partial p_0}{\partial x} - \frac{\partial p_c}{\partial x} - \rho_w g \sin\alpha\right)\right] + \frac{\partial}{\partial x}\left[\frac{AA_0}{\mu_0}\left(\frac{\partial p_0}{\partial x} - \rho_0 g \sin\alpha\right)\right] = 0$$

$$\Longrightarrow A\left[\frac{\partial p_0}{\partial x}\left(\frac{A_w}{\mu_w} + \frac{A_0}{\mu_0}\right) - \frac{A_w}{\mu_w}\frac{\partial p_c}{\partial x} - g\sin\alpha\left(\frac{A_w}{\mu_w}\rho_w + \frac{A_0}{\mu_0}\rho_0\right)\right] = C \tag{10}$$

$$\frac{\partial p_0}{\partial x} = \frac{C + K\frac{A_w}{\mu_w}\frac{\partial p_c}{\partial x} - g\sin\alpha A\left(\frac{A_w}{\mu_w}\rho_w + \frac{A_0}{\mu_0}\rho_0\right)}{A\left(\frac{A_w}{\mu_w} + \frac{A_0}{\mu_0}\right)} \tag{11}$$

Here,C = Integration constant. Integrating Equations (11) and (9), and then we get that

$$\phi \frac{\partial A_w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{A_w}{\mu_w}A\left(\frac{C + A\frac{A_w}{\mu_w}\frac{\partial p_c}{\partial x} - g\sin\alpha A\left(\frac{A_w}{\mu_w}\rho_w + \frac{A_0}{\mu_0}\rho_0\right)}{A\left(\frac{A_w}{\mu_w} + \frac{A_0}{\mu_0}\right)} - \frac{\partial p_c}{\partial x} - \rho_w g \sin\alpha\right)\right] \tag{12}$$

Also we have

$$p_0 = \mathcal{P} + \frac{p_c}{2}, where \mathcal{P} = \frac{1}{2}(p_0 + p_w) \tag{13}$$

where $\mathcal{P} = \frac{1}{2}(p_0 + p_w)$ says about a mean pressure. By integrating equation (13) and (10)

$$C=\frac{1}{2}A\frac{A_0}{\mu_0}\frac{\partial p_c}{\partial x}-\frac{1}{2}A\frac{A_w}{\mu_w}\frac{\partial p_c}{\partial x}-g\sin\alpha A\left(\frac{A_w}{\mu_w}\rho_w+\frac{A_0}{\mu_0}\rho_0\right) \tag{14}$$

After simplified and swapping the value of C from equation (14) into equation (12), we inevitably obtain at as the equation of continuity in a porous matrix.

$$\frac{\partial M_w}{\partial t}\phi+\frac{\partial}{\partial x}\left(\frac{A}{2}\frac{A_w}{\mu_w}\frac{\partial p_c}{\partial x}+gA\frac{A_w}{\mu_w}\rho_w\sin\alpha\right)=0 \tag{15}$$

Now, we use [24] as our frame of reference for the capillary pressure and relative permeability function for water.

$$p_c=\left(M_w^{\frac{-1}{2}}-C_1\right)\beta \tag{16}$$

$$A_w=M_w^3 \tag{17}$$

The following results are obtained from equation (15) after entering all the values from equation (16)

$$\frac{\partial M_w}{\partial t}+\frac{A}{\phi\mu_w}\frac{\partial}{\partial x}\left(-\frac{\beta}{4}M_w^{\frac{3}{2}}\frac{\partial M_w}{\partial x}+M_w^3\rho_w g\sin\alpha\right)=0 \tag{18}$$

The dimensionless variable is used to make the aforementioned equation dimensionless.

$$X=\frac{x}{L}\qquad and \qquad T=\frac{A\beta}{\phi\mu_w L^2}t$$

The dimensionless equation is now transformed as:

$$\frac{\partial M_w}{\partial T}-\frac{1}{4}\frac{\partial}{\partial x}\left(M_w^{\frac{3}{2}}\frac{\partial M_w}{\partial X}\right)+B\sin\alpha S_w^2\frac{\partial M_w}{\partial X}=0 \tag{19}$$

$$where \qquad B=\frac{3\rho_w g}{\beta}$$

The movement of saturation of the fluid medium in a homogeneous porous material with an elevation and gravitational action is described by the equation (19).

3 Variational Iteration Method

The approach of He's Variational Iteration Method [25-31] can be described by assuming the non-linear partial differential equation.

$$L\left(M(x,t)\right)+N\left(M(x,t)\right)+R\left(M(x,t)\right)=f(x) \tag{20}$$

M(x,0)=g(x)

where $f(x)$ depicts an inhomogeneous term, $L(x)$ depicts a linear term, $N(x)$ depicts a non-linear term, and R denotes a linear operator with partial derivatives with respect to x. We may put up a correction functional for the equation(4) in accordance with the method:

$$M_{n+1}(x)=M_n(x)+\int_0^x\lambda(x)\left[L(M_n(x))+R(\overbrace{M_n}(x))+N(\overbrace{M_n}(x))-f(x)\right]d\tau \tag{21}$$

The VIM can be recognized by the λ and λ is a Lagrange multiplier [32,33]. $M_0(x)$ signifies an initial approximation with unknowns, H_n implies the nth approximation, and $\overbrace{M_n}$ is in view of a restriction variation i.e. $\delta\overbrace{M_n}=0$. The Lagrange Multiplier and the beginning approximation M_0 can easily determine the solution M's consecutive approximation $M_{n+1}, n\geq 0$ of the solution $M=\lim_{n\to+\infty}M_n$. Equation (5) can be used to solved iteratively using $M_0(x,t)=g(x)$.

4 Application of method

A proposed method is applied to solve Equation(19) with the initial condition equation(20) is taken.

Applying VIM in the above equation,

$$(M_w)_{n+1}=(M_w)_n+\int_0^x\lambda(x)\left[\frac{\partial(M_w)_n}{\partial T}-\frac{1}{4}\frac{\partial}{\partial x}\left((M_w^{\frac{3}{2}})_n\frac{\partial(M_w)_n}{\partial X}\right)+A\sin\alpha(M_w^2)_n\frac{\partial(M_w)_n}{\partial X}\right] \tag{22}$$

Its stationary condition will come as follow:

$$\lambda^{'}(\tau)=0$$

$$1+\lambda(\tau)|_{\tau=t}=0$$

The Lagrange Multiplier will be recognized as the $\lambda=-1$. Consequently, the correction functional will be revised and the initial condition (25) will be treated as the initial estimate

$$(M_W)_0 = e^{-X}$$

After using the initial approximation in equation (26) we can get our first approximation solution:

$$(M_w)_1 = \mathrm{e}^{-X} + (5/8)\,T\mathrm{e}^{-(5/2)\,X} + A\sin(\alpha)\,\mathrm{e}^{-2\,X}T$$

then second approximation or iteration solutions obtained as:

$$\begin{aligned}(M_w)_2 =& 5/8\,\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}A^2(\sin(a))^2\,\mathrm{e}^{-4\,X}T^2 + \frac{45\,A\sin(a)\,\mathrm{e}^{-9/2\,X}T}{16}\\
&+ 4\,A^2(\sin(a))^2\,\mathrm{e}^{-4\,X}T + \frac{75\,A\sin(a)\,\mathrm{e}^{-6\,X}T^2}{32} + 5\,A^3(\sin(a))^3\,\mathrm{e}^{-5\,X}T^2\\
&+ \frac{175\,A^2(\sin(a))^2\,T^3\mathrm{e}^{-7\,X}}{64} + \frac{65\,A^3(\sin(a))^3\,T^3\mathrm{e}^{-13/2\,X}}{16} + \mathrm{e}^{-X} - 5/8\,\mathrm{e}^{-5/2\,X}\\
&+ \frac{5\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}\mathrm{e}^{-2\,X}}{32} + 5/8\,T\mathrm{e}^{-5/2\,X} - A\sin(a)\,\mathrm{e}^{-2\,X}\\
&+ \frac{3125\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}T^2\mathrm{e}^{-5\,X}}{8192} + A\sin(a)\,\mathrm{e}^{-3\,X}\\
&+ \frac{11\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}\mathrm{e}^{-3\,X}A\sin(a)\,T}{16}\\
&+ \frac{505\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}T^2\mathrm{e}^{-9/2\,X}A\sin(a)}{512} + \frac{55\,A^2(\sin(a))^2\,\mathrm{e}^{-11/2\,X}T^2}{8}\\
&+ A\sin(a)\,\mathrm{e}^{-2\,X}T + \frac{295\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}\mathrm{e}^{-7/2\,X}T}{512}\\
&+ \frac{625\,A\sin(a)\,T^3\mathrm{e}^{-15/2\,X}}{1024} + 2\,A^4(\sin(a))^4\,\mathrm{e}^{-6\,X}T^3\end{aligned}$$

$$\vdots$$

and so on. Hence, we can get the approximate analytical solution for equation ( )is obtained as:

$$\begin{aligned}M_w =& 5/8\,\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}A^2(\sin(a))^2\,\mathrm{e}^{-4\,X}T^2 + \frac{45\,A\sin(a)\,\mathrm{e}^{-9/2\,X}T}{16}\\
&+ 4\,A^2(\sin(a))^2\,\mathrm{e}^{-4\,X}T + \frac{75\,A\sin(a)\,\mathrm{e}^{-6\,X}T^2}{32} + 5\,A^3(\sin(a))^3\,\mathrm{e}^{-5\,X}T^2\\
&+ \frac{175\,A^2(\sin(a))^2\,T^3\mathrm{e}^{-7\,X}}{64} + \frac{65\,A^3(\sin(a))^3\,T^3\mathrm{e}^{-13/2\,X}}{16} + \mathrm{e}^{-X} - 5/8\,\mathrm{e}^{-5/2\,X}\\
&+ \frac{5\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}\mathrm{e}^{-2\,X}}{32} + 5/8\,T\mathrm{e}^{-5/2\,X} - A\sin(a)\,\mathrm{e}^{-2\,X}\\
&+ \frac{3125\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}T^2\mathrm{e}^{-5\,X}}{8192} + A\sin(a)\,\mathrm{e}^{-3\,X}\\
&+ \frac{11\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}\mathrm{e}^{-3\,X}A\sin(a)\,T}{16}\\
&+ \frac{505\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}T^2\mathrm{e}^{-9/2\,X}A\sin(a)}{512} + \frac{55\,A^2(\sin(a))^2\,\mathrm{e}^{-11/2\,X}T^2}{8}\\
&+ A\sin(a)\,\mathrm{e}^{-2\,X}T + \frac{295\sqrt{16\,\mathrm{e}^{-X} + 10\,T\mathrm{e}^{-5/2\,X} + 16\,A\sin(a)\,\mathrm{e}^{-2\,X}T}\mathrm{e}^{-7/2\,X}T}{512}\\
&+ \frac{625\,A\sin(a)\,T^3\mathrm{e}^{-15/2\,X}}{1024} + 2\,A^4(\sin(a))^4\,\mathrm{e}^{-6\,X}T^3 + ...\end{aligned}$$

5 Results and Discussions

The effects of taking inclination and concentration rates into consideration on the fingering phenomena in a porous matrix have been examined. The varying parameters that were employed are shown in Table 1. The numerical results for concentration rates at various inclined planes are

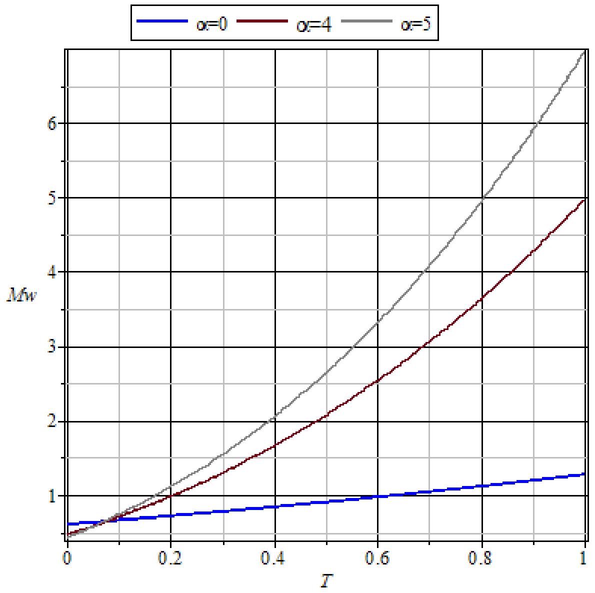

Fig. 2: Concentration rates for the fingering occurrence for various inclination angle α for $X = 0.1$.

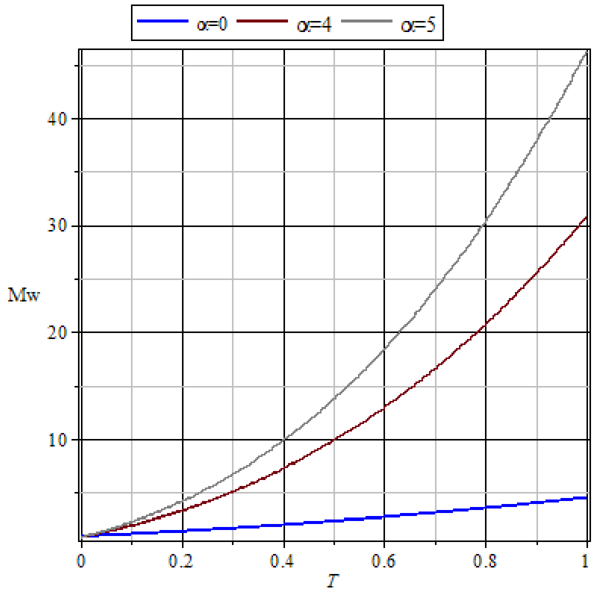

Fig. 3: Concentration rates for the fingering occurrence for various inclination angle α for $X = 0.5$

shown in Table 2. Figure 2 displays the rate of water saturation in the porous medium at different inclined angles at the entry, X = 0.1 and X = 0.5. The results show that as the angle of inclination increases, the water saturation rate will decrease and be greater at the zero inclined plane close to the inlet X = 0.1. Figures 3 and 4 illustrate the effect of the concentration rates in relation to dimensionless time, indicating that the concentration rates will be greater for dimensionless time $T = 1$.

Parameters	Values
β	$6895N/m^2$
g	$9.8m/s^2$
ρ_w	$1000kg/m^2$
μ_w	$0.894Nm/s^2$
ρ_0	$982kg/m^3$
K	$10^{-12}m^2$

Table 1 The numerical values used in this calculation are given as follows

	X=0.1			
T	$\alpha = 0$	$\alpha = 5$	$\alpha = 10$	$\alpha = 15$
0.1	1.134059084	2.230216575	1.611639529	0.973513334
0.2	1.389935531	4.161167501	2.563324695	0.976803266
0.3	1.673519932	6.674033205	3.724841844	0.978774502
0.4	1.985829554	9.838028536	5.114226127	0.979455218
0.5	2.327849035	13.72111949	6.749103249	0.978873706
0.6	2.700533215	18.39024481	8.646749358	0.97705838
0.7	3.104809597	23.9114797	10.82413785	0.974037784
0.8	3.541580479	30.35016046	13.29797666	0.969840593
0.9	4.011724845	37.7709821	16.08473874	0.964495639
1	4.516100026	46.23807626	19.20068692	0.958031867

Table 2 Comparison of the concentration rates for the fingering occurrence at $X = 0.1$.

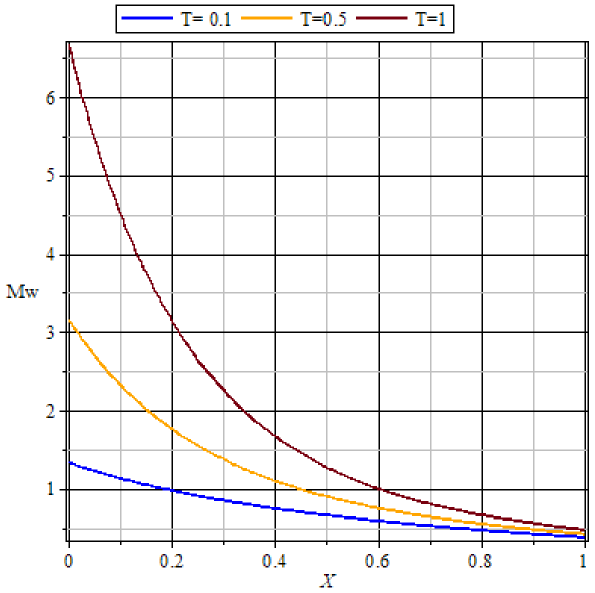

Fig. 4: Concentration rates for the fingering occurrence at $\alpha = 0$
.

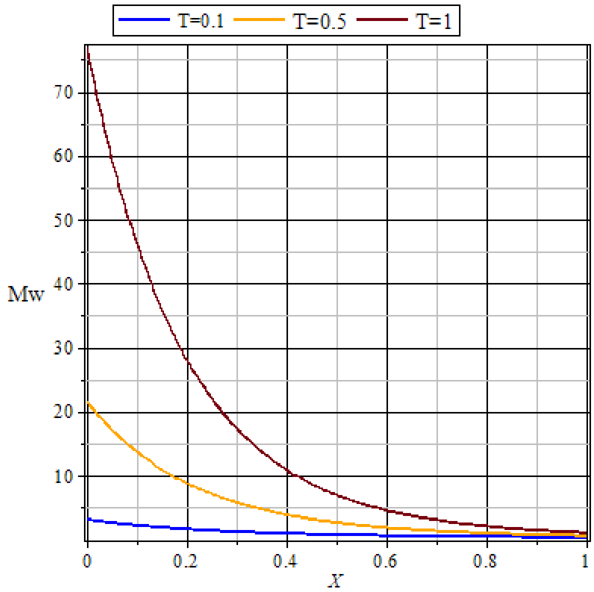

Fig. 5: Concentration rates for the fingering occurrence at $\alpha = 5$

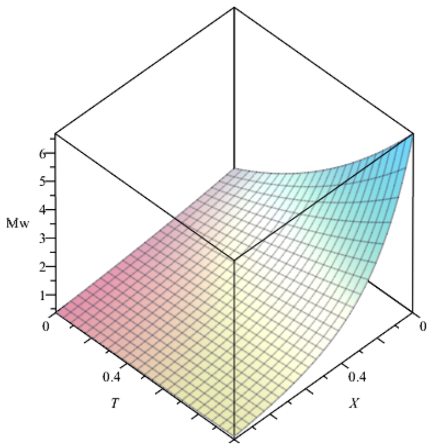

Fig. 6: 3D behaviour of the concentration rates for the fingering occurrence at $\alpha = 0$.

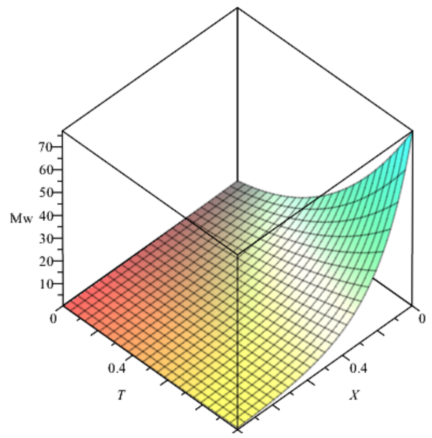

Fig. 7: 3D behaviour of the concentration rates for the fingering occurrence at $\alpha = 5$

	X=0.5			
T	$\alpha = 0$	$\alpha = 5$	$\alpha = 10$	$\alpha = 15$
0.1	0.659698629	0.736652051	0.677681198	0.715372382
0.2	0.715766235	1.104620484	0.8714704	0.705042806
0.3	0.774799108	1.540678479	1.089781041	0.69452472
0.4	0.836861564	2.050405004	1.333999803	0.683826017
0.5	0.902016685	2.639288944	1.605487911	0.672954537
0.6	0.970326375	3.312741245	1.905583819	0.661918079
0.7	1.041851426	4.076104612	2.235605441	0.650724381
0.8	1.116651573	4.934661371	2.596852076	0.639381133
0.9	1.194785543	5.893639946	2.990606043	0.627895969
1	1.276311102	6.958220277	3.418134076	0.616276463

Table 3 Comparison of the concentration rates for the fingering occurrence at $X = 0.5$.

6 Conclusion

Throughout this work, we presented an analytical solution for the wetting phase concentration profile during the fingering occurrence using the variational iteration method. The variation of the water concentration profile with dimensionless length (X) and dimensionless time is also discussed (T). Since the zero inclined planes is physically comparable to the occurrence in real life, we can conclude that the concentration rates are higher there. The precision and efficiency of VIM have been illustrated graphically as well as through tables to verify the method's accuracy and efficiency, which is extremely effective when dealing with highly nonlinear challenges. The VIM has the capability of giving an iterative solution utilizing rapidly convergent consecutive approximations. For nonlinear operators, the VIM has no objective requirement, such as linearization, limited parameters, Adomian polynomials, and many more.

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Weakly nonlinear analysis of combined effect of g-jitter and thermal difference on a Rivlin-Ericksen nanofluid in Hele-Shaw cell

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Abstract: We have studied the heat and mass transfer phenomena by using weakly non-linear stability analysis on the combined effect of g-jitter and thermal difference of a Rivlin-Ericksen nano-fluid in Hele-Shaw cell, when there is temperature difference between fluid and nano-particle. For non-linear stability analysis, we used double Fourier series technique for studying amount of heat and mass transfer. From analysis it is concluded that convection sets in earlier for in situation of temperature difference as compared to similar temperature between fluid and nano-particles.

Keywords: Gravity modulation, Hele-Shaw cell, Non-linear instability, Rivlin-Ericksen nanofluid, Thermal difference.

1 Introduction

Revlin-Ericksen fluid, one type of viscoelastic fluid, is theoretically introduced by Rivilin-Ericksen [1]. Srivastava [2] examined the unsteady flow of Rivlin-Ericksen fluid with uniform distribution of dust particles through channels of different cross sections in the presence of time dependent pressure gradient. Bhadauria et al. [3] examined the effect of gravity modulation of thermal instability in a Revlin-Ericksen fluid Saturated anisotropic porous medium. They have used Hill's equation and the Floquet theory, for obtaining the convective threshold. It is found that gravity modulation can significantly affect the stability limits of the system. Rana and Kumar [4] studied the thermal instability of a Rivlin-Ericksen elastico-viscous rotating fluid permeated with suspended particles and variable gravity field in porous medium. These investigation of gravity modulation of thermal instability with Rivlin-Ericksen simple fluid. Sheu [5] investigated the linear stability of convection in a viscoelastic nanofluid layer. He used Oldroyd B model to describe the rheological behavior of a viscoelastic nanofluid. The model used for the nanofluid is incorporated the effects of Brownian motion and thermophoresis. They showed that there was competition among the processes of thermophoresis, Brownian diffusion, and viscoelasticity which caused oscillatory rather than stationary convection to occur. Rana and Thakur [6], investigated the effect of suspended particles on thermal convection in Rivlin-Ericksen elastico-viscous fluid in a Brinkman porous medium. Chand and Rana [7] studied the thermal instability of Rivlin-Ericksen elastico-viscous nanofluid saturated by a porous medium. Chand et al. [8] investigated the revised model of thermal instability in a Rivlin-Ericksen elastico-viscous nanofluid in a porous medium. Rana et al. [9] studied the more realistic model of thermal instability of a Rivlin-Ericksen nanofluid saturated by a Darcy-Brinkman porous medium. Saini and Sharma [10] studied the effect of vertical throughflow in Rivlin-Ericksen elastico-viscous nanofluid in a non-Darcy porous medium.

A situation when a vessel containing a heavy liquid vibrates vertically with constant frequency and amplitude, known as Gravity modulation. Benjamin and Ursell [11], investigated the effect of gravity modulation on the existence of standing waves on the free surface of a liquid in a vessel. They used Mathieu's equation to discuss the stability criteria. The stability of a horizontal layer of fluid heated from above or below is examined by Gresho and Sani [12] for the case of a time-dependent buoyancy force which is generated by shaking the fluid layer, thus causing a sinusoidal modulation of the gravitational field. They found that gravity modulation can significantly affect the stability limits of the system. Several works [13]-[17] have been devoted for analysing the linear and non-linear stability analysis of onset of convection under the influence of gravity modulation. Out of many, some of closely related work is reported here.

For visualizing the hydrodynamical circulation penetration into earth, a laboratory apparatus was set up and studying the two dimensional convection patterns. Firstly, its mathematical representation is given by Hele-Shaw in 1898, [18]. He studied a slow two-dimensional flow in a uniformly porous medium and laminar flow in a narrow slot sandwiched between parallel walls i.e. Hele-Shaw cell. Incipient, Straus [19], investigated thermal instability in a porous medium. He reported that thermal convection is two dimensional and roll convection persists to a Rayleigh number nearly ten times the critical value of Rayleigh number. While calculating the Rayleigh number for onset of convection in Hele-Shaw cell, the conduction heat transfer form walls of Hele-Shaw cell is also taken into account. Lapwood [20], derived the correct Rayleigh number for the thin Hele-Shaw cell also verified with experimental data for critical Rayleigh number equal to $4\pi^2$. In 1976, Hartline and Lister [21], derived the Rayleigh number for the thermal convection in a Hele-Shaw cell with gap d and full width (gap plus walls) Y. They concluded that the system of equations are identical to that describing flow through porous medium for marginal stability. They also calculated the vertical flow velocity.

Most of the above cited papers related to nano-fluid, authors considered the thermal equilibrium between nano particles and fluid particles. In his work, Vadasz explained the higher thermal conductivity of nano-fluids due to thermal differences between the particle and fluid phases. Agrawal et al. in (2014) have been investigated Rayleigh Benard convection in a nano fluid layer using a thermal non-equilibrium model. They considered the temperature difference as local thermal non-equilibrium (LTNE) for fluid and nano particles. But in this paper we considered it temperature difference between nano-fluid and fluid particle because LTNE is standard terminology used for temperature difference between fluid and porous matrix. Here, we investigated the effect of temperature difference between fluid and nano particles and gravity modulation on the heat and mass transfers on a non-Newtonian nano-fluid in Hele-Shaw cell.

2 Problem Definition

Assuming a Rivlin-Ericksen nano-fluid layer of depth d_1 between two free-free permeable boundaries at $\tilde{z}=0$ and $\tilde{z}=d_1$, with temperature on boundaries, denoted by $\tilde{T}_h$ and $\tilde{T}_u$ ($\tilde{T}_h > \tilde{T}_u$) of lower and upper boundaries respectively. We consider such a system, which is extended in x-direction but in the y-direction, it is supposed to be bounded by vertical impermeable boundaries (side walls) at $\tilde{y}=0$ and $\tilde{y}=b\ (<< d_1)$.

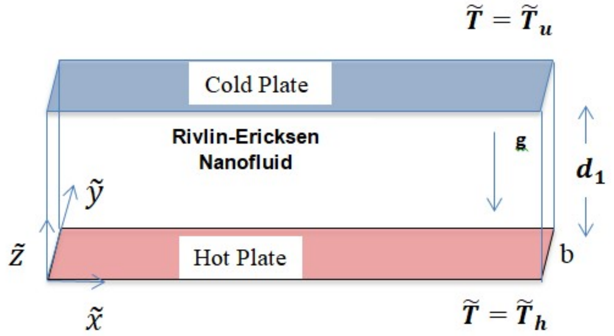


Fig. 1: Physical presentation of the problem

2.1 Mathematical-Equations

For Mathematical equation in a HS-cell filled saturated by nanofluid with porous material, the Brinkman model has been applied. Therefore, the Mathematical equations are as per (Agarwal et al. [22] and Bhadauria and Kumar [24]):

$$\tilde{\nabla}\cdot\tilde{\mathbf{q}} = 0 \tag{1}$$

$$\rho_{nl}\left[\frac{\partial}{\partial \tilde{t}} + (\tilde{\mathbf{q}}\cdot\tilde{\nabla})\right]\tilde{\mathbf{q}} = -\tilde{\nabla}\tilde{p} - \frac{1}{K_{11}}(\mu + \tilde{\mu}\frac{\partial}{\partial \tilde{t}})\tilde{\mathbf{q}} + \mu\tilde{\nabla}^2\tilde{\mathbf{q}} + [\tilde{\phi}\rho_p + \rho_{nl}(1-\tilde{\phi})\{1-\beta(\tilde{T_{nl}} - \tilde{T_u})\}]\mathbf{g} \tag{2}$$

$$\left[\frac{\partial}{\partial \tilde{t}} + (\tilde{\mathbf{q}}\cdot\tilde{\nabla})\right]\tilde{T}_{nl} = \alpha_{nl}\tilde{\nabla}^2\tilde{T}_{nl} + \frac{(\rho c)_p}{(\rho c)_{nl}}[D_B\tilde{\nabla}\tilde{\phi}\cdot\tilde{\nabla}\tilde{T}_{nl} + \left(\frac{D_T}{\tilde{T}_{nl}}\right)\tilde{\nabla}\tilde{T}_{nl}\cdot\tilde{\nabla}\tilde{T}_{nl}] + \frac{h}{(1-\phi_0)(\rho c)_{nl}}(\tilde{T}_p - \tilde{T}_{nl}) \tag{3}$$

$$\left[\frac{\partial}{\partial \tilde{t}} + (\tilde{\mathbf{q}}\cdot\tilde{\nabla})\right]\tilde{T}_p = \frac{k_p}{(\rho c)_p}\tilde{\nabla}^2\tilde{T}_p + \frac{h}{\phi_0(\rho c)_p}(\tilde{T}_{nl} - \tilde{T}_p) \tag{4}$$

$$\left[\frac{\partial}{\partial \tilde{t}} + (\tilde{\mathbf{q}}\cdot\tilde{\nabla})\right]\tilde{\phi} = D_B\tilde{\nabla}^2\tilde{\phi} + \left(\frac{D_T}{\tilde{T}_u}\right)\tilde{\nabla}^2\tilde{T}_{nl} \tag{5}$$

Here volumetric fraction and temperature for nano-particles on the boundaries are assumed as constant. The boundary conditions are

$$\left.\begin{aligned}\tilde{\mathbf{q}} = \mathbf{0},\ \tilde{T}_{nl} = \tilde{T}_h,\ \tilde{T}_p = \tilde{T}_h,\ \tilde{\phi} = \tilde{\phi}_0\ \ at\ \ \tilde{z} = 0,\\ \tilde{\mathbf{q}} = \mathbf{0},\ \tilde{T}_{nl} = \tilde{T}_u,\ \tilde{T}_p = \tilde{T}_u,\ \tilde{\phi} = \tilde{\phi}_1\ \ at\ \ \tilde{z} = d_1.\end{aligned}\right\} \tag{6}$$

where, $\tilde{t}$ is time, $\tilde{\mathbf{q}}$ is Rivlin-Ericksen nanofluid velocity, $\tilde{\phi}$ is nanoparticle-volume fraction, $K_{11} = b^2/12$ is the fluid-flow permeability, D_B is brownian diffusion-coefficient, $\tilde{p}$ is pressure, d_1 is dimensional layer-depth, h is interface heat-transfer coefficient between the fluid and particle phases, D_T is thermophoresis diffusion-coefficient, $(\rho c)_{nl}$ and $(\rho c)_p$ is heat-capacities, β is proportionality-factor, $\tilde{\mu}$ is kinematic visco-elasticity, μ is viscosity, ρ_{nl} is fluid-density, ρ_p is nanoparticle-density, $\tilde{T_{nl}}$ is temperature of fluid, $\tilde{T_p}$ is temperature of nanoparticle, $\mathbf{g}$ is gravitational acceleration[$\mathbf{g} = g_0\mathbf{g_1}, \mathbf{g_1} = (1+\epsilon_1\cos(\Omega t))\hat{\mathbf{e_z}}$], g_0 is mean-gravity, ϵ_1 is amplitude of modulation, Ω is modulating-frequency, ϕ_0 is reference nanoparticle volume-fraction, k_{nl} is effective thermal conductivity of fluid, k_p is effective thermal conductivity of particle, α_{nl} is thermal diffusivity of the Rivlin-Ericksen nanofluid and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is operator.
For non-dimensionlizing the Mathematical equations, the following scale has been utilized:

$$\left.\begin{aligned} \frac{1}{d_1}(\tilde{x},\tilde{y},\tilde{z}) = (x^*,y^*,z^*),\ \ \frac{\alpha_{nl}}{d_1^2}\tilde{t} = t^*,\ \ \alpha_{nl} = \frac{k_{nl}}{(\rho c)_{nl}},\ \ \frac{d_1}{\alpha_{nl}}\tilde{q} = q^*, \\ \frac{d_1^2}{\mu\alpha_{nl}}\tilde{p} = p^*,\ \ \phi^* = \frac{(\tilde{\phi}-\tilde{\phi}_0)}{(\tilde{\phi}_1-\tilde{\phi}_0)},\ \ T_{nl}^* = \frac{(\tilde{T_{nl}}-\tilde{T}_u)}{(\tilde{T}_h-\tilde{T}_u)},\ \ T_p^* = \frac{(\tilde{T}_p-\tilde{T}_u)}{(\tilde{T}_h-\tilde{T}_u)},\ \ d_1\tilde{\nabla} = \nabla^* \end{aligned}\right\} \tag{7}$$

Putting eqn.(7) into the eqns.(1-5), after non-dimensionlizing and dropping $*$ for convenient, the mathematical equations become:

$$\nabla\cdot\mathbf{q} = 0 \tag{8}$$

$$\frac{Hs}{Pr}\left[\frac{\partial}{\partial t}+(\mathbf{q}\cdot\nabla)\right]\mathbf{q} = -Hs\,\nabla p-(1+F\frac{\partial}{\partial t})\mathbf{q}+Hs\,\nabla^2\mathbf{q}-\mathbf{g_1}\,Rm\,\hat{\mathbf{e_z}}+\mathbf{g_1}Rh\,T_{nl}\,\hat{\mathbf{e_z}}-\mathbf{g_1}Rn\,\phi\,\hat{\mathbf{e_z}} \tag{9}$$

$$\left[\frac{\partial}{\partial t}+(\mathbf{q}\cdot\nabla)\right]T_{nl} = \nabla^2 T_{nl}+\left(\frac{N_B}{Le}\right)\nabla\phi\cdot\nabla T_{nl}+\left(\frac{N_A\,N_B}{Le}\right)\nabla T_{nl}\cdot\nabla T_{nl}+N_H(T_p-T_{nl}) \tag{10}$$

$$\left[\frac{\partial}{\partial t}+(\mathbf{q}\cdot\nabla)\right]T_p = \epsilon\nabla^2 T_p+\gamma N_H(T_{nl}-T_p) \tag{11}$$

$$\left[\frac{\partial}{\partial t}+(\mathbf{q}\cdot\nabla)\right]\phi = \left(\frac{1}{Le}\right)\nabla^2\phi+\left(\frac{N_A}{Le}\right)\nabla^2 T_{nl} \tag{12}$$

The boundary conditions become

$$\left.\begin{aligned} \mathbf{q}=\mathbf{0},\ \ T_{nl}=1,\ \ T_p=1,\ \ \phi=0,\ \ at\ z=0, \\ \mathbf{q}=\mathbf{0},\ \ T_{nl}=0,\ \ T_p=0,\ \ \phi=1,\ \ at\ z=1. \end{aligned}\right\} \tag{13}$$

where,

$Rh = \dfrac{\rho_{nl}g_0\beta K_{11}d_1(\tilde{T}_h-\tilde{T}_u)}{\mu\alpha_{nl}}$ is HS-Rayleigh number,
$Rn = \dfrac{(\rho_p-\rho_{nl})(\tilde{\phi}_1-\tilde{\phi}_0)g_0K_{11}d_1}{\mu\alpha_{nl}}$ is nanoparticle-concentration HS-Rayleigh number,
$Rm = \dfrac{\{\rho_p\tilde{\phi}_1-\rho_c(1-\tilde{\phi}_1)\}g_0K_{11}d_1}{\mu\alpha_{nl}}$ is basic density HS-Rayleigh number,
$Pr = \dfrac{\nu}{\alpha_{nl}}$ is Prandtl number,
$Hs = \dfrac{K_{11}}{d_1^2}$ is HS-number,
$Le = \dfrac{\alpha_{nl}}{D_B}$ is Lewis number,
$N_A = \dfrac{D_T}{D_B\tilde{T}_u}\dfrac{(\tilde{T}_h-\tilde{T}_u)}{(\tilde{\phi}_1-\tilde{\phi}_0)}$ is modified diffusivity ratio,
$N_B = \dfrac{(\rho c)_p}{(\rho c)_{nl}}(\tilde{\phi}_1-\tilde{\phi}_0)$ is modified particle density increment,
$F = \dfrac{\tilde{\mu}}{\mu}\dfrac{\alpha_{nl}}{d_1^2}$ is kinematic viscoelasticity permeability,
$N_H = \dfrac{hd_1^2}{(1-\phi_0)k_{nl}}$ is the Nield number or the interphase heat transfer parameter,
$\gamma = \dfrac{(1-\phi_0)(\rho c)_{nl}}{\phi_0(\rho c)_p}$ is the modified thermal capacity ratio and
$\epsilon = \dfrac{k_p(\rho c)_{nl}}{k_{nl}(\rho c)_p}$ is the thermal diffusivity ratio.

2.2 *Basic-Solution*

At the basic state the quantities depends on z only:

$$\mathbf{q}=\mathbf{0},\ \phi=\phi_b(z),\ p=p_b(z),\ T_{nl}=T_{nlb}(z),\ T_p=T_{p_b}(z) \tag{14}$$

Applying eqn.(14) into eqns.(8-12), we obtain:

$$\frac{d^2T_{nlb}}{dz^2} = 0 \tag{15}$$

$$\frac{d^2T_{p_b}}{dz^2} = 0 \tag{16}$$

$$\frac{d^2\phi_b}{dz^2}+N_A\frac{d^2T_{nlb}}{dz^2} = 0 \tag{17}$$

Applying the boundary conditions eqn.(13) in eqns.(15-17), we get the basic state solutions as:

$$T_{nlb} = 1 - z \tag{18}$$

$$T_{p_b} = 1 - z \tag{19}$$

$$\phi_b = z \tag{20}$$

3 Stability Exploration

3.1 Perturbation-State

Now, we superimposed very small perturbation on the basic state, then

$$\mathbf{q} = \mathbf{0} + {\epsilon_1}'\mathbf{q}', \;\; p = p_b + {\epsilon_1}'p', \;\; T_{nl} = T_{nlb} + {\epsilon_1}'T'_{nl}, \;\; T_p = T_{p_b} + {\epsilon_1}'T'_p, \;\; \phi = \phi_b + {\epsilon_1}'\phi'. \tag{21}$$

Applying eqn.(21) into the eqns.(8-12), we get:

$$\nabla \cdot \mathbf{q} = 0 \tag{22}$$

$$\frac{Hs}{\mathrm{Pr}}\frac{\partial \mathbf{q}'}{\partial t} = -Hs\,\nabla p' - (1 + F\frac{\partial}{\partial t})\mathbf{q}' + Hs\,\nabla^2\mathbf{q}' - \mathbf{g_1}\;\left[-Rh\,T'_f\,\hat{\mathbf{e_z}} + Rn\,\phi'\,\hat{\mathbf{e_z}}\right] \tag{23}$$

In view of eliminating pressure-term from eqn.(23), taking curl two-times on both side, then we obtain:

$$\frac{Hs}{Pr}\frac{\partial(\nabla^2 w')}{\partial t} = -\nabla^2 w' - F\frac{\partial(\nabla^2 w')}{\partial t} + Hs\nabla^4 w' + \mathbf{g_1}Rh\nabla_2^2 T'_{nl} - \mathbf{g_1}Rn\nabla_1^2\phi' \tag{24}$$

$$\frac{\partial T'_{nl}}{\partial t} + (\mathbf{q}'\cdot\nabla)T_{nlb} = \nabla^2 T'_{nl} + N_H(T'_p - T'_{nl}) \tag{25}$$

$$\frac{\partial T'_p}{\partial t} + (\mathbf{q}'\cdot\nabla)T_{p_b} = \epsilon\nabla^2 T'_p + \gamma N_H(T'_{nl} - T'_p) \tag{26}$$

$$\frac{\partial \phi'}{\partial t} + (\mathbf{q}'\cdot\nabla)\phi_b = \frac{1}{Le}\nabla^2\phi' + \left(\frac{N_A}{Le}\right)\nabla^2 T'_{nl} \tag{27}$$

Where, $\nabla_2^2 \equiv \nabla^2 - \frac{\partial^2}{\partial z^2}$.

3.2 Non-Linear Exploration

We implement stream function Ψ such that $u = \frac{\partial \Psi}{\partial z}$ and $w = -\frac{\partial \Psi}{\partial x}$ and apply into the eqns.(23,25,26 and 27), then we yield:

$$\frac{Hs}{Pr}\frac{\partial(\nabla^2\Psi)}{\partial t} = Hs\nabla^4\Psi - \nabla^2\Psi - F\frac{\partial(\nabla^2\Psi)}{\partial t} - \mathbf{g_1}Rh\frac{\partial T_{nl}}{\partial x} + \mathbf{g_1}Rn\frac{\partial \phi}{\partial x} \tag{28}$$

$$\frac{\partial T_{nl}}{\partial t} - \frac{\partial \Psi}{\partial x}\frac{dT_{nlb}}{dz} = \epsilon\nabla^2 T_{nl} + \frac{\partial(\Psi, T_{nl})}{\partial(x,z)} + N_H(T_p - T_{nl}) \tag{29}$$

$$\frac{\partial T_p}{\partial t} - \frac{\partial \Psi}{\partial x}\frac{dT_{p_b}}{dz} = \nabla^2 T_p + \frac{\partial(\Psi, T_p)}{\partial(x,z)} + \gamma N_H(T_{nl} - T_p) \tag{30}$$

$$\frac{\partial \phi}{\partial t} - \frac{\partial \Psi}{\partial x}\frac{d\phi_b}{dz} = \frac{1}{Le}\nabla^2\phi + \frac{N_A}{Le}\nabla^2 T_{nl} + \frac{\partial(\Psi, \phi)}{\partial(x,z)} \tag{31}$$

Where, $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$. Here we assume all physical quantities are independent of y and hence for non-linear stability exploration, we utilize the Fourier-series expansions:

$$\Psi = A_{11}(t)\sin(ax)\sin(\pi z) \tag{32}$$

$$T_{nl} = B_{11}(t)\cos(ax)\sin(\pi z) + B_{02}(t)\sin(2\pi z) \tag{33}$$

$$T_p = C_{11}(t)\cos(ax)\sin(\pi z) + C_{02}(t)\sin(2\pi z) \tag{34}$$

$$\phi = D_{11}(t)\cos(ax)\sin(\pi z) + D_{02}(t)\sin(2\pi z) \tag{35}$$

Putting the expressions (32-35) into the eqns.(28-31) and leaving second and third term of RHS of eqn.(29) and (30) as they are small according to Agrawal.et.al [22] and utilizing the orthogonalization procedure of Galerkin's method, we get:

$$A'_{11}(t) = -\frac{\Pr\left(a\left(\epsilon_1\cos(t\Omega)+1\right)\left(RhB_{11}(t)-RnD_{11}(t)\right)+\delta^2 A_{11}(t)\left(\delta^2 Hs+1\right)\right)}{\delta^2(F\Pr+Hs)} \tag{36}$$

$$B'_{11}(t) = A_{11}(t)(-(\pi a B_{02}(t)+a)) - B_{11}(t)\left(\delta^2+N_H\right) + C_{11}(t)N_H \tag{37}$$

$$B'_{02}(t) = \frac{1}{2}\pi a A_{11}(t)B_{11}(t) - B_{02}(t)\left(N_H+4\pi^2\right) + C_{02}(t)N_H \tag{38}$$

$$C'_{11}(t) = -A_{11}(t)\left(\pi a C_{02}(t)+a\right) + \gamma B_{11}(t)N_H - C_{11}(t)\left(\delta^2\epsilon+\gamma N_H\right) \tag{39}$$

$$C'_{02}(t) = \frac{1}{2}\pi a A_{11}(t)C_{11}(t) + \gamma B_{02}(t)N_H - C_{02}(t)\left(\gamma N_H+4\pi^2\epsilon\right) \tag{40}$$

$$D'_{11}(t) = \frac{aLeA_{11}(t)\left(1-\pi D_{02}(t)\right)-\delta^2\left(N_A B_{11}(t)+D_{11}(t)\right)}{Le} \tag{41}$$

$$D'_{02}(t) = \frac{\pi\left(aLeA_{11}(t)D_{11}(t)-8\pi\left(N_A B_{02}(t)+D_{02}(t)\right)\right)}{2Le} \tag{42}$$

The above described system of autonomous concurrent ODE is solved via numerical technique.

3.3 Heat and Nanoparticle Concentration Transportation

The nanoliquide phase thermal Nusselt-number is declared as:

$$Nu_f(t) = \frac{\text{Heat transport by (conduction+convection)}}{\text{Heat transport by conduction}}$$

$$Nu_{nl}(t) = 1 + \left[\frac{\int_0^{2\pi/a}\left(\frac{\partial T_{nl}}{\partial z}\right)dx}{\int_0^{2\pi/a}\left(\frac{\partial T_{nlb}}{\partial z}\right)dx}\right]_{z=0} \tag{43}$$

Applying eqn.(18) and (33) into the eqn.(43), then

$$Nu_f(t) = 1 - 2\pi B_{02}(t) \tag{44}$$

The particle phase thermal Nusselt-number is declared as:

$$Nu_p(t) = 1 + \left[\frac{\int_0^{2\pi/a}\left(\frac{\partial T_p}{\partial z}\right)dx}{\int_0^{2\pi/a}\left(\frac{\partial T_{pb}}{\partial z}\right)dx}\right]_{z=0} \tag{45}$$

Applying eqn.(19) and (34) into the eqn.(45), then

$$Nu_p(t) = 1 - 2\pi C_{02}(t) \tag{46}$$

$$Nu_\phi(t) = \frac{\text{Solute transport by (diffusion+advection)}}{\text{Solute transport by molecular diffusion}}$$

$$Nu_\phi(t) = 1 + \left[\frac{\int_0^{2\pi/a}\left(\frac{\partial \phi}{\partial z}+N_A\frac{\partial T_{nl}}{\partial z}\right)dx}{\int_0^{2\pi/a}\left(\frac{\partial \phi_b}{\partial z}\right)dx}\right]_{z=0} \tag{47}$$

Applying eqns.(20,33 and 35) into the eqn.(47), then

$$Nu_\phi(t) = 1 + 2\pi\left(N_A B_{02}(t)+D_{02}(t)\right) \tag{48}$$

4 Results and Discussion

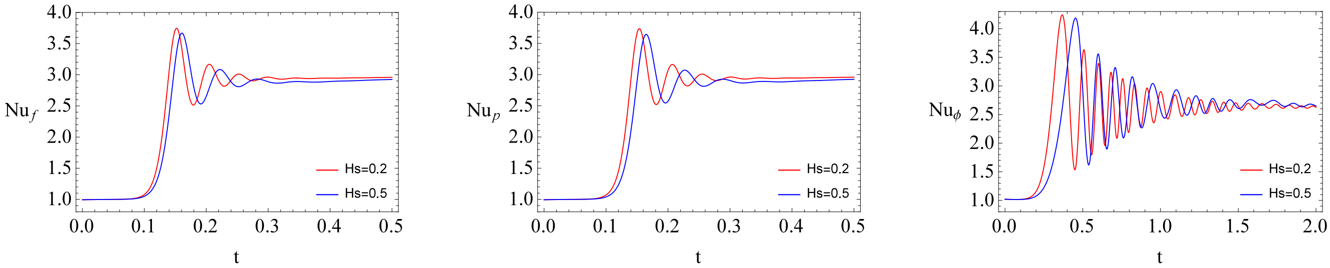
In this manuscript, a weakly non-linear stability exploration has been done by investigating the combined effect of g-giter and thermal difference on heat and mass transportation. Here, Nu_f, Nu_p and Nu_ϕ shows fluid-phase, particle-phase and concentration nusselt-number respectively. Figs. (2a) and (2b), shows that on increasing Hs then decreases nusselt number that is heat-transfer in both fluid/particle phases in the system decreases. This happens because on increasing the Hs number permeability will increase. Thus it takes more time for convection to start. Fig.(2c), give information about mass-transfer and shows similar behavior to that of fluid/particle phases. In fig.(3a), we see clearly on increasing the value of HS-number Hs, heat-transfer reduces in the fluid-phase and convection gets delayed. In fig.(3b), we see clearly heat-transfer in particle-phase is maximum as compared to fluid-phase. This excess heat-transfer is possibly due to the enhanced thermal conductivity of nanoparticles. Fig.(3c), give information about mass-transfer and shows similar behavior to that of fluid phase.

Figs. (4a),(4b) and (4c), shows that an increment in the amount of kinematic-viscoelastic parameter F, then decreases heat-transfer in both fluid/particle phases as well as mass-transfer in the system. In figs.(5a), (5b) and (5c), we see clearly similar behavior as figs. (4a), (4b) and (4c), but in case of particle-phase heat-transfer is maximum as compared to fluid-phase in the system.

In figs.(6a),(6b) and (6c), we see clearly an increment in the amount of amplitude of modulation(ϵ_1), heat-transfer increases in both fluid/particle phases. But mass-transfer decreases in the system. In figs.(7a),(7b) and (7c), we see clearly similar behavior as figs. (6a), (6b) and (6c), but in case of particle-phase heat-transfer is maximum as compared to fluid-phase in the system.

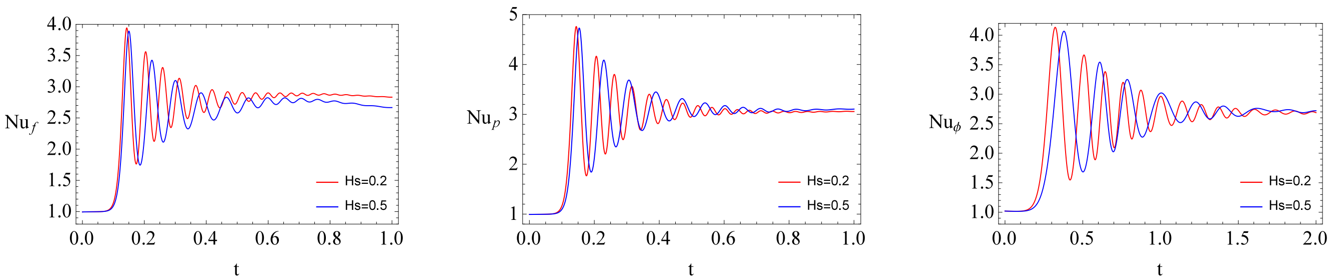
In figs.(8a),(8b) and (8c), we see clearly an increment in the amount of modulating-frequency (Ω), heat-transfer decreases in both fluid/particle phases. But mass-transfer increases in the system. In figs.(9a), (9b) and (9c), we see clearly similar behavior as figs. (8a),(8b) and (8c), but in case of particle-phase heat-transfer is maximum as compared to fluid-phase in the system.

In figs. (10a),(10b) and (10c), we see clearly convection sets in earlier for $N_H = 50$ as compared to $N_H = 0$. In case of $N_H = 50$ heat-transport in particle-phase is more than the heat-transport in the fluid-phase and in case of $N_H = 50$ mass-diffusion (advection) earlier than $N_H = 0$.



(a) (b) (c)

Fig. 2: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) Hs When $N_H = 0, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, F = 1.2, \epsilon_1 = 0.4, \epsilon = 1, \Omega = 10$.



(a) (b) (c)

Fig. 3: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) Hs When $N_H = 50, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, F = 1.2, \epsilon_1 = 0.4, \epsilon = 1, \Omega = 10, \gamma = 0.5$.

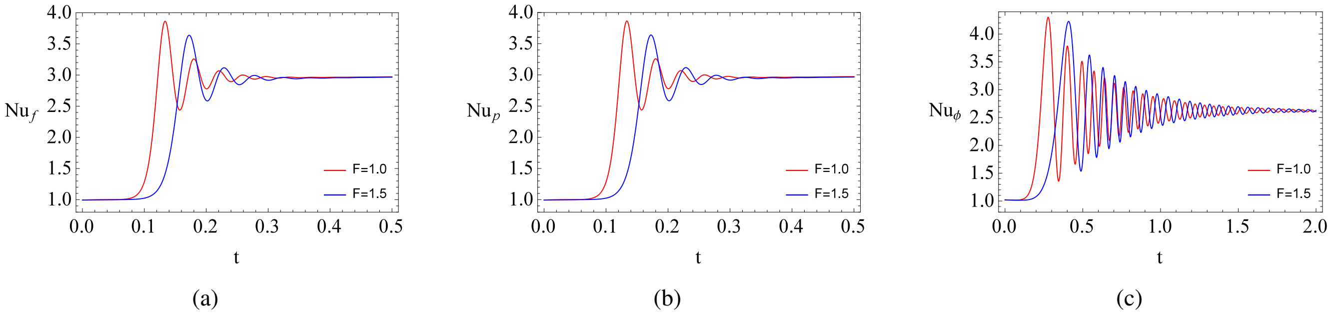


Fig. 4: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) F When $N_H = 0, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, Hs = 0.1, \epsilon_1 = 0.4, \epsilon = 1, \Omega = 10$.

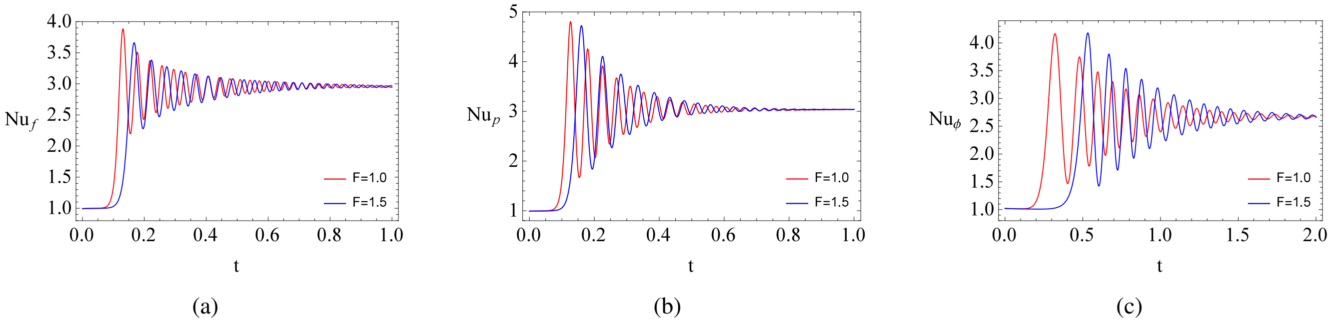


Fig. 5: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) F When $N_H = 50, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, Hs = 0.1, \epsilon_1 = 0.4, \epsilon = 1, \Omega = 10, \gamma = 0.5$.

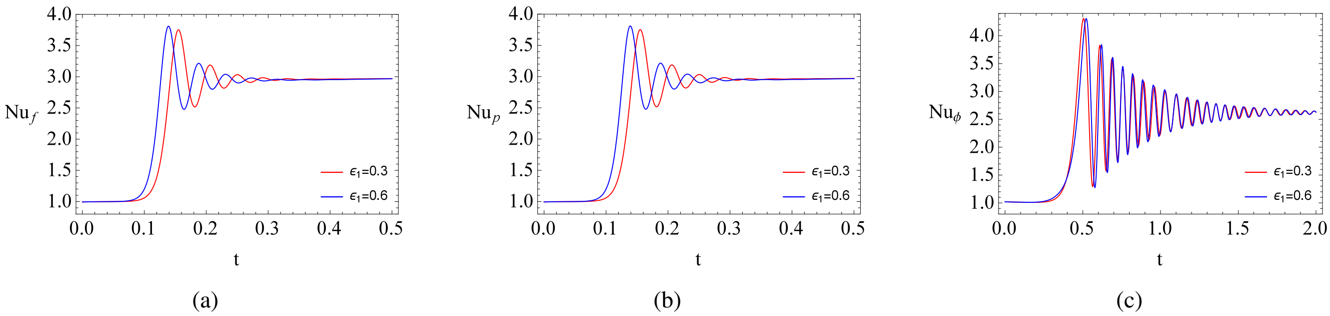


Fig. 6: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) ϵ_1 When $N_H = 0, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, Hs = 0.1, F = 1.2, \epsilon = 1, \Omega = 10$.

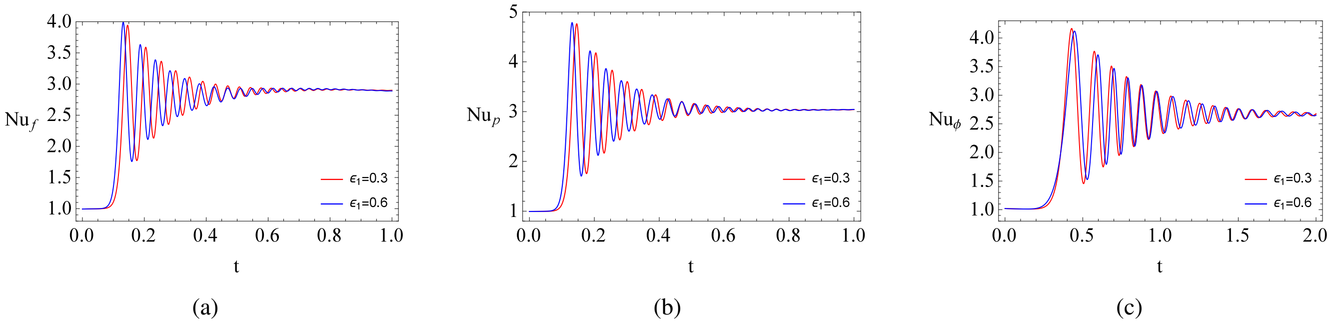


Fig. 7: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) ϵ_1 When $N_H = 50, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, Hs = 0.1, F = 1.2, \epsilon = 1, \Omega = 10, \gamma = 0.5$.

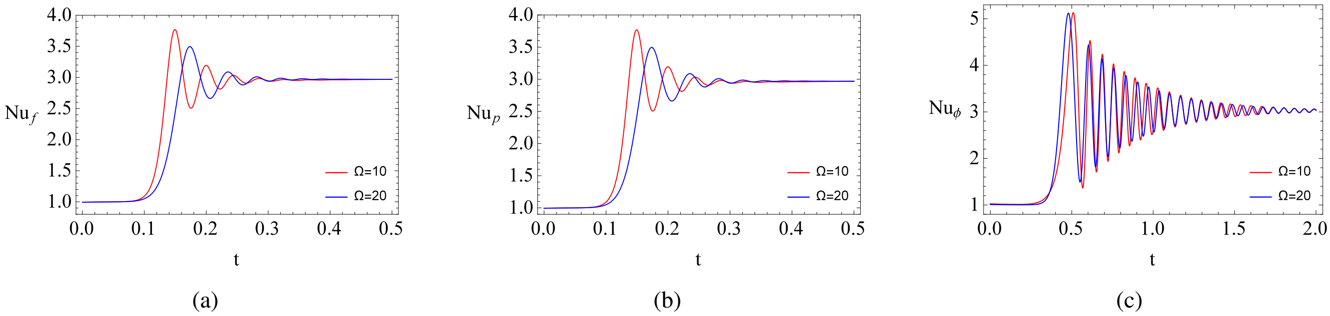


Fig. 8: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) Ω When $N_H = 0, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, Hs = 0.1, F = 1.2, \epsilon = 1, \epsilon_1 = 0.4$.

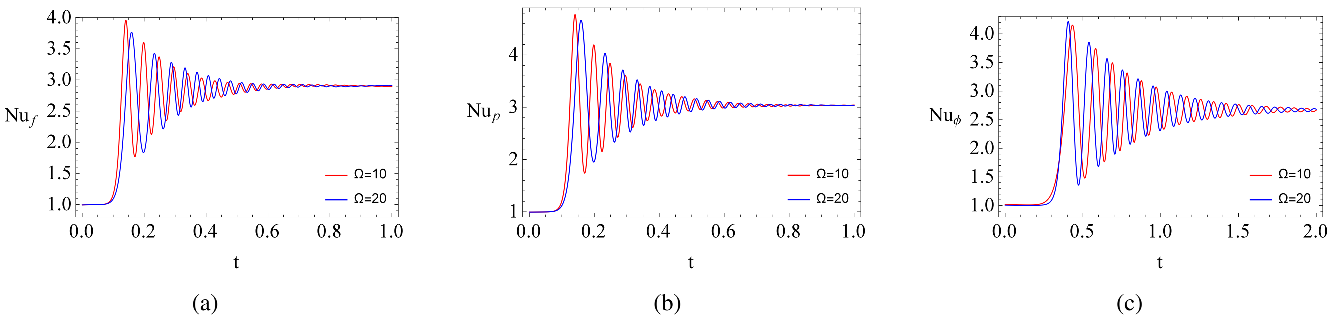


Fig. 9: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) Ω When $N_H = 50, Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, Hs = 0.1, F = 1.2, \epsilon = 1, \epsilon_1 = 0.4, \gamma = 0.5$.

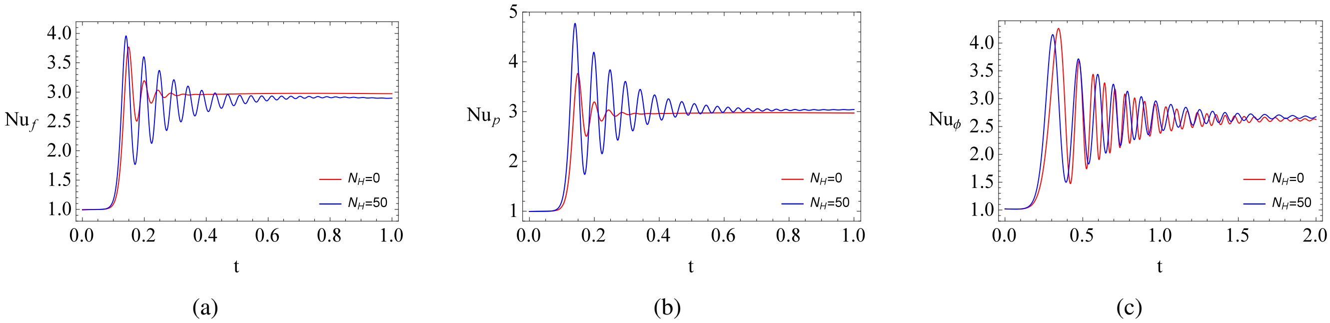


Fig. 10: Curves of Nu_f, Nu_p and Nu_ϕ versus time t for various values of (a),(b) and (c) N_H When $Rn = 2, \Pr = 10, Le = 10, N_A = 0.2, a = 2.22, Hs = 0.1, F = 1.2, \epsilon = 0.04, \gamma = 0.5, \epsilon_1 = 0.4, \Omega = 10$.

5 Conclusions

In the present manuscript, weakly nonlinear exploration of combined influence of g-jitter and thermal difference on a Rivlin-Ericksen nanofluid in Hele-Shaw cell has been examined by performing nonlinear exploration. Outcomes of the present analysis have been depicted graphically and the following observations (conclusions) are scripted as below:

1. An increment in the amount of HS-number decreases heat-transfer as well as mass-transfer in the system.
2. On increasing the amount of amplitude of g-jitter, heat-transfer increases but mass-transfer decreases in the system.
3. On increasing the amount of modulating-frequency, heat-transfer decreases but mass-transfer increases in the system.
4. An increment in the amount of kinematic-viscoelasticity parameter F, decreases heat as well as mass transportation in the system.
5. Convection sets in earlier for in situation of temperature difference as compared to similar temperature between fluid and nano-particle.
6. In case of temperature difference, heat-transport in particle-phase is more than the heat-transport in fluid-phase.

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Control over the Performance of Quantum Neuromorphic Computing with Reservoir Computing Networks

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Abstract: The evaluation of unknown states for a given quantum system is one of the key problems in quantum information processing. The most efficient method of state characterization is quantum state tomography, where the full density matrices are reconstructed from the experimental measurements or numerical simulations performed on quantum states. The improvement of the computational performance in quantum state tomography and its related problems is a challenging task for modern theoretical physics. The general scheme of computing deals with the input information which goes into a quantum reservoir through a recurrent evolution. After the evolution, the final output is obtained as the linear combination of the readout elements. In the proposed approach, the quantum reservoir is modeled with the Lindbladian equation. The control over performance is made by the coherent coupling parameter between the input quantum state and the reservoir. The control feedback algorithm is represented with the set of the Kolesnikov target attractor (TA) algorithm to drive certain parameters of quantum state tomography, for instance, the outputs for the density matrix. TA feedback is formulated in a discrete form and discuss its possible development and applications.

Keywords: Non-linear feedback algorithms, Quantum informatics, Quantum neuromorphic computing, Reservoir computing networks.

1 Introduction

The concept of reservoir computing (RC) is originated in algorithms based on stable learning at a real-time lower computational cost, such as echo state networks and liquid state machines [1]. The main part of RC is a so-called 'reservoir': a high-dimensional dynamical system forms a neural network with fixed random connections, such architecture allows to avoid the overhead of controlling a large number of connections [2]. The reservoir gets the temporal input data, adjusts the weights of the readout signal by training, and approximates the target output signal [3]. Different physical realizations of reservoirs are based on electronic circuits, photonic systems, spintronic systems, mechanical machines (soft and compliant robots), and even biological networks (*in-vitro* cultured cells) [4].

Quantum reservoir computing systems have their own features that could not be simulated on conventional classical computers [5]. The evaluation of unknown states for a given quantum system is one of the key problems of quantum information processing [6]. The information about the input system enters the quantum reservoir, which goes through a recurrent evolution. After the evolution, the final output is taken as the linear combination of the readout elements.

The weights in the system are of two types: the weights representing the coupling of the reservoir with the input and output layers, and the bidirectional recurrent weights connecting the reservoir nodes.

The most efficient method of state characterization is quantum state tomography, where full density matrices are reconstructed from experimental measurements or numerical simulations performed on quantum states [7]. The improvement of the computational performance in quantum state tomography and its related problems is a challenging task for modern theoretical physics. The general scheme of computing deals with the input information which goes into a quantum reservoir through a recurrent evolution. After the evolution, the final output is obtained as the linear combination of the readout elements [6].

In our approach, the quantum reservoir is modeled with the Lindbladian equation. The control over performance is made by the coherent coupling parameter between the input quantum state and the reservoir [8]. Usually, the control during the quantum state tomography is performed as an open-loop (feedforward) scheme [6, 9], but here we discuss the closed-loop (feedback) algorithm. The control feedback algorithm is represented with the discrete analog of the Kolesnikov Target Attractor approach [10] to drive certain parameters of quantum state tomography, for instance, the outputs for the density matrix. We discuss the pros and cons of our proposed control approach and its possible development and applications.

2 Quantum State Tomography

Quantum state tomography (QST) is a method to reconstruct the density matrix of a given quantum system. In finite D-dimensional Hilbert space, the density matrix is described by $D^2 - 1$ independent real-valued parameters [9].

The initial phase of the QST tomography process is the training to recognize the given quantum state. It is based on the set of experimental or numerically generated data. Then the trained set-up can be used for analyzing the unknown states of the given quantum system.

The training states in the reservoir need to be linearly independent, by the set of D^2 randomly generated quantum states can be sufficient [9].

The reservoir dynamics are described by the equation [6]:

$$i\hbar \frac{d\rho}{dt} = \left[\hat{H}_R, \rho\right] + \frac{i\gamma}{2}\hat{L}(\rho) + \hat{T}_{\text{int}}(\rho) \ , \tag{1}$$

where $\hat{H}_R$ is the reservoir Hamiltonian, $\hat{L}(\rho)$ is a Lindbladian operator describing the dissipation in the system; and $\hat{T}_{\text{int}}(\rho)$ is the operator activating the coupling between the input modes and the reservoir. The process of quantum state tomography can be performed in the following steps [9]:

1. Initially the reservoir stays in the vacuum state or excited only with the uniform field.
2. A coupling between the input modes and the reservoir is activated through a cascade coupling as a set of the Heaviside delta-functions or through a coherent coupling, see [6, 9] for details.
3. The vector $\mathbf{n}$ consisting of the occupation number n_j of each readout mode is measured.
4. The desired output is evaluated:

$$\mathbf{Y}^{\text{out}} = \mathbf{M}\mathbf{n} + \mathbf{m} \ . \tag{2}$$

In (2) the matrix $\mathbf{M}$ and the constant vector $\mathbf{m}$ are determined through the training process.

For the purpose of tomography let's chose: $\mathbf{Y}^{\text{out}} = \rho_{\text{in}}$ (here the density matrix is arranged in a column vector format). Then in the process of tomography, we expect that: $\rho_{\text{in}} = \mathbf{M}\mathbf{n} + \mathbf{m}$.

In reality, RHS(2) is not exactly equal to the real density matrix, such that the vector representation of the density matrix reconstructed in the process of tomography is given by:

$$\rho_{\text{in}}^{\text{tomo}} = \mathbf{M}\mathbf{n} + \mathbf{m} \ , \tag{3}$$

with some error as a result of the experimental or numerical observation of the density matrix.

To evaluate the error of QST, the fidelity is defined as [9, 11]:

$$F = \left(\text{Tr}\left[\sqrt{\sqrt{\rho_{\text{in}}} \cdot \rho_{\text{in}}^{\text{tomo}} \cdot \sqrt{\rho_{\text{in}}}}\right]\right)^2 \ . \tag{4}$$

For the multiple inputs $j = 1, ..., N$, the fidelity should be computed for each input separately as F_j, and then the average fidelity is calculated:

$$\bar{F} = \frac{1}{N}\sum_{j=1}^{N} F_j \ . \tag{5}$$

In the case of ideal error-free quantum tomography, the fidelity (4)-(5) must be equal to 1: $F = 1$, otherwise: $F < 1$ [9].

3 Control over the Performance in Quantum Tomography

To make a control over the quantum tomography performance we use here the concept of creating in the dynamical system an artificial target attractor locking the trajectories in the neighborhood of the control goal. Such a method has been proposed by Kolesnikov in [10] for continuous -time systems.

In the continuous formulation of Target Attractor (TA) feedback, we define a function ψ to form an artificial target attractor in the dynamical system as: $\psi(t) = v(t) - v_t(t)$. Here $v(t)$ represents a controlled variable, while $v_t(t)$ stays for the target variable function, i.e. we make a control tracking for the variable v. Then we demand the exponential convergence of the goal function:

$$T\frac{d\psi}{dt} = -\psi \ . \tag{6}$$

An arbitrary positive constant T corresponds to the typical time scale of the TA control. Eq.(6) has the solution:

$$\psi(t) = e^{-t/T}\psi(0) \ . \tag{7}$$

Thus, by (7) the system dynamics for $v(t)$ come exponentially fast to the target function $v_t(t)$ and then stays locked in its neighborhood.

For the purpose of quantum tomography we develop here a discrete analog of the Kolesnikov algorithm. Let's define the matrix analog of the target attractor (5) in the form:

$$\psi_k = \mathbf{Y}_k^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}} \tag{8}$$

with the target vector outcome:

$$\mathbf{Y}_t^{\mathrm{out}} = \mathbf{M}_t \mathbf{n} + \mathbf{m}_t \ . \tag{9}$$

The outcome is a sample in the process of training. The target outcome contains the target matrix $\mathbf{M}_t$ and target vector $\mathbf{m}_t$. We need the exponential convergence of the control procedure, like in (7), as the discrete step k is increases:

$$\psi_k = e^{-\gamma k}\psi_0 \ ; \ \gamma = \mathrm{const} > 0 \ . \tag{10}$$

The training procedure for the discrete Kolesnikov algorithm looks at the k-th and $(k+1)$-th steps as:

$$\begin{aligned} \mathbf{Y}_k^{\mathrm{out}} &= \mathbf{Y}_t^{\mathrm{out}} + e^{-\gamma k}\left(\mathbf{Y}_0^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}}\right) = \mathbf{M}_k \mathbf{n} + \mathbf{m}_k \ ; \\ \mathbf{Y}_{k+1}^{\mathrm{out}} &= \mathbf{Y}_t^{\mathrm{out}} + e^{-\gamma (k+1)}\left(\mathbf{Y}_0^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}}\right) = \mathbf{M}_{k+1} \mathbf{n} + \mathbf{m}_{k+1} \ . \end{aligned} \tag{11}$$

By (10)-(11), we can evaluate the difference for the vector $\mathbf{m}$ at the k-th and $(k+1)$-th steps as:

$$\mathbf{m}_{k+1} - \mathbf{m}_k = \left[e^{-\gamma(k+1)} - e^{-\gamma k}\right](\mathbf{Y}_0^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}}) \ . \tag{12}$$

With the first equation in (11), we express the vector $\mathbf{n}$:

$$\mathbf{n} = (\mathbf{M}_{k+1} - \mathbf{M}_k)^{-1}\left[(\mathbf{Y}_{k+1}^{\mathrm{out}} - \mathbf{Y}_k^{\mathrm{out}}) - (\mathbf{m}_{k+1} - \mathbf{m}_k)\right] \ . \tag{13}$$

Then, after the substitution (12) into (13), we find the vector $\mathbf{n}$ explicitly:

$$\begin{aligned} \mathbf{n} &= (\mathbf{M}_{k+1} - \mathbf{M}_k)^{-1}\mathbf{B}_k \ ; \\ \mathbf{B}_k &= (\mathbf{Y}_{k+1}^{\mathrm{out}} - \mathbf{Y}_k^{\mathrm{out}}) - \left[e^{-\gamma(k+1)} - e^{-\gamma k}\right](\mathbf{Y}_0^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}}) \ . \end{aligned} \tag{14}$$

Now let's substitute the vector $\mathbf{n}$ from (14) back to the second equation (11) and finally get:

$$\mathbf{M}_{k+1}\mathbf{M}_k^{-1}(\mathbf{Y}_k^{\mathrm{out}} - \mathbf{m}_k) = (\mathbf{Y}_k^{\mathrm{out}} - \mathbf{m}_k) + \mathbf{B}_k \ . \tag{15}$$

Eq.(15) is the main result of our discrete analog for the Kolesnikov algorithm (5)-(7). Practically, by (13) we reduced the learning process for quantum tomography to the problem of finding the eigenvectors and eigenvalues for the matrix $\mathbf{M}$ at the $(k+1)$-th step.

Asymptotically, we can evaluate:

$$\begin{aligned} \mathbf{Y}_k^{\mathrm{out}} &\to \mathbf{Y}_t^{\mathrm{out}} \ \text{ as } \ k \to \infty \ ; \\ \mathbf{M}_{k+1}\mathbf{n} + \mathbf{m}_{k+1} &\simeq \mathbf{M}_k \mathbf{n} + \mathbf{m}_k \ , \\ \text{then } \ \mathbf{B}_k &\simeq \mathbf{m}_{k+1} - \mathbf{m}_k \ . \end{aligned} \tag{16}$$

This simplification could be useful for the fast numerical evaluation of the auxiliary matrix $\mathbf{B}$.

4 Finalization of the Control Algorithm

In this section we formulate the final form of our novel algorithm with the Kolesnikov-type feedback:
0. At the 0-th step, get the density matrix arranged in a column vector format: $\mathbf{Y}_0^{\mathrm{out}} = \rho_{\mathrm{in},0}$ (from experimental data or numerical simulations). Then perform the algorithmic cycle:

1. Start from the density matrix arranged in a column vector format for the k-th step:

$$\mathbf{Y}_k^{\mathrm{out}} = \rho_{\mathrm{in},k} \; . \tag{17}$$

2. Compute the auxiliary vector $\mathbf{B}$ for the k-th step:

$$\mathbf{B}_k = (\mathbf{Y}_{k+1}^{\mathrm{out}} - \mathbf{Y}_k^{\mathrm{out}}) - \left[e^{-\gamma(k+1)} - e^{-\gamma k}\right](\mathbf{Y}_0^{\mathrm{out}} - \mathbf{Y}_t^{\mathrm{out}}) \; . \tag{18}$$

3. Solve the eigenvector problem (15) for the $(k+1)$-th step and compute the vector $\mathbf{m}$ and the matrix $\mathbf{M}$:

$$\begin{aligned} \mathbf{m}_{k+1} &= \mathbf{m}_k - \mathbf{B}_k \; ; \\ \mathbf{M}_{k+1} &= \mathbf{b}_k \mathbf{a}_k^{-1} \; ; \text{ where :} \\ \mathbf{a}_k &= \mathbf{M}_k^{-1}(\mathbf{Y}_k^{\mathrm{out}} - \mathbf{m}_k) \; ; \\ \mathbf{b}_k &= (\mathbf{Y}_k^{\mathrm{out}} - \mathbf{m}_k) + \mathbf{B}_k \; . \end{aligned} \tag{19}$$

4. Find the vector $\mathbf{n}$, as in (14):

$$\mathbf{n} = (\mathbf{M}_{k+1} - \mathbf{M}_k)^{-1}\mathbf{B}_k \; . \tag{20}$$

5. Finally, by (11), get $\mathbf{Y}^{\mathrm{out}}$ for the $(k+1)$-th step:

$$\mathbf{Y}_{k+1}^{\mathrm{out}} = \mathbf{M}_{k+1}\mathbf{n} + \mathbf{m}_{k+1} \; . \tag{21}$$

6. Repeat all these procedures 1-6 for the next algorithmic cycle up to the necessary level of accuracy. For the evaluation of the error use the fidelity (4)-(5).

5 Main Result

The training process is the most time-consuming phase to analyze the given system in quantum tomography.

The usage of our novel approach when each vector and matrix variable is computed in the frame of an exponentially converging discrete-step Kolesnikov-type algorithm drastically increases the speed of training for the reservoir computing system and optimizes the training.

6 Conclusions

The feedback control algorithm over the performance of quantum state tomography proposed here demonstrates a set of advantages:
– The invented algorithm works for the optimization of the computational sources and decreases the computational cost in the real-time numerical process.
– The algorithm is robust and stable under relatively small external perturbations.
– The algorithm is valid for different types of interaction between the input modes and the reservoir: cascade coupling, and coherent coupling.

The algorithm also can be easily extended to different control goals: preparation, estimation and reconstruction of quantum states, quantum computing, compressing quantum circuits, and others [11].

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Positive solutions for a singular Caputo-Fabrizio fractional differential equations

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Abstract: In this paper, we prove the existence and uniqueness of solutions for a singular Caputo-Fabrizio fractional differential equation boundary value problem. The main results of this paper are obtained by constructing the monotone iterative sequences of upper and lower solutions and applying a fixed point theorem. We also present an example supporting our theoretical results.

Keywords: Caputo-Fabrizio fractional derivative, Positive solution, Singular Dirichlet fractional boundary value problems.

1 Introduction

Recently, there has been a growing interest on the applications of fractional boundary value problems. These applications contain various scientific areas such as engineering, physics, aviscoelasticity, electrochemistry and electromagnetics; [1, 4] and the references therein. This paper deals with the positive solutions of the fractional boundary value problems (FBVP) involving the Caputo-Fabrizio fractional derivative. In the last two decades, there are many works on FBVPs [5, 8].

In [7], the authors consider the following singular fractional differential equation involving the Caputo fractional derivative

$$^{c}D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad u(0) = u'(1) = u''(0) = 0, \quad 0 < t < 1, \alpha \in (2, 3],$$

where $\lim_{t\to 0^+} f(t, u(t)) = \infty$. It is shown that the problem has at least one positive solution with the help of nonlinear alternative of Leray-Schauder and a fixed point theorem in a cone. In [9], the following singular fractional Dirichlet boundary value problem is considered:

$$^{R}D^{\alpha}_{0+}u(t) + f(t, u(t), {}^{R}D^{\beta}_{0+}) = 0, \quad u(0) = u(1) = 0, \quad 0 < t < 1, \alpha \in (1, 2], \quad \beta > 0,$$

where $^{R}D^{\alpha}_{0+}$ is the standard Riemann-Liouville fractional derivative, $\alpha - \beta \geq 1$ and f satisfies the CarathÃl'odory conditions on $[0, 1](0, \infty) \times \mathbb{R}$,, f is positive and $f(x, \cdot, \cdot)$ is singular at the origin.

By the Caratheodory conditions on $[0, 1] \times (0, \infty) \times \mathbb{R}$, we mean that f satisfies the following

(i) $f(\cdot, x, y) : [0, 1] \to \mathbb{R}$ is measurable for all $(x, y) \in (0, \infty) \times \mathbb{R}$,
(ii) $f(t, \cdot, \cdot) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in [0, 1]$.
(iii) for each compact set $\mathcal{K} \subset (0, \infty) \times \mathbb{R}$ there is a function $m_{\mathcal{K}} \in L^p[0, 1]$ such that

$$|f(t, x, y)| \leqslant m_{\mathcal{K}}(t) \text{ for a.e. } t \in [0, 1] \text{ and all } (x, y) \in \mathcal{K}.$$

Inspired by the above studies, in this paper, we investigate the existence and uniqueness of positive solution for a singular FBVP of the Caputo-Fabrizio fractional derivative with the Dirichlet boundary conditions

$$^{CF}D^{\alpha}_{0+}u(t) - f(t, u(t), u'(t)) = 0, \quad u(0) = u(1) = 0, \quad 0 < t < 1, \alpha \in (1, 2],, \tag{1}$$

where $^{CF}D^{\alpha}_{0+}$ is the Caputo-Fabrizio fractional derivative, f satisfies the Caratheodory conditions on $[0, 1](0, \infty) \times \mathbb{R}$,, f is positive and $f(x, \cdot, \cdot)$ is singular at the origin.

We say that the solution $u \in C[0, 1]$ is a positive solution of problem (1) provided that $u > 0$ on $(0, 1)$, $^{CF}D^{\alpha}_{0+}u \in L^1[0, 1]$, u satisfies the boundary conditions.

Most of the works on FBVPs in the literature are devoted to fractional differential equation involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative. The main drawback of these fractional derivatives is that they have singular kernel at the starting point of the domain. This drawback leads to investigate some new definitions for fractional derivatives in the literature. The Caputo-Fabrizio fractional derivative is a newly defined fractional derivative which has exponential decay in the kernel while the Caputo fractional derivative has a singular kernel based on a power law.

The organization of this paper is as follows. In Section 2, we recall the definition of the Caputo-Fabrizio fractional derivative and integration and its properties. In Section 3, the existence and uniqueness of the solutions of the problem are investigated. We give an example to demonstrate the applicability of the results in the last section.

2 Preliminaries

This section introduces some definitions and preliminaries that will be needed in the following.

Definition 1. *[10] Let $f \in H^1(a,b), \quad a < b$ and $\alpha \in (0,1]$. The fractional Caputo-Fabrizio derivative is defined as*

$$ {}^{CF}D^{\alpha}_{0^+}u(x) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^x \exp\Big(-\frac{\alpha}{1-\alpha}(x-t)\Big)u'(t)\,dt, \quad t \geq 0, \tag{2} $$

where $M(\alpha)$ is a normalization function with $M(0) = M(1) = 1$.

Definition 2. *The Caputo-Fabrizio fractional integral of order $\alpha \in (0,1)$ is defined as*

$$ {}^{CF}\mathcal{I}^{\alpha}_0 u(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(x) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^x u(s)\,ds. \tag{3} $$

Imposing $\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1$, we can have an explicit expressing for $M(\alpha), \quad \alpha \in (0,1]$ given as

$$ M(\alpha) = \frac{2}{2-\alpha}. $$

The high order Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ is defined as

$$ {}^{CF}D^{\alpha+n}_{0^+}u(x) := {}^{CF}D^{\alpha}_{0^+}({}^{CF}D^{n}_{0^+}u(x)). $$

$AC[0,1]$ denotes the space of absolutely continuous functions on the interval $[0,1]$ and $AC_{loc}(0,1]$ be the space consisting of functions that are absolutely continuous on every interval $[a,1] \subset (0,1]$. We always assume $p > 1, \quad q(p-1) = p$.

Lemma 1. *[11] Assume $\alpha > 0$, for $u \in L^1(0,1)$,*
(1) ${}^{CF}\mathcal{I}^{\alpha}_0 {}^{CF}D^{\alpha}_0 u(x) = u(x) - \sum_{k=0}^{n} \frac{u^{(k)}(0)}{k!}x^k, \quad x \in [0,1], u \in L^p[0,1]$.
(2) ${}^{CF}D^{\alpha}_0 {}^{CF}\mathcal{I}^{\alpha}_0 u(x) = u(x)$, where $n = [\alpha] + 1$. The Laplace transform of the Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ is given by

$$ \mathcal{L}\Big\{{}^{CF}D^{\sigma}_0 u(x)\Big\}(s) = \frac{s^{n+1}\mathcal{L}\{f(x)\}(s) - s^n f(0) - s^{n-1}f'(0)\cdots - f^{(n)}(0)}{s + \alpha(1-s)}. $$

We need the following lemma in the sequel.

Lemma 2. *Given $h \in C[0,1]$, and $1 < \alpha \leqslant 2$, the unique solution of the following FBVP*

$$ \begin{aligned} &{}^{CF}D^{\alpha}_0 u(x) + g(x) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = 0, \end{aligned} \tag{4} $$

is given by $u(x) = \int_0^1 G(x,s)g(s)ds$ where

$$ G(t,x) = \begin{cases} (1-\alpha)t(1-x), & 0 \leq t \leq x \leq 1, \\ (1-\alpha)x(1-t), & 0 \leq x \leq t \leq 1. \end{cases} $$

Proof: Applying the Laplace operator to the equation (4), we get

$$\mathcal{L}\Big\{{}^{CF}D_0^{\alpha}(u)(x)\Big\}(s) = \mathcal{L}\Big\{g(x)\Big\}(s).$$

Appealing Laplace operator on both sides of the above equation, we find that

$$\frac{s^2U(s) - su(0) - u'(0)}{s + \alpha(1-s)} = -G(s),$$

where $U(s) = \mathcal{L}\Big\{(u)(x)\Big\}(s)$ and $G(s) = \mathcal{L}\Big\{g(x)\Big\}(s)$. The last equation can be rewritten as

$$U(s) = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) - \frac{1-\alpha}{s}G(s) - \frac{\alpha}{s^2}G(s).$$

The inverse Laplace operator is now applied to above equation to arrive at

$$u(x) = u(0) + xu'(0) - (1-\alpha)\int_0^x (x-t)g(t)\,dt. \tag{5}$$

Taking into account the Dirichlet boundary conditions

$$u(x) = (1-\alpha)x\int_0^1 (1-t)g(t)dt - (1-\alpha)\int_0^x (x-t)g(t)\,dt,$$

or, equivalently we have

$$\begin{aligned} u(x) &= \int_0^x (1-\alpha)t(1-x)g(t)dt + \int_x^1 (1-\alpha)x(1-t)g(t)\,dt \\ &= \begin{cases} (1-\alpha)t(1-x), & 0 \le t \le x \le 1, \\ (1-\alpha)x(1-t), & 0 \le x \le t \le 1, \end{cases} \end{aligned}$$

which gives the desired result.

□

Consider the Banach space $X = \Big\{u : u \in C[0,1] \cap C^1(0,1]\Big\}$ with the norm $\|u\|_X = \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty\right\}$, where $\|\cdot\|_\infty$ is the sup-norm and we set $Z = L^p[0,1]$ with the usual norm denoted by $\|\cdot\|_p$. We apply the Leray-Schauder Continuation Principle :

Theorem 1. *Let X be a Banach space and $T : X \to X$ be a compact map. Suppose that there exists an $R > 0$ such that if $u = \lambda Tu$ for $\lambda \in (0,1)$, then $\|u\|_X < R$. Then T has a fixed point.*

Let us define a mapping $T : X \to X$ by

$$Tu(x) = (1-\alpha)x\int_0^1 (1-t)f(t,u(t),u'(t))dt - (1-\alpha)\int_0^x (x-t)f(t,u(t),u'(t))\,dt, \quad x \in [0,1].$$

Then, for $x \in (0,1]$,

$$(Tu)'(x) = (1-\alpha)\int_0^1 (1-t)f(t,u(t),u'(t))dt - (1-\alpha)\int_0^x f(t,u(t),u'(t))\,dt.$$

It is clear that $Tu \in C[0,1] \cap C^1(0,1]$. The next result establishes the desired properties of T in the Banach space that will be used for proving the positive solution of the problem (1).

Lemma 3. *Assume that $(i)-(iii)$ and $p > 1, q(p-1) = p$ hold. Then the mapping $T : X \to X$ is compact.*

Proof: Let $E \subset X$ be bounded subset and put $r = \sup\{\|u\|_X : u \in E\}$. Then, by Assumption (iii), there exists a nonnegative function $\phi_r \in L^p[0,1]$, such that, for all $u \in E$ and a.e. $t \in [0,1]$,

$$\left|f\left(x,u(x),u'(x)\right)\right| \le \phi_r(x).$$

By Holder's inequality, with the assumption that $p > 1, q(p-1) = p$, for all $u \in E$, we obtain

$$\|Tu\|_\infty \le \frac{2}{(q+1)^{1/q}}\|\phi_r\|_p\,.$$

Thus, the set $T(E) \subset X$ is bounded. Let $x_1, x_2, \in [0,1]$ with $x_1 < x_2$ and $u \in E$. In what follows, the generic constant C_1 depends only on the parameters α and q.

We find that

$$\begin{aligned}|(Tu)(x_2)-(Tu)(x_1)| =&(1-\alpha) \mid \int_0^{x_2} (x_2-s) f\left(s, u(s), u'(s)\right) ds \\ &-\int_0^{x_1} (x_1-s) f\left(s, u(s), u'(s)\right) ds \\ &-(x_2-x_1) \int_0^1 (1-s) f\left(s, u(s), u'(s)\right) ds \mid \\ &\leq C(x_2-x_1) \|\phi_r\|_1 .\end{aligned}$$

This shows that T is equicontinuous on $(0,1]$. It follows, by the Arzela-Ascoli theorem, that $T(E)$ is compact. The proof of the continuity of T is a routine application of the Lebesgue Dominated Convergence Theorem. □

3 Main result

To obtain our main result, we need the following conditions.

(iv) There exist $r, s, k : [0,1] \to [0,\infty)$ such that $r, s, k \in L^p[0,1]$ and

$$|f(x,u,v)| \leq r(x)|u| + s(x)|v| + k(x), \quad \text{a.e. } x \in [0,1]$$

Theorem 2. *Assume that (i), (ii), (iv), and $p>1, q(p-1)=p$ hold. Suppose that the functions r, s satisfy*

$$\frac{2}{(q+1)^{1/q}} \|r\|_p + \|s\|_p < 1. \tag{6}$$

Then the boundary value problem (1) has at least one solution.

Proof: We consider, for $\lambda \in (0,1)$, the following FBVP

$$^{CF}D_0^\alpha u(t) = \lambda f\left(t, u(t), u'(t)\right), \quad t \in (0,1),$$

subject to the Dirichlet boundary conditions. We would like to verify that the set of all possible solutions is a priori bounded in X by a constant independent of $\lambda \in (0,1)$. We obtain at once

$$\begin{aligned}\left\|^{CF}D_0^\alpha u\right\|_p &= \lambda \left\|f\left(t,u,u'\right)\right\|_p \\ &< \left\|f\left(t,u,u'\right)\right\|_p \\ &\leq \|ru\|_p + \left\|su'\right\|_p + \|k\|_p \\ &\leq \|r\|_p \|u\|_0 + \|s\|_p \left\|u'\right\|_\infty + \|k\|_p \\ &\leq A\|r\|_p \left\|^{CF}D_0^\alpha u\right\|_p + B\|s\|_p \left\|^{CF}D_0^\alpha u\right\|_p + \|k\|_p.\end{aligned}$$

As a result, we arrive at

$$\left\|^{CF}D_0^\alpha u\right\|_p \leq \frac{\|\gamma\|_p}{1 - A\|\alpha\|_p - B\left\|t^{\alpha-2}\beta\right\|_p}.$$

This means that the solution set is a priori bounded in $L^p[0,1]$ by a constant independent of $\lambda \in (0,1)$. It follows, by Lemma 3, that the solution set is bounded in X by a constant independent of $\lambda \in (0,1)$ in view of

$$\|u\| \leq \max\{A, B\} \left\|^{CF}D_0^\alpha u\right\|_p$$

It readily follows from the results above that the function $u \in X$ is a solution of the boundary value problem ((1) if $u \in X$ is a fixed point of the mapping T. The mapping T is compact. Since the a priori estimate condition of Theorem 1 is verified by virtue of the above inequality, the assertion follows. We complete the proof. □

4 An example

Let us consider the following boundary value problem

$$
\begin{aligned}
{}^{CF}D_0^{\frac{3}{2}}u(x) &= f(x,u(x),u'(x)), \quad 0<x<1, \\
u(0) &= 0, \quad u(1)=0.
\end{aligned} \tag{7}
$$

$$
\frac{2}{(q+1)^{1/q}} = \left(\frac{4}{7}\right)^{2/3}.
$$

Here, we take

$$
f(x,u,v) = \frac{1}{x^{2/7}}\frac{u^2}{1+|x|} + \frac{4}{5\pi x^{4/5}}y\arctan v + \frac{1}{x^{1/4}}.
$$

We observe that the assumption (iv) is fulfilled with $r=\frac{1}{x^{2/7}}$, $s=1$, and $k=\frac{1}{x^{1/4}}$ satisfying $\|r\|_3 = \frac{7^{1/3}}{5}$. Elementary calculations show that the inequality (6) is satisfied. Therefore, we conclude that the problem (7) has one positive solution on $(0,1)$.

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The Problem of the Irregular Element

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Abstract: In a set of $n-1$ identical elements (regular elements) is inserted another element (*irregular element*) which in all observable characteristics has the same as those of regular elements but, according to an unobservable feature, it differs from them. So we obtain a set $\mathbf{S}$ with n elements. The irregular element can have positive or negative deviation of a specific feature compared to the regular elements of the set $\mathbf{S}$. A characteristic of the feature of the irregular element is that its presence in any subset $\mathbf{A}$ of $\mathbf{S}$ gives it a deviation compared with any subset B with the same number of elements, but that all its elements are regular. Of course, this deviation has the direction of the set that contains the irregular element. Suppose that a comparison operation is available, for each pair of subsets of $\mathbf{S}$ with the same number of elements, to indicate whether or not one of them has a positive deviation from the other, which means that the irregular element belongs to one of these two sets, but without knowing which of them. Under these conditions, it is clear that with at most $n-2$ *comparison operations* between two subsets with one element each, it is possible to identify the irregular element. But, the problem we will pose here is that of identifying this element with as few comparison operations as possible between the subsets of $\mathbf{S}$, where $\mathbf{S}$ is a set with n elements.

Keywords: Irregular element, Comparison operation, Positive deviation.

1 Solving the problem for small values of n

• $\mathbf{n=1}$
Here the solution is evident. The only element of the set is also the irregular element. So, we have zero *comparison operations*.

• $\mathbf{n=2}$
In this case the problem has no solution. This numbers of elements makes it impossible to identify the irregular element.

• $\mathbf{n=3}$
We do the *comparison operation* for a pair of them and are two possible results:

a) There is no deviation according to specific feature of the irregular element. So, the comparing elements are regular and consequently the remaining element is the irregular element. In this case, only one *comparison operation* was used.
b) There is deviation according to specific feature of the irregular element. In this case we do the second *comparison operation* between an element from the used pair and the third element. If there is a deviation, then the element comparing twice is the irregular element. Otherwise, the element of the first pair compared, that is not used in the second comparison will be the irregular element. As can be seen, for $n=3$, the maximum number of *comparison operations* that guarantees the identification of the irregular element is 2.

• $\mathbf{n=4}$
Acting as in the case of $n=3$, after two *comparison operations* for subsets with a single element, where one element of the first compared pair is used again in the second, the identification of the irregular element is achieved. So, the maximum number of *comparison operations* that guarantees the identification of the irregular element is 2. Indeed, if we have positive deviation in both two *comparison operations*, then the element used twice will be the irregular element. If we have no positive deviation in both two *comparison operations*, then the fourth element (not used) is the irregular element.

2 Consideration of the problem for a natural number n

We will distinguish two cases:

(I) The number of elements of the set S is a term of the sequence $a_p=4\cdot 3^p$, namely

$$n=4\cdot 3^p, p=1,2,...$$

We denote by $\mathcal{P}_1$ the following procedure:

"*Divide the set* $\mathbf{S}$ *into three disjoint subsets* $\mathbf{A}, \mathbf{B}, \mathbf{C}$, *with the same number of elements* $4 \cdot 3^{p-1}$. *So,* $\mathbf{S} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$. *After that we do the comparison operation for* $\mathbf{A}$ *and* $\mathbf{B}$".

We observe that the implementation of the procedure $\mathcal{P}_1$ requires only one *comparison operation*. Consider the possible comparison results as follows:

(Ia) The sets $\mathbf{A}$ and $\mathbf{B}$ do not have deviation.

It is clear that the irregular element in this case belongs to subset $\mathbf{C}$. It is the same case as (I), but now, the number of elements is $4 \cdot 3^{p-1}$. We do $\mathbf{S} := \mathbf{C}$ and repeat the procedure $\mathcal{P}_1$

If we obtain the case (I) after the implementation of the procedure $\mathcal{P}_1$, and this result occurs at any repeat of that procedure, then after p-steps we will obtain the set $\mathbf{C}$ with $4 \cdot 3^0 = 4$ elements, to which belongs the irregular element. It is now known that for such sets the individualization of the irregular element requires only two comparison operations. So, when in any *comparison operation* we obtain the case (Ia), the solution of the problem arises with:

$$N = 2 + p = 2 + \log_3 \frac{4 \cdot 3^p}{4} = 2 + \log_3 \frac{n}{4} \tag{1}$$

comparison operations.

(Ib) After $k(k = 1, 2, 3, ..., p-1)$ consecutive executions of the procedure $\mathcal{P}_1$, comes out for the first time that, the compared subsets have a deviation. So the k-th execution of the procedure $\mathcal{P}_1$ has shown that one of the subsets $\mathbf{A}$ or $\mathbf{B}$, with $4 \cdot 3^{p-k}$ elements, has *positive deviations* from the other. We always mark by $\mathbf{A}$ the subset with *positive deviation* from $\mathbf{B}$. It is clear that all the elements of the third subset $\mathbf{C}$ are regular as the irregular element is located in either of the subsets $\mathbf{A}$ or $\mathbf{B}$. In case (Ib) we will execute the procedure $\mathcal{P}_2$ as follows:

- *Divide each of the subsets* $\mathbf{A}$ *and* $\mathbf{B}$, *where each of them has* $4 \cdot 3^{p-k}$ *elements, in three disjoint subsets with the same number of elements* $4 \cdot 3^{p-k-1}$ *and mark them* $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$, *and* $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$
- *Construct the combined sets* $B_1 \cup A_2$ *and* $A_1 \cup B_2$ *for comparison between them*
- *Execute the comparison operation between the* $B_1 \cup A_2$ *and* $A_1 \cup B_2$

It is evident that the procedure $\mathcal{P}_2$ has only one *comparison operation*. It is clear that, when there are *positive deviations* between the subsets A_i and B_i, for $i = 1, 2, 3$, the subest A_i has *positive deviation* from the set B_i, because according to the agreement that we made above, the subsets $\mathbf{A}$ have positive deviations from the subsets $\mathbf{B}$. Let us consider now the three possible outcomes of the comparison operation:

a) The subset $B_1 \cup A_2$ has *positive deviation* from the subset $A_1 \cup B_2$ and we obtain these statements:

- Subsets $\mathbf{A_3}$ and $\mathbf{B_3}$ contain only regular elements;
- Subset $\mathbf{A_2}$ has *positive deviations* from the subset $\mathbf{B_2}$
- The irregular element is located in one of the subsets $\mathbf{A_2}$ and $\mathbf{B_2}$;
- Subsets $\mathbf{A_1}$ and $\mathbf{B_1}$ contain only regular elements.

For further investigation we consider the pair of subsets $\mathbf{A_2}$ and $\mathbf{B_2}$, where one of them contains the irregular element, and we do:

$$\mathbf{A} := \mathbf{A_2} \quad \text{and} \quad \mathbf{B} := \mathbf{B_2}.$$

Now, each of the subsets $\mathbf{A}$ and $\mathbf{B}$ has $4 \cdot 3^{p-k-1}$ elements. So, we repeat the procedure $\mathcal{P}_2$ for these two sets, where the three subsets $\mathbf{A_i}$ and the three subsets $\mathbf{B_i}$ that we obtain again from the division of $\mathbf{A}$ and $\mathbf{B}$ now will have $4 \cdot 3^{p-k-2}$ elements.

b) The subsets $B_1 \cup A_2$ and $A_1 \cup B_2$ have no deviation. This result shows that the irregular element belong to $A_3 \cup B_3$ and we repeat the procedure $\mathcal{P}_2$ doing $\mathbf{A} := \mathbf{A_3}; \mathbf{B} := \mathbf{B_3}$.

c) The subset $A_1 \cup B_2$ has *positive deviation* from the subset $B_1 \cup A_2$ and reasoning as in case a) we will have:

- The irregular element belongs to one of the subsets $\mathbf{A_1}$ and $\mathbf{B_1}$
- For further investigation we consider the pair of subsets $\mathbf{A_1}$ and $\mathbf{B_1}$
- Repeat the procedure $\mathcal{P}_2$ doing: $\mathbf{A} := \mathbf{A_1}; \mathbf{B} := \mathbf{B_1}$, where each of these subsets has $4 \cdot 3^{p-k-1}$ elements.

We note that in all the above cases, the analysis of the results of the implementation of procedure $\mathcal{P}_2$ leads to its repetition (and not to the implementation of procedure $\mathcal{P}_1$) on a pair of subsets with a number of elements three times smaller than their direct predecessors. Therefore after $p-k$ repetitions of procedure $\mathcal{P}_2$, we obtain a pair of subsets each of them with $4 \cdot 3^0 = 4$ elements, that contains the irregular element.

Let now $\mathbf{A} := \{a_1, a_2, a_3, a_4\}$ and $\mathbf{B} := \{b_1, b_2, b_3, b_4\}$ be subsets with the same assumption that $\mathbf{A}$ has *positive deviation* from $\mathbf{B}$. In this final situation we divide each of them in two subsets with two elements:

$$\mathbf{A_1} = \{a_1, a_2\}, \mathbf{A_2} = \{a_3, a_4\}, \mathbf{B_1} = \{b_1, b_2\} \, \mathbf{B_2} = \{b_3, b_4\}.$$

After that, we consider the subsets $B_1 \cup A_2$ and $A_1 \cup B_2$. Now execute the *comparison operation* between them, which has only two outcomes:

1) $B_1 \cup A_2$ has *positive deviation* from the subset $A_1 \cup B_2$.
This outcome shows that the irregular element belongs to $A_2 \cup B_2 = \{a_3, a_4, b_3, b_4\}$.

2) $A_1 \cup B_2$ has *positive deviation* from the subset $B_1 \cup A_2$.
In this case the irregular element belongs to $A_1 \cup B_1 = \{a_1, a_2, b_1, b_2\}$.

As can be seen, in each of the above two cases it remains to individualize the irregular element in a subset of four elements. It is already known that two *comparison operations* are sufficient to determine the irregular element. Thus the total number of *comparison operations* executed in the three problem-solving phases in case (Ib) is:

$$N = k + (p - k) + 2 = 2 + p = 2 + \log_3 \frac{n}{4} \tag{2}$$

(II) The number of elements of the set $\mathbf{S}$ is a natural number n that does not belong to the sequence $\{4 \cdot 3^p\}$. In this case, there exists a natural number p such that:

$$4 \cdot 3^p < n < 4 \cdot 3^{p+1} \tag{3}$$

To examine the problem in such sets we will distinguish two cases according to the number n:

(IIa) $\frac{2}{3} 4 \cdot 3^{p+1} < n < 4 \cdot 3^{p+1}$

As above, the number n can be written in the form $n = 2 \cdot 4 \cdot 3^p + q$, where $0 < q < 4 \cdot 3^p$. In this case we divide the set $\mathbf{S}$ in three disjoint subsets $\mathbf{A}, \mathbf{B}, \mathbf{C}$ with number of elements $4 \cdot 3^p$, $4 \cdot 3^p$ and q respectively. Execute the *comparison operation* for the subsets $\mathbf{A}, \mathbf{B}$ and consider the outcomes:

1) The subsets $\mathbf{A}$ and $\mathbf{B}$ have no deviation. So, the subsets $\mathbf{A}$ and $\mathbf{B}$ contain only regular elements and consequently the irregular element belongs to the subset $\mathbf{C}$. Complete the subset $\mathbf{C}$ with elements from $A \cup B$, until it becomes with $4 \cdot 3^p$ elements and denote this new set by C'. From the outcomes of the case (I), we will identify the irregular element with $2 + p$ comparison operations. So, including the first comparison, the total number of them will be

$$N = 1 + 2 + p \tag{4}$$

Now, dividing by 4 all sides of inequality (3) and taking their logarithms with base 3 will have: $p < \log_3 \frac{n}{4} < p + 1$. So, $p + 1$ is the first natural number greater than $\log_3 \frac{n}{4}$. Now, if we denote in this case

$$p + 1 = [\log_3 \frac{n}{4}] \tag{5}$$

then, from (4) and (5) we can express the number of *comparison operations* N in terms of n as below:

$$N = 2 + [\log_3 \frac{n}{4}] \tag{6}$$

2) Subsets $\mathbf{A}$ and $\mathbf{B}$ have deviation and we keep the agreement that $\mathbf{A}$ has *positive deviation* from $\mathbf{B}$. With this result of the comparison operation between $\mathbf{A}$ and $\mathbf{B}$, we are in the same conditions as in case (Ib). So, each of $\mathbf{A}$ and $\mathbf{B}$ will be divided in three disjoined subsets with $4 \cdot 3^{p-1}$ elements. According to the mode of operation used in case (Ib), in our conditions where $k = 1$, the number of *comparison operations* to identify the irregular element in $\mathbf{A}$ or $\mathbf{B}$ will be $1 + (p - 1) + 2 = 2 + p$. Now, adding the first *comparison operation*, from which, we found that $\mathbf{A}$ and $\mathbf{B}$ had deviation between them, the total number of *comparison operations* to individualize the irregular element will be $2 + p + 1$. So, for the number $\mathbf{N}$, we obtain again the formula (6).

(IIb) $4 \cdot 3^p < n < \frac{2}{3} 4 \cdot 3^{p+1}$

As in (IIa), even in this case we will have $n = 4 \cdot 3^p + q$ where $0 < q < 4 \cdot 3^p$. Under these conditions, we divide the set $\mathbf{S}$ into three disjoined subsets $\mathbf{A}, \mathbf{B}, \mathbf{C}$ with a number of elements respectively $2 \cdot 3^p$, $2 \cdot 3^p$ and q. Execute the *comparison operation* for the subsets $\mathbf{A}, \mathbf{B}$ and consider the following possible cases:

1) The subsets $\mathbf{A}$ and $\mathbf{B}$ have no deviation. In this case the irregular element belongs to $\mathbf{C}$. We complete the subset $\mathbf{C}$ with regular elements taken from $A \cup B$ until it becomes with $4 \cdot 3^p$ elements. As we have shown in the case (IIa), we identify the irregular element in this subset by $2 + p$ *comparison operations*, and including the first *comparison operation* between the subsets $\mathbf{A}$ and $\mathbf{B}$, the total number of them will be $2 + p + 1$ which is again the same as formula (6).

2) The subset $\mathbf{A}$ has *positive deviation* from the set $\mathbf{B}$. So, the irregular element belongs to the set $A \cup B$ that will have $2 \cdot 3^p + 2 \cdot 3^p = 4 \cdot 3^p$ elements. For this subset with such a number of elements, as shown in point (I), the identification of the irregular element requires a number of $2 + p$ *comparison operations*. Adding the first *comparison operation* between subsets A and B, this number becomes again:

$$N = 2 + (p + 1) = 2 + [\log_3 \frac{n}{4}] \tag{7}$$

(IIc) $n = \frac{2}{3} 4 \cdot 3^{p+1} = 8 \cdot 3^p$

In this case we divide the set $\mathbf{S}$ into four disjoined subsets $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ with $2 \cdot 3^p$ elements. Execute the *comparison operation* between $\mathbf{A}$ and $\mathbf{B}$. If come out a deviation, the irregular element belongs to $A \cup B$, which has $2 \cdot 3^p + 2 \cdot 3^p = 4 \cdot 3^p$ elements, otherwise the irregular element belongs to $C \cup D$, with the same number of elements $4 \cdot 3^p$.
So we obtain again the same formula (7) for the number N of *comparison operations* that is needed to identify the irregular element in terms of the number n of elements of $\mathbf{S}$, in which is this irregular element.

3 Conclusion

The way of searching for the irregular element in a set with n elements, presented above, guarantees its identification with $N = 2 + [\log_3 \frac{n}{4}]$ *comparison operations*, where the symbol $[\log_3 \frac{n}{4}]$ means $\log_3 \frac{n}{4}$, if it is an integer, or the first integer greater than $\log_3 \frac{n}{4}$, if it is not an integer.

There are indications that lead us to believe that this number is the **minimum possible** of the appropriate operations to solve the problem of the irregular element.

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Some Properties of the Z-Symmetric Manifold Admitting Conharmonic Curvature Tensor

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Abstract: The object of the present paper is to study the Z-symmetric manifold with the conharmonic curvature tensor. In the first section, the definition of the conharmonic curvature tensor is given. In the second section, some properties of the Z-symmetric tensor are mentioned. In the third section, Z-symmetric manifold admitting conharmonic curvature tensor is considered and some theorems about this manifold are proved. In the last section, an example for the existence of these manifolds is given.

Keywords: Codazzi tensor, Conharmonic curvature tensor, Recurrent tensor, Z-symmetric tensor.

1 Introduction

Conformal geometry has deep importance in pure mathematics, such as complex analysis, Riemann surface theory, differential geometry and algebraic topology, [1]-[3]. Computational conformal geometry is important in digital geometry processing. Discrete conformal geometry has been presented to compute conformal mapping which has been broadly applied in numerous practical fields, including computer vision and graphics, visualization, medical imaging, etc. In medical imaging, conformal geometry has been applied to surface parametrization and extract intrinsic features for natural objects like the brain, colon, spleen and other human organs.

Historically, conformal mappings have been considered in many monographs, surveys and papers. Also, the theory of conformal mappings has very important applications in general relativity.

Let (M, g) and $(\bar{M}, \bar{g})$ be two n-dimensional Riemannian manifolds with metric tensors g_{ij} and $\bar{g}_{ij}$, respectively. Both metrics are defined by a common coordinate system (x^i). The correspondence between (M, g) and $(\bar{M}, \bar{g})$ is conformal, if the fundamental tensors g_{ij} and $\bar{g}_{ij}$ of two manifolds M and $\bar{M}$ are related by the relation

$$\bar{g}_{ij}(x) = e^{2\sigma(X)} g_{ij}(x) \tag{1}$$

where $\sigma(x)$ is a scalar function of x's.

By the transformation (1), it also follows that the relation between the Christoffel symbols Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ compatible with the metrics g_{ij} and $\bar{g}_{ij}$, respectively, is given by

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij} \tag{2}$$

where $\sigma_i = \frac{\partial \sigma}{\partial x^i}$, $\sigma^h = g^{hi}\sigma_i$, g^{ij} are the components of the inverse matrix to g_{ij}, and δ^h_i is the Kronecker delta.

A conformal mapping is called homothetic if the function σ is constant, that is, $\bar{g}_{ij}(x) = c g_{ij}(x)$. The condition is equivalent to $\sigma_i = 0$, hence, the mapping is also an affine one.

Denoting R^h_{ijk} and $\bar{R}^h_{ijk}$ are the Riemann tensors of the manifolds M and $\bar{M}$, respectively, then we have ([4, 5])

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{hl}(\sigma_{lk} g_{ij} - \sigma_{li} g_{jk}) + (\delta^h_k g_{ij} - \delta^h_j g_{ik})\Delta_1 \sigma$$

$$\bar{S_{ij}} = S_{ij} + (n-2)\sigma_{ij} + (\Delta_2 \sigma + (n-2)\Delta_1 \sigma) g_{ij}$$

$$\bar{r} = e^{-2\sigma}(r + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma) \tag{3}$$

where $\sigma_i = \partial_i \sigma$, $\Delta_1 \sigma = g^{ij}\sigma_i \sigma_j$, $\Delta_2 \sigma = g^{ij}\sigma_{i,j}$, $\sigma_{ij} = \sigma_{i,j} - \sigma_i \sigma_j$. We denote that $S_{ij} = R^h_{ijh}$ and $\bar{S}_{ij} = \bar{R}^h_{ijh}$ are the Ricci tensors and $r = S_{ij} g^{ij}$ and $\bar{r} = \bar{S}_{ij} \bar{g}^{ij}$ are the scalar curvatures with respect to M and $\bar{M}$, respectively.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general, a harmonic function is not invariant under the conformal transformation. In [6], Ishii obtained the conditions at which a harmonic function remains invariant and he introduced the conharmonic transformation as a subgroup of the conformal transformation (1) satisfying the condition [6]

$$\sigma^{h}_{,h} + \sigma^{h}_{,h}\sigma^{h}_{,} = 0 \tag{4}$$

where comma denotes the covariant differentiation with respect to the metric g.

Thus, we can say that the conharmonic transformation which is a special type of conformal transformation preserves the harmonicity of smooth functions. It is well known that such transformations have invariant tensors, so-called the conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor, that is, it possesses the classical symmetry properties of the Riemannian curvature tensor.

A rank-four tensor L that remains invariant under the conharmonic transformation of a Riemannian manifold (M, g) is given by

$$L(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(X,U)S(Y,Z) - g(Y,U)S(X,Z)] \tag{5}$$

where R and S denote the Riemannian curvature tensor of type (0,4) defined by $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type (0,2), respectively. The curvature tensor defined by (5) is known as the conharmonic curvature tensor. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called a conharmonically flat. Thus, this tensor represents the deviation of the manifold from the conharmonic flatness.

Q denotes the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S of type (0,2), that is

$$g(QX,Y) = S(X,Y). \tag{6}$$

Let $\{e_i, \quad i = 1,2,...,n\}$ be an orthonormal basis of the tangent space at each point of the manifold. From (5), we have

$$\begin{aligned}\overline{L}(X,Y) &= \sum_{i=1}^{n} L(X,e_i,e_i,Y) = \sum_{i=1}^{n} L(e_i,X,Y,e_i) \\ &= -\frac{r}{n-2} g(X,Y)\end{aligned} \tag{7}$$

and

$$\begin{aligned}\sum_{i=1}^{n} L(e_i,e_i,X,Y) &= \sum_{i=1}^{n} L(X,Y,e_i,e_i) \\ &= 0\end{aligned} \tag{8}$$

where r is the scalar curvature of the manifold. Also, from (5) it follows that [7]

$$\begin{aligned}&L(X,Y,Z,U) = -L(Y,X,Z,U) \\ &L(X,Y,Z,U) = -L(X,Y,U,Z) \\ &L(X,Y,Z,U) = L(Z,U,X,Y) \\ &L(X,Y,Z,U) + L(X,Z,U,Y) + L(X,U,Y,Z) = 0.\end{aligned} \tag{9}$$

In [7], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetries and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. This tensor has valuable applications in general relativity. In [8], Abdussatter investigated its physical significance in the theory of general relativity. The conharmonic transformation has also been studied by Siddique and Ahsan [9], Ghosh, De and Taleshian [10], and many others.

A non-flat Riemannian manifold which is called a recurrent manifold [11] if the curvature tensor of this manifold satisfies the relation

$$(\nabla_W R)(X,Y,Z,U) = A(W)R(X,Y,Z,U) \tag{10}$$

where A is a non-zero 1-form. A non-flat Riemannian manifold which is called a Ricci-recurrent manifold if the Ricci tensor of this manifold satisfies the relation([12]-[14])

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) \tag{11}$$

where A is a non-zero 1-form.

A Riemannian manifold has a Ricci tensor of Codazzi type if the Ricci tensor S of type (0,2) is non-zero and satisfies the following condition, [15]

$$(\nabla_X S)(Y,W) = (\nabla_Y S)(X,W). \tag{12}$$

2 Z-Tensor on a Riemannian Manifold

In 2012, Mantica and Molinari defined a generalized symmetric tensor of type $(0,2)$ which is called Z-tensor and given by, [16]

$$Z_{kl} = S_{kl} + \phi g_{kl}, \tag{13}$$

where ϕ is an arbitrary scalar function. The scalar $\bar{Z}$ is the trace of the Z-tensor and from (13), it can be written as

$$\bar{Z} = g^{kl} Z_{kl} = r + n\phi. \tag{14}$$

The classical Z tensor is defined with the choice $\phi = -\frac{1}{n}r$. Shortly, the generalized Z-tensor is called as the Z-tensor. In some cases, the Z-tensor gives the several well known structures on Riemannian manifolds. For example, i) If $Z_{kl} = 0$ (i.e, Z-flat) then this manifold reduces to an Einstein manifold, [17]; ii) If $\nabla_j Z_{kl} = \lambda_j Z_{kl}$ (Z-recurrent) then this manifold reduces to a generalized Ricci recurrent manifold [18]; iii) If $\nabla_j Z_{kl} = \nabla_k Z_{jl}$, (Codazzi tensor) then we find $\nabla_j S_{kl} - \nabla_k S_{jl} = \frac{1}{2(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)r$, [19]. This result gives us that this manifold is a nearly conformal symmetric manifold ($(NCS)_n$), [20]; iv) The relation between the Z-tensor and the energy-stress tensor of Einstein' s equation, [21], with the cosmological constant Λ is $Z_{jl} = kT_{jl}$ where $\phi = -\frac{1}{2}r + \Lambda$ and k is the gravitational constant. In this case, the Z-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ. The vacuum solution (Z=0) determines an Einstein space $\Lambda = \left(\frac{n-2}{2n}\right)r$; the conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla_j Z_{kl} = 0$ then this spacetime gives the conserved enery-momentum density.

This manifold has received a great deal of attention, and is studied in considerable details by many authors ([16], [22]-[28]), etc. Motivated by the above studies, in the present, we examine the properties of the Z-symmetric manifold with the conharmonic curvature tensor.

The present paper is organized as follows: We give some definitions in section 1 and section 2. In section 3, we study the Z-symmetric manifold with the conharmonic curvature tensor. In this section, we prove some theorems related by the properties of these manifolds. In section 4, we give an example for the existence of these manifolds.

3 Z-Symmetric Manifold with the Conharmonic Curvature Tensor

In this section, we consider Z-symmetric manifold with the conharmonic curvature tensor. In local coordinates, from (5) and (13), the relation between the Z-tensor and the conharmonic curvature tensor is found as

$$L_{hijk} = R_{hijk} - \frac{1}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] + \frac{2\phi}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}] \tag{15}$$

By taking the covariant derivative of (15), we can find

$$L_{hijk,l} = R_{hijk,l} - \frac{1}{n-2}[g_{ij}Z_{hk,l} - g_{ik}Z_{hj,l} + g_{hk}Z_{ij,l} - g_{hj}Z_{ik,l}] + \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{16}$$

Now, suppose that our manifold is Z-recurrent. Considering the equation (11) for the Z-tensor, we can write $Z_{ij,l} = \lambda_l Z_{ij}$. Hence, we see from (16) that

$$L_{hijk,l} = R_{hijk,l} - \frac{\lambda_l}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] + \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{17}$$

It is obtained by the equations (15) and (16)

$$\frac{1}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] = R_{hijk} - L_{hijk} + \frac{2\phi}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{18}$$

By the aid of (18), the expression (17) can be written as

$$L_{hijk,l} - \lambda_l L_{hijk} = R_{hijk,l} - \lambda_l R_{hijk} + \frac{2}{n-2}(g_{ij}g_{hk} - g_{ik}g_{hj})(\phi_l - \lambda_l\phi). \tag{19}$$

Theorem 1. *Let (M,g) be a Z-recurrent Riemannian manifold. If the conharmonic curvature tensor of (M,g) is recurrent with the recurrence vector field λ_l then the scalar function ϕ satisfies the relation $\phi_l = \lambda_l\phi$.*

Proof: Assume that (M,g) a is Z-recurrent manifold admitting the recurrence vector field λ_l. If (M,g) is also conharmonically recurrent manifold admitting the recurrence vector field λ_l, we have from (17) and (19),

$$R_{hijk,l} = \lambda_l R_{hijk} + \frac{2}{n-2}(\phi_l - \lambda_l\phi)(g_{ij}g_{hk} - g_{ik}g_{hj}). \tag{20}$$

Multiplying (20) by $g^{ij}g^{hk}$, we get

$$r_{,l} = \lambda_l r + \frac{2n(n-1)}{n-2}(\phi_l - \lambda_l\phi). \tag{21}$$

On the other hand, if we assume that (M,g) is also conharmonically recurrent then we have from (10) for the conharmonic curvature tensor

$$L_{hijk,l} = \lambda_l L_{hijk}. \tag{22}$$

Multiplying (22) by $g^{ij}g^{hk}$ and using the relation (7), we can easily see that

$$r_{,l} = \lambda_l r. \tag{23}$$

Finally, we see from (23) that $\phi_l = \lambda_l\phi$. This completes the proof. □

Theorem 2. *If the Z-tensor of (M,g) is covariantly constant, then the trace of the conharmonic curvature tensor is also constant.*

Proof: Assume that the Z-tensor of (M,g) is covariantly constant, i.e., we have

$$Z_{ij,k} = 0. \tag{24}$$

If we use (24) in (16), we find

$$L_{hijk,l} = R_{hijk,l} + \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{25}$$

Multiplying (25) by $g^{ij}g^{hk}$, we get

$$\bar{L}_{,l} = r_{,l} + \frac{2n(n-1)}{n-2}\phi_l. \tag{26}$$

Now, taking the covariant derivative of (13), we obtain

$$Z_{ij,l} = S_{ij,l} + \phi_l g_{ij}. \tag{27}$$

Because M is of the covariantly constant Z-tensor, the equation (27) reduces to following form

$$S_{ij,l} = -\phi_l g_{ij}. \tag{28}$$

Thus, multiplying (28) by g^{ij}, the equation (28) reduces to

$$r_{,l} = -n\phi_l. \tag{29}$$

Changing the indices j and l in the equation (28) and then substracting these two equations, we obtain

$$S_{ij,l} - S_{il,j} = -\phi_l g_{ij} + \phi_j g_{il}. \tag{30}$$

Multiplying the equation (30) by g^{ij}, we get

$$r_{,l} - S^j_{l,j} = (1-n)\phi_l. \tag{31}$$

Now, using the expression $S^j_{l,j} = \frac{1}{2}r_{,l}$, known as the Ricci Identity, in (31), we find

$$r_{,l} = 2(1-n)\phi_l. \tag{32}$$

Thus, comparing the equations (29) and (32), one can obtain $\phi_l = 0$ and $r_{,l} = 0$, i.e., the scalar function ϕ and the scalar curvature r must be constants.

Finally, considering the equation (26), we obtain

$$\bar{L}_{,l} = 0. \tag{33}$$

In this case, the equation (33) shows that the trace of the conharmonic curvature tensor is constant. Thus, the proof is completed. □

Theorem 3. *Let the conharmonic curvature tensor of (M,g) be covariantly constant. The trace of the Z-tensor is harmonic function if and only if the 1-from ϕ_l generated by the scalar function ϕ is divergence-free.*

Proof: Assume that the conharmonic curvature tensor of (M,g) is covariantly constant. In this case, we have

$$R_{hijk,l} = \frac{1}{n-2}[g_{ij}S_{hk,l} - g_{ik}S_{hj,l} + g_{hk}S_{ij,l} - g_{hj}S_{ik,l}]. \tag{34}$$

Multiplying (34) by g^{hk}, we get

$$r_{,l} = 0. \tag{35}$$

Thus, the scalar curvature r of our manifold must be constant. Now, taking the covariant derivative of (14), we find

$$\bar{Z}_{,l} = r_{,l} + n\phi_l. \tag{36}$$

And, putting the equation (35) in (36), we obtain

$$\bar{Z}_{,l} = n\phi_l. \tag{37}$$

If we take the covariant derivative of (37), we have

$$\bar{Z}_{,lm} = n\phi_{l,m}. \tag{38}$$

Multiplying (38) by g^{lm}, we find

$$\Delta\bar{Z} = g^{lm}\bar{Z}_{,lm} = n\phi^{l}_{,l}. \tag{39}$$

In this case, if the trace of the Z-tensor is harmonic, the vector field ϕ_l is divergence-free. The converse is also true. Thus, the proof is completed. □

Theorem 4. *In a conharmonically recurrent manifold, if the Z-tensor of (M,g) is Codazzi type then the relation between the vector fields ϕ_l and λ_l is obtained as*

$$\phi_l = \frac{r}{2(1-n)}\lambda_l.$$

Proof: Let (M,g) be given as a Riemannian manifold with the recurrent conharmonic curvature tensor admitting the recurrence vector field λ_l. By the aid of the equations (10) and (16), we get

$$\lambda_l L_{hijk} = R_{hijk,l} - \frac{1}{n-2}[g_{ij}Z_{hk,l} - g_{ik}Z_{hj,l} + g_{hk}Z_{ij,l} - g_{hj}Z_{ik,l}] + \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{40}$$

Thus, multiplying (40) by g^{hk}, it can be found that

$$\lambda_l L_{ij} = S_{ij,l} - \frac{1}{n-2}[g_{ij}\bar{Z}_{,l} + (n-2)Z_{ij,l}] + \frac{2(n-1)}{(n-2)}\phi_l g_{ij}. \tag{41}$$

Again, multiplying (41) by g^{ij}, we find

$$\lambda_l \bar{L} = r_{,l} - \frac{2(n-1)}{(n-2)}\bar{Z}_{,l} + \frac{2n(n-1)}{(n-2)}\phi_l. \tag{42}$$

From (7) and (14), the equation (42) reduces to

$$r_{,l} = \lambda_l r. \tag{43}$$

Now, we assume that the Z-symmetric tensor is Codazzi type. In this case, considering the Z-tensor for the equation (12), it can be written

$$Z_{ij,l} - Z_{il,j} = 0. \tag{44}$$

Thus, from (13) and (44), we conclude

$$S_{ij,l} - S_{il,j} = \phi_j g_{il} - \phi_l g_{ij}. \tag{45}$$

Multiplying (45) by g^{ij}, we get

$$r_{,l} - S^{j}_{l,j} = (1-n)\phi_l. \tag{46}$$

By the Ricci identity, (46) takes the form

$$r_{,l} = 2(1-n)\phi_l. \tag{47}$$

So, comparing the equations (43) and (47), it can be seen that

$$\phi_l = \frac{r}{2(1-n)}\lambda_l. \tag{48}$$

Thus, the proof is completed. □

In the following theorems, our manifold (M,g) that admits recurrent conharmonic curvature tensor and Codazzi type Z-tensor is shown by $(\bar{M}, g)$.

Theorem 5. *A necessary and sufficient condition the recurrent vector field λ_l of $(\bar{M}, g)$ to be divergence-free is that the divergence of the generated vector field ϕ_l to be*

$$\phi^l_{,l} = \frac{2(1-n)}{r}||\phi||^2.$$

Proof: Differentiating covariantly of the equation (48), we obtain

$$\phi_{l,m} = \frac{r_{,m}}{2(1-n)}\lambda_l + \frac{r}{2(1-n)}\lambda_{l,m}. \tag{49}$$

If we put the relation (47) in (49) instead of $r_{,m}$ then we get

$$\phi_{l,m} = \phi_m\lambda_l + \frac{r}{2(1-n)}\lambda_{l,m}. \tag{50}$$

Again from (48), (50) takes the form

$$\phi_{l,m} = \frac{2(1-n)}{r}\phi_m\phi_l + \frac{r}{2(1-n)}\lambda_{l,m}. \tag{51}$$

Now, multiplying (51) by g^{lm}, we get

$$\phi^l_{,l} = \frac{2(1-n)}{r}||\phi||^2 + \frac{r}{2(1-n)}\lambda^l_{,l}. \tag{52}$$

It is clear from (52) that if we assume λ_l is divergence-free then $\phi^l_{,l} = \frac{2(1-n)}{r}||\phi||^2$. The converse is also true. This completes the proof. □

Theorem 6. *A necessary and sufficient condition the generated vector field ϕ_l of $(\bar{M}, g)$ to be divergence-free is that the recurrent vector field λ_l be of negative value and*

$$\lambda^l_{,l} = -||\lambda||^2$$

where $||\lambda||$ is the length of the vector field λ_l.

Proof: By putting (48) in (50), we find

$$\phi_{l,m} = \frac{r}{2(1-n)}(\lambda_l\lambda_m + \lambda_{l,m}). \tag{53}$$

Thus, multiplying (53) by g^{lm} then we get

$$\phi^l_{,l} = \frac{r}{2(1-n)}(||\lambda||^2 + \lambda^l_{,l}). \tag{54}$$

If we assume that $\phi^l_{,l} = 0$ then we can see that the divergence of λ_l is of negative value and it satisfies the relation

$$\lambda^l_{,l} = -||\lambda||^2.$$

The converse is also true. Thus, the proof is completed. □

4 An Example for the Existence of These Manifolds

We define a Riemannian metric on 4-dimensional real number space $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^2}(dx^2)^2 + (dx^3)^2 + (\sin x^3)^2(dx^4)^2 \tag{55}$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are found as, respectively,

$$\Gamma^1_{11} = \Gamma^2_{22} = \frac{1}{2}, \quad \Gamma^3_{44} = -\frac{\sin 2x^3}{2}, \quad \Gamma^4_{34} = \cot x^3,$$

$$R_{3434} = (sinx^3)^2,$$

$$S_{33} = -1, \quad S_{44} = -(\sin x^3)^2. \tag{56}$$

and the components obtained by the symmetry properties. In this case, from (56), the scalar curvature is r=-2. Then the only non-zero components of the Z-symmetric tensor are found as from (13) and (56)

$$Z_{11} = \phi e^{x^1}, \quad Z_{22} = \phi e^{x^2}, \quad Z_{33} = -1 + \phi, \quad Z_{44} = -(\sin x^3)^2 + \phi(\sin x^3)^2. \tag{57}$$

In view of the above relations, the only non-zero components of the covariant derivatives of the Z-symmetric tensor are obtained as follows:

$$\begin{aligned} &Z_{11,j} = e^{x^1}\phi_j = 0 \qquad j = 1,2,3,4 \\ &Z_{22,j} = e^{x^2}\phi_j = 0 \qquad j = 1,2,3,4 \\ &Z_{33,j} = \phi_j = 0 \qquad j = 1,2,3,4 \\ &Z_{44,j} = -(sinx^3)^2\phi_j = 0 \qquad j = 1,2,3,4 \end{aligned} \tag{58}$$

It can be seen that the scalar function ϕ is independent of x^1, x^2, x^3, x^4 coordinates. Thus, the scalar function ϕ must be constant.

Hence, from the equation (14), we can say that the trace of the conharmonic curvature tensor must be constant. Thus, this is an example satisfying Theorem2.

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A Finite Difference Method to Solve the Linear Lane-Emden Equations

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Abstract: The purpose of this paper is to present an effective a numerical method to solve the linear Lane-Emden equation using finite difference method. This numerical method base on the Lagrange polynomial interpolation to obtain finite difference equation. Firstly, we obtain the finite difference formulas of $y'(x)$ and $y''(x)$, then we transform the linear Lane-Emden equation into the finite difference equation. Using boundary conditions, we get the desired numerical results. Finally, some illustrative examples are included to show the validity and applicability of the given method.

Keywords: Error analysis, Finite difference equation, Lagrange interpolation, Lane-Emden equation, Numerical method.

1 Introduction

The Lane-Emden type equations are singular boundary value problems relating to ordinary differential equations, which is second order. These equations used in mathematical physics, astrophysics to model several phenomena, such as, stellar structure [1], thermal behavior of gas spheres and thermionic currents [2]-[4]. Lane-Emden equations have the following form:

$$y^{''}(x) + \frac{\alpha}{x} y^{'}(x) + q(x)y(x) = r(x), \quad 0 < x < 1, \; \alpha \geq 0 \tag{1}$$

with the following conditions,

$$y(0) = 0, \;\; y(1) = \beta \tag{2}$$

where $q(x)$ is a real value function and $r(x)$ is an analytical function. Nowadays, Lane-Emden equations have been drawn interest by many more researchers, especially, to find numerical solution values. For this propose, many more researchers have presented some papers to find numerical solutions of Eq.(1). We give some examples Legendre wavelet [5], Taylor wavelet method [6], Chebyshev operational matrix method [7], B-spline method [8], Hermite polynomials [9], modified decomposition method [10], Bernstein operational matrix method [11], Pade series [12], homotopy perturbation method [13], variational iteration method [14], Adomian decomposition method [15], Haar wavelet collocation method [16], continuous polynomial wavelet [17], a variational iteration method [18], Chebyshev spectral methods [19], modified Mickens-type NSFD schemes [20]. In this paper, we investigate a finite difference method with Lagrange polynomials for solving a special type of singular value problem, Lane-Emden equation. Firstly, Algorithm of finite difference is devised by using Taylor's Theorem a means of constructing approximations [21]. Leonhard Euler have applied this method to initial value problems in 1768-1769. Thus, Euler made a breakthrough in applied mathematics and constructed numerical mathematics. Recently, some researchers extended this idea for more problems, partial differential equations and more [22]-[25]. In this paper, we shall apply the second order Lagrange polynomial approaches were made instead of first and second order derivatives in the Eq.(1). Thus, it gives us a difference equation which includes approximate solutions of the exact solution at some points. If the equation is rearranged and the matrix form is obtained and solved, the approximate solutions at each point were found. In section 2 we describe the method that is formed by combining finite differences and interpolation. In section 3, we examined under what conditions the method is stable. In the last section, we applied this method to the Lane-Emden equation and evaluated the results.

2 Method of Solution

In this section, we try to find formulas for approximating the derivatives using polynomials interpolation. For this purpose, we use the second order interpolation polynomials. Let take the Lagrange interpolating polynomial form

$$y(x) \approx p_2(x) = y(x_0)L_0^{(2)}(x) + y(x_1)L_1^{(2)}(x) + y(x_2)L_2^{(2)}(x) \tag{3}$$

Then the approximate derivatives are

$$y^{'}(x_0) \approx p_2^{'}(x_0) = y(x_0)(L_0^{(2)})^{'}(x_0) + y(x_1)(L_1^{(2)})^{'}(x_0) + y(x_2)(L_2^{(2)})^{'}(x_0) \tag{4}$$

$$y^{''}(x_0) \approx p_2^{''}(x_0) = y(x_0)(L_0^{(2)})^{''}(x_0) + y(x_1)(L_1^{(2)})^{''}(x_0) + y(x_2)(L_2^{(2)})^{''}(x_0) \tag{5}$$

where $(L_j^{(2)})(x)$ is a polynomial of degree 2 which is called Lagrange interpolation polynomials of degree 2 and and $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$. We evaluate the values of $(L_j^{(2)})^{'}(x_0)$ and $(L_j^{(2)})^{''}(x_0)$ in Eqs.4. and 5. If we compute these values, we obtain,

$$(L_0^{(2)})^{'}(x_0) = -\frac{3}{2h}, \quad (L_1^{(2)})^{'}(x_0) = \frac{2}{h}, \quad (L_2^{(2)})^{'}(x_0) = -\frac{1}{2h}, \tag{6}$$

$$(L_0^{(2)})^{''}(x_0) = \frac{1}{h^2}, \quad (L_1^{(2)})^{''}(x_0) = -\frac{2}{h^2}, \quad (L_2^{(2)})^{''}(x_0) = \frac{1}{h^2}, \tag{7}$$

Thus, we have the approximate of the first and second derivatives

$$y^{'}(x) = \frac{1}{2h}(-y(x+2h) + 4y(x+h) - 3y(x)) \tag{8}$$

$$y^{''}(x) = \frac{1}{h^2}(y(x+2h) - 2y(x+h) + y(x)) \tag{9}$$

If we show $y(x_k) = y_k$, $y(x_k + h) = y_{k+1}$ and $y(x_k + 2h) = y_{k+2}$ the above relations can be written as

$$y^{'}(x) = \frac{1}{2h}(-y_{k+2} + 4y_{k+1} - 3y_k) \tag{10}$$

$$y^{''}(x) = \frac{1}{h^2}(y_{k+2} - 2y_{k+1} + y_k) \tag{11}$$

where $1 \leq k \leq n-1$. Now, we want to write the Eq.(1) as a finite difference equation. Firstly, we convert the Eq.(1) into the following form:

$$xy^{''}(x) = \alpha y^{'}(x) + xq(x)y(x) = xr(x), \quad 0 < x < 1, \quad \alpha \geq 0 \tag{12}$$

If Eq.(10) and Eq.(11) are put into Eq.(12), we have the difference equation

$$\frac{x_k}{h^2}(y_{k+2} - 2y_{k+1} + y_k) + \frac{\alpha}{2h}(-y_{k+2} + 4y_{k+1} - 3y_k) + x_k q(x_k) y_k = x_k r(x_k) \tag{13}$$

Simply, Eq.(13) can be written

$$(\frac{x_k}{h^2} - \frac{\alpha}{2h})y_{k+2} + (-\frac{2x_k}{h^2})y_{k+1} + (\frac{x_k}{h} - \frac{3\alpha}{2h} + x_k q_k)y_k = f(x_k)$$

where $q = q(x_k)$. And we have the final difference equation

$$(x_k - \frac{h\alpha}{2})y_{k+2} + (2\alpha h - 2x_k)y_{k+1} + (x_k - \frac{3\alpha h}{2} + x_k q_k)y_k = h^2 f(x_k) \tag{14}$$

where $k = 0, 1, ..., N-2$. The matrix-vector form of Eq.(14) is

$$\begin{bmatrix} 2\alpha h - 2x_0 & x_0 - \alpha h/2 & 0 & \dots & 0 \\ x_1 - 3\alpha h/2 + x_1 q_1 & 2\alpha h - 2x_1 & 0 & \dots & 0 \\ 0 & x_2 - 3\alpha h/2 + x_2 q_2 & 2\alpha h - 2x_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2\alpha h - 2x_{N-2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \end{bmatrix}$$

$$= \begin{bmatrix} h^2 f(x_0) - (x_0 - 3\alpha h/2 + x_0 q_0)y_0 \\ h^2 f(x_1) \\ \vdots \\ h^2 f(x_{N-2}) - (x_{N-2} - \alpha h/2) \end{bmatrix} \tag{15}$$

where $y_k \approx y(x_k)$ and $x_k = kh$. Thus, the desired numerical results $y_k \approx y(x_k)$ can be properly obtained above finite difference scheme Eq.(15).

2.1 Existence and uniqueness of the solution

It is well known that a tridiagonal nxn matrix W is called diagonally dominant if $| d_i |>| l_i | + | u_i |$, $1 \leq i \leq n$. If a tridiagonal matrix W is diagonally dominant, $det(W) \neq 0$ which is implied that a linear system with the augmented matrix W has unique solution (For details see [21], page 78). From Eq.(15), it must be satisfied

$$| 2\alpha h - 2x_k |>| x_k - \alpha h/2 | + | x_k - 3\alpha h/2 + x_k q_k | \tag{16}$$

Case I: Let assume $x_k < \alpha h$, we have;

- If $x_k - 3\alpha h/2 + x_k q_k > 0$, $h > \frac{4k+2kq_k}{5\alpha}$
- $x_k - 3\alpha h/2 + x_k q_k < 0$, $q_k < 0$.

Case II: Let assume $x_k > \alpha h$, we have;

- If $x_k - 3\alpha h/2 + x_k q_k > 0$, $q_k < 0$,
- If $x_k - 3\alpha h/2 + x_k q_k < 0$, $\frac{k(1+q_k)}{3\alpha} > 1$.

2.2 *Error Analysis*

Now, we shall investigate the error analysis and convergence of the mention method. It is well known that $y(x)$ is a sufficiently smooth function on $[0, 1]$ and $I_N(x)$ is the interpolating polynomial to y at x_i, then we have [20],[21],

$$y(x) - I_N(x) = \frac{y^{(N+1)}(\xi_x)}{(n+1)!} w_n(x), \ \xi_x \in [0, 1]$$

which is called the interpolation error theorem, where

$$w_n(x) = \prod_{i=0}^{N} (x - x_i)$$

Since our interpolation polynomial is the second order polynomial, we have the following equation

$$y(x) - p_2(x) = \frac{1}{6} w_2(x) y'''(\xi_x) \tag{17}$$

Carefully, If we take the derivative of the Eq.(17) and compute for $x = x_0$, we have

$$y'(x) - p_2(x_0) = \frac{1}{6} w_2'(x_0) y'''(\xi_0) \tag{18}$$

and so

$$y'(x_0) - p_2'(x_0) = \frac{1}{3} h^2 y'''(\xi_0) \tag{19}$$

Similarly, we have the error term for the second derivatives

$$y''(x_0) - p_2''(x_0) = -h y'''(\xi_0) + \frac{2}{3} h^2 \frac{d}{dx} y'''(\xi_0) \tag{20}$$

If we combine the Eqs.(8-9) and (19-20), we can write the errors

$$y'(x) - [\frac{1}{2h}(-y(x+2h) + 4y(x+h) - 3y(x))] = O(h^2) \tag{21}$$

$$y''(x) - [\frac{1}{h^2}(y(x+2h) - 2y(x+h) + y(x))] = O(h) \tag{22}$$

If Eqs.(21)-(22) are put into Eq.(1), we get

$$\frac{x_k}{h^2}(y_{k+2} - 2y_{k+1} + y_k) + \frac{\alpha}{2h}(-y_{k+2} + 4y_{k+1} - 3y_k) + x_k q(x_k) y_k - x_k r(x_k) = O(h) \tag{23}$$

which is stated that the estimation error is $O(h)$ accurate.

3 Numerical Examples

In this section, some numerical examples are presented to illustrate the accuracy and effectiveness properties of the method. To study the behavior of the present method, we applied the following law: Absolute error is defined by the $\mid y_k - y(x_k) \mid$ where $y(x_k)$ are the exact solutions and y_k denote the approximate solution obtained by the present method.

Example 1. Consider the following singular boundary value problem

$$y''(x) - \frac{2}{x} y' + (1 + x^2) y = x^4 - 2x^2 + 7 \tag{24}$$

The exact solution is $y = 1 - x^2$. If we write approximations (10) and (11) in place of y'',y' in Eq.(23), we obtain the following difference equation.

$$y_k[-\frac{x_k}{h^2} + \frac{3}{h} + x_k - x_k^3] + y_{k+1}[\frac{2x_k}{h^2} - \frac{4}{h}] + y_{k+2}[-\frac{x_k}{h^2} + \frac{1}{h}] = x_k^5 - 2x_k^3 + 7x_k \tag{25}$$

where $h = \frac{1}{N}$. We obtain $(N-1)$ difference equations for $k = 0, ..., N-2$. Numerical results are shown in Table 1 and Table 2. If we choose $h = 1/20$, our errors is so small for some values of k.

k	Exact Solution	Numerical Solution	Error
1	0.9975000000	0.9975000000	0
2	0.9900000001	0.9900000000	0.1e-11
3	0.9775000852	0.9775000000	0.852e-10
4	0.9600000000	0.9600000000	0
5	0.9375000002	0.9375000000	0.2e-11
6	0.9100000002	0.9100000000	0.2e-11
7	0.8775000001	0.8775000000	0.1e-11
8	0.8400000001	0.8400000000	0.1e-11
9	0.7975000000	0.7975000000	0

Table 1 Exact and numerical solutions for $h = \frac{1}{20}$

.

k	Exact Solution	Numerical Solution	Error
0.1	0.9900000000	0.9900000000	0
0.2	0.9600000001	0.9600000000	0
0.3	0.9100000000	0.9100000073	0.73e-8
0.4	0.8400000000	0.8400000000	0
0.5	0.7500000000	0.7500000001	0.1e-9
0.6	0.6400000000	0.6400000001	0.1e-9
0.7	0.5100000000	0.5100000000	0
0.8	0.3600000000	0.3600000000	0
0.9	0.1900000000	0.1900000000	0

Table 2 Exact and numerical solutions for $h = \frac{1}{10}$

.

h	1/2	1/4	1/8	1/16	1/32	1/64
E_{max}	0.037036213	0.046968636	0.023127895	0.0003764443	0.011674832	0.019164548

Table 3 Maximum errors for different h

.

Example 2. Consider the following singular boundary value problem

$$y''(x) - \frac{1}{x}y' + 4x^2 y = 0 \tag{26}$$

with the boundary conditions $y(0) = 0$, $y(1) = sin(1)$. The exact solution is $y(x) = sin(x^2)$. If we reorder and write approximations (10) and (11) in place of y'',y' in Eq.(26), we obtain the following difference equation.

$$y_k[\frac{x_k}{h^2} + \frac{3}{2h} + 4x_k^2] + y_{k+1}[\frac{-2x_k}{h^2} - \frac{2}{h}] + y_{k+2}[\frac{x_k}{h^2} + \frac{1}{2h}] = 0 \tag{27}$$

where $h = 1/N$. We obtain $(N-1)$ difference equations for $k = 0, , N-2$. Maximum errors for different h are shown in Table 3.

4 Conclusion

In this study, numerical solutions were tried to be obtained by taking Lane-Emden equation in singular value problems. Lagrange polynomials are used in addition to finite differences to obtain numerical solutions. Equation was made discrete and its values were examined as points. Approximate solution, exact solution and error values at each point were calculated and the obtained results were given in tables. It was seen that the more points were examined, the closer the exact values were approached, in other words, the larger the N value, the smaller the error values. This result is supported by table 4. One of the advantages of this method, which is obtained by combining finite differences with Lagrange polynomials interpolations, is the short runtime of the program Maple 13.

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A Note on the Order of Convergence of Kantorovich Type Operators Including Apostol-Genocchi Polynomials I

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Abstract: In this study, under some assumptions, an operator is constructed with the help of certain polynomials whose generating function is known. Firstly, the moments are obtained to prove the uniform convergence of this operator. Secondly, the central moments of the operator are given to be used in the error estimation, and then the rate of convergence is investigated with the help of various tools such as the first modulus of continuity, the Lipschitz class, Peetre's K-functional, the second modulus of continuity. In addition, a numerical example is given by using the first modulus of continuity with the help of the Maple software.

Keywords: Apostol-Genocchi polynomials, modulus of smoothness, moments.

1 Introduction

In [1], Luo defined an extension of higher-order Genocchi polynomials based on the idea of Apostol, known as Apostol-Genocchi polynomials. Apostol-Genocchi polynomials $G_k^{(\alpha)}(x;\lambda)$ with α th order $(\alpha \in \mathbb{N}\cup\{0\})$ $G_k^{(\alpha)}(x;\lambda)$ are given by the generating functions

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} G_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!} \ \ (t\in\mathbb{C}, \ \ |t|+|\log(-\lambda)|<\pi) \tag{1}$$

where $\lambda = |\lambda|\, e^{i\varphi}, -\pi \leq \varphi < \pi$ and $\log(\lambda) = \log(|\lambda|) + i\varphi$. It is clear that $G_k^{(\alpha)}(x;\lambda)$ polynomials are an extension of the classical Genocchi polynomial, i.e., when $\alpha = 1$ and $\lambda = 1$, $G_k^{(\alpha)}(x;\lambda)$ is the classical Genocchi polynomial $G_k(x)$. For $x = 0$, the resulting form is the Genocchi number G_k, i.e., $G_k := G_k(0)$. When $\lambda \neq -1$, $G_k^{(\alpha)}(x;\lambda)$ must be implicitly limited to non-negative integer values [3].

For the various types of the operators defined according to Apostol-Genocchi polynomials and their typical approximation properties, see [2], [3], and [4]. In the above-mentioned studies, the typical convergence properties of these operators were investigated. In this study, motivated by [5] and [2], we investigate some properties of convergence of generalization of Szasz-Kantorovich type operator based on Apostol-Genocchi polynomials are as follows:

$$T_{n,\alpha}^{\beta_n,\gamma_n}(f;x) = \gamma_n \left(\frac{2}{\lambda e+1}\right)^{-\alpha} e^{-\beta_n x}\sum_{k=0}^{\infty}\frac{G_k^{(\alpha)}(\beta_n x;\lambda)}{k!}\int_{\frac{k}{\gamma_n}}^{\frac{k+1}{\gamma_n}} f(t)\,dt, \tag{2}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are strictly increasing sequences of positive numbers such that

$$\lim_{n\to\infty}\frac{1}{\gamma_n} = 0, \ \frac{\beta_n}{\gamma_n} = 1 + O\left(\frac{1}{\gamma_n}\right). \tag{3}$$

Throughout the paper, we consider the following:
Let $C[0,\infty)$ be the set of all real-valued continuous functions on $[0,\infty)$, $E := \{f : \forall x \in [0,\infty), |f(x)| \leq ae^{bx}, a \in \mathbb{R}, b \in \mathbb{R}^+\}$ and also $C_E[0,\infty) := C[0,\infty) \cap E$.

2 Main Results

To investigate the convergence of the sequence $T_{n,\alpha}^{\beta_n,\gamma_n}$ to the uniform continuous and bounded function f; we give some results.

Lemma 1. *For operators in* (2)*, we have the following equalities:*

$T_{n,\alpha}^{\beta_n,\gamma_n}(1;x) = 1$

$T_{n,\alpha}^{\beta_n,\gamma_n}(t;x) = \frac{\beta_n}{\gamma_n}x + \frac{1}{\gamma_n}\frac{\lambda e+2\alpha+1}{2(\lambda e+1)},$

$T_{n,\alpha}^{\beta_n,\gamma_n}(t^2;x) = \frac{\beta_n^2}{\gamma_n^2}x^2 + \frac{\beta_n}{\gamma_n^2}\frac{2\lambda e+2\alpha+2}{\lambda e+1}x + \frac{1}{\gamma_n^2}\frac{(-3\alpha+2)\lambda e+(-3\alpha+1)\lambda^2e^2+3\alpha^2+3\alpha+1}{3(\lambda e+1)^2}.$

Proof: Taking the derivative of both sides of the equation eq. (1) with respect to t, i.e., using the generating functions of the Apostol-Genocchi polynomials, we obtain the following equality:

$\sum_{k=0}^{\infty} kp_k(x)t^{k-1} = \frac{-e^{xt}t^{\alpha-1}2^{\alpha}(((\alpha-x)t-\alpha)\lambda e^t-xt-\alpha)}{(\lambda e^t+1)^{\alpha+1}}.$

For $t = 1$ and $x = \beta_n x$ in above equality, we get the following terms:

$\sum_{k=0}^{\infty} kp_k(\beta_n x) = \frac{2^{\alpha}e^{\beta_n x}}{(\lambda e+1)^{\alpha+1}}(e\lambda\beta_n x + \beta_n x + \alpha),$

$\sum_{k=0}^{\infty} k^2p_k(\beta_n x) = \frac{2^{\alpha+1}}{(\lambda e+1)^2}(\beta_n^2x^2 + \beta_n x(\alpha+1) - \alpha)\lambda e + \alpha\beta_n x)$

$+\frac{2^{\alpha}e^{\beta_n x}}{(\lambda e+1)^{\alpha+2}}\left((\beta_n^2x^2 + \beta_n x - \alpha)\lambda^2e^2 + \beta_n^2x^2 + \beta_n x + \alpha^2)\right).$

By using above equalities and the definition of the operator $T_n^{\beta_n,\gamma_n}$, we obtain the desired result. □

Lemma 2. *For operators in* (2)*, the central moments are obtained as follows:*

i. $T_{n,\alpha}^{\beta_n,\gamma_n}(t-x;x) = (\frac{\beta_n}{\gamma_n} - 1)x + \frac{1}{\gamma_n}\frac{\lambda e+2\alpha+1}{2(\lambda e+1)},$

ii. $T_{n,\alpha}^{\beta_n,\gamma_n}((t-x)^2;x) = (\frac{\beta_n^2}{\gamma_n^2} - \frac{2\beta_n}{\gamma_n} + 1)x^2 + \left(\frac{\beta_n}{\gamma_n^2}\frac{2\lambda e+2\alpha+2}{\lambda e+1} - \frac{1}{\gamma_n}\frac{\lambda e+2\alpha+1}{\lambda e+1}\right)x$

$+ \frac{1}{\gamma_n^2}\frac{(-3\alpha+1)\lambda^2e^2+(-3\alpha+2)\lambda e+3\alpha^2+3\alpha+1}{3(\lambda e+1)^2}$

Proof: It is enough to apply Lemma 1 to obtain desired results. □

Theorem 1. *Let* $f \in C_E[0,\infty)$*. Then the sequence* $\left\{T_{n,\alpha}^{\beta_n,\gamma_n}\right\}_{n\geq 1}$ *converges uniformly to the function* f *on every closed subset of* $[0,\infty)$*.*

Proof: Under the assumptions of (3) and by using the Lemma 1, we have

$$\lim_{n\to\infty} T_{n,\alpha}^{\beta_n,\gamma_n}\left(t^i;x\right) = x^i,\ i = 0,1,2$$

which converge uniformly in each closed subset of $[0,\infty)$. On account of Korovkin's theorem, $\left(T_{n,\alpha}^{\beta_n,\gamma_n}\right)$, converges uniformly with respect to each closed subset of $[0,\infty)$. □

Theorem 2. *Let* $f \in C_E[0,\infty)$*. Then*

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x) - f(x)\right| \leq 2\omega(f;\delta_n),$$

where $\delta_n := \sqrt{T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right)}$ *is in Lemma 2 and* ω *is the modulus of continuity of* f *defined by*

$$\omega(f;\delta) := \sup\{|(f(t) - f(x)| : |t-x| \leq \delta, t,x \in [0,\infty)\}.$$

Proof: It is clear that the operator $T_{n,\alpha}^{\beta_n,\gamma_n}$ is linear and has the property $T_n^{\beta_n,\gamma_n}(1;x)=1$, and also for the modulus of the continuity, we know the following relation:

$$|f(x)-f(y)|\le\omega(f;\delta)\left(\frac{|x-y|}{\delta}+1\right).$$

If all these three facts mentioned above are used, the following inequality is obtained:

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right|\le\omega(f;\delta)\left\{1+\frac{\gamma_n}{\delta}\left(\frac{2}{\lambda e+1}\right)^{-\alpha}e^{-\beta_n x}\sum_{k=0}^{\infty}\frac{G_k^{(\alpha)}(\beta_n x;\lambda)}{k!}\int_{\frac{k}{\gamma_n}}^{\frac{k+1}{\gamma_n}}|t-x|\,dt\right\}. \tag{4}$$

When we use the Cauchy-Schwarz inequality for integration, we get the following inequality

$$\int_{\frac{k}{\gamma_n}}^{\frac{k+1}{\gamma_n}}|t-x|\,dt\le\frac{1}{\sqrt{\gamma_n}}\left(\int_{\frac{k}{\gamma_n}}^{\frac{k+1}{\gamma_n}}|t-x|^2\,dt\right)^{\frac{1}{2}},$$

which gives the following:

$$\sum_{k=0}^{\infty}\frac{G_k^{(\alpha)}(\beta_n x;\lambda)}{k!}\int_{\frac{k}{\gamma_n}}^{\frac{k+1}{\gamma_n}}|t-x|\,dt\le\frac{1}{\sqrt{\gamma_n}}\sum_{k=0}^{\infty}\frac{G_k^{(\alpha)}(\beta_n x;\lambda)}{k!}\left(\int_{\frac{k}{\gamma_n}}^{\frac{k+1}{\gamma_n}}|t-x|^2\,dt\right)^{\frac{1}{2}}. \tag{5}$$

After a simple operation on equation (5), we get the following inequality:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{G_k^{(\alpha)}(\beta_n x;\lambda)}{k!}\int_{\frac{k}{\gamma_n}}^{\frac{k+1}{\gamma_n}}|t-x|\,dt&\le\sqrt{\left(\frac{2}{\lambda e+1}\right)^{\alpha}\frac{e^{\beta_n x}}{\gamma_n}}\left[\left(\frac{2}{\lambda e+1}\right)^{\alpha}\frac{e^{\beta_n x}}{\gamma_n}T_n^{\beta_n,\gamma_n}\left((t-x)^2;x\right)\right]^{\frac{1}{2}}\\&\le\left(\frac{2}{\lambda e+1}\right)^{\alpha}\frac{e^{\beta_n x}}{\gamma_n}\sqrt{T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right)}.\end{aligned}$$

Now, if we take into account the (4), then we get the below:

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right|\le\left\{1+\frac{1}{\delta}\sqrt{T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right)}\right\}\omega(f;\delta).$$

By taking $\delta:=\delta_n(x)=\sqrt{T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right)}$, we obtain the desired result. □

For $0<\alpha\le1$ and $K>0$, the Lipschitz class of order α is defined as:

$$Lip_\alpha(K)=\left\{f\in C_E[0,\infty):|f(t)-f(x)|\le K|t-x|^{\alpha},t,x\in[0,\infty)\right\}$$

with the property $\omega(f,\delta)\le K\delta_n^{\alpha}$ for all $\delta>0$.

Theorem 3. *Let $f\in Lip_\alpha(K)$. Then we have $\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right|\le K\delta_n^{\alpha}(x)$, where $\delta:=\delta_n(x)=\sqrt{T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right)}$.*

Proof: Since $f\in Lip_\alpha(K)$,

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right|\le T_{n,\alpha}^{\beta_n,\gamma_n}(|f(s)-f(x)|;x)\le KT_{n,\alpha}^{\beta_n,\gamma_n}\left(|t-x|^{\alpha};x\right) \tag{6}$$

After applying the Hölder inequality to (6) with $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$, we obtain the following:

$$\begin{aligned}\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right|&\le KT_{n,\alpha}^{\beta_n,\gamma_n}\left(|t-x|^2 1;x\right)\le K\left(T_{n,\alpha}^{\beta_n,\gamma_n}\left(|t-x|^2;x\right)\right)^{\frac{\alpha}{2}}\left(T_{n,\alpha}^{\beta_n,\gamma_n}(1;x)\right)^{\frac{2-\alpha}{\alpha}}\\&=K\left(T_{n,\alpha}^{\beta_n,\gamma_n}\left(|t-x|^2;x\right)\right)^{\frac{\alpha}{2}}=K(\delta_n(x))^{\alpha}\end{aligned}$$

with this result, the proof is complete. □

n	$\alpha=0$	$\alpha=1$	$\alpha=2$
2.10^2	0.0176465752	0.0176476362	0.0176487292
2.10^3	0.0017672196	0.0017672206	0.0017672222
2.10^4	0.0001767710	0.0001767710	0.0001767710
2.10^5	0.0000176782	0.0000176782	0.0000176782
2.10^6	0.0000017684	0.0000017684	0.0000017684
2.10^7	0.0000001772	0.0000001772	0.0000001772

Table 1 The error estimation of function $f(x)=\frac{x^3}{\sqrt{1+x^2}}$ by using modulus of continuity.

Example 1. *When we choose $\gamma_n=n^2+10$, $\beta_n=n^2$ for the operator $T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)$ with α order, the approximation of $T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)$ to $f(x)=\frac{x^3}{\sqrt{1+x^2}}$ on the interval $[0,\infty)$ is given in Table 1. The error gets smaller as n increases.*

Now, we shall use the second order modulus of continuity and Peetre's K- functional for the estimation of the rate of convergence of the operator $T_{n,\alpha}^{\beta_n,\gamma_n}$. For this purpose, recall that the second order modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$\omega_2(f;\delta):=\sup_{0<t\leq\delta}\|f(.+2t)-2f(.+t)+f(.)\|_{C_B},$$

where $C_B[0,\infty)$ is the set of real-valued functions on $[0,\infty)$ which are uniformly continuous and bounded with the norm $\|f\|_{C_B}=\sup_{x\in[0,\infty)}|f(x)|$. For $\delta>0$, the Peetre's K-functional is defined as

$$K(f;\delta):=\inf_{g\in C^2[0,\infty)}\left\{\|f-g\|_{C_B}+\delta\|g\|_{C_B^2}\right\},$$

where $C_B^2[0,\infty):=\left\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\right\}$, with the norm

$$\|g\|_{C_B^2}:=\|g\|_{C_B}+\left\|g'\right\|_{C_B}+\left\|g''\right\|_{C_B}.$$

Theorem 4. *Let $f\in C_B[0,\infty)$ and $x\in[0,\infty)$. Then*

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right|\leq 2K(f;\delta),$$

where $\delta:=\xi_n(x)=\frac{1}{2}\left[T_{n,\alpha}^{\beta_n,\gamma_n}(t-x;x)+T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right)\right]$.

Proof: Using the Taylor expansion of g and the linearity of the operators $T_{n,\alpha}^{\beta_n,\gamma_n}$, we can write the following equality. Let $c\in(x,t)$,

$$T_{n,\alpha}^{\beta_n,\gamma_n}(g;x)-g(x)=g'(x)T_{n,\alpha}^{\beta_n,\gamma_n}(t-x;x)+\frac{1}{2}g''(c)T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right). \tag{7}$$

For $t\geq x$, we get

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(g;x)-g(x)\right|\leq\left[\left|T_{n,\alpha}^{\beta_n,\gamma_n}(t-x;x)\right|+\left|T_{n,\alpha}^{\beta_n,\gamma_n}\left((t-x)^2;x\right)\right|\right]\|g\|_{C_B^2[0,\infty)}. \tag{8}$$

Using Lemma 1 and (8), we can write the following:

$$\begin{aligned}\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right| &\leq\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f-g;x)\right|+\left|T_{n,\alpha}^{\beta_n,\gamma_n}(g;x)-g(x)\right|+|f(x)-g(x)|\\ &\leq 2\|f-g\|_{C_B[0,\infty)}+\left|T_{n,\alpha}^{\beta_n,\gamma_n}(g;x)-g(x)\right|\\ &\leq 2\left[\|f-g\|_{C_B[0,\infty)}+\xi_n(x)\|g\|_{C_B^2[0,\infty)}\right]\end{aligned}$$

If we take the infimum over all $g\in C_B^2[0,\infty)$, then we get the concept of Peetre's K-functional verifies the following

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x)-f(x)\right|\leq 2K(f;\xi_n(x)).$$

□

In the next theorem, we use the concepts of the second order Steklov function and the second modulus of continuity to obtain an estimation for $T_{n,\alpha}^{\beta_n,\gamma_n}$. See [5] and [7] for similar theorems established using different operators. Also, for details about Steklov function, see [6] and [9]. But first, let's give a lemma to use in the aforementioned theorem:

Lemma 3. *[8] Let $f \in C^2[0,\infty)$ and (L_n) be a sequence of positive linear operators with $L_n(1;x) = 1$. Then we have*

$$|L_n(f;x) - f(x)| \leq \sqrt{L_n((t-x)^2;x)}\,\|f'\| + \frac{1}{2}L_n((t-x)^2;x)\left\|f''\right\|.$$

Theorem 5. *Let $f \in C[0,a]$ and $h := h_n(x) = \sqrt[4]{T_{n,\alpha}^{\beta_n,\gamma_n}((t-x)^2;x)}$. Then we have the following inequality:*

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x) - f(x)\right| \leq \frac{2}{a}\|f\|\,h_n^2 + \frac{3}{4}(h_n^2 + a + 2)\omega_2(f;h_n),$$

where $\|f\| = \sup\limits_{x\in[0,\infty)} |f(x)|$.

Proof: Assume that f_h is the second-order Steklov function corresponding to f and $f \in C[0,a]$ and $h \in (0, \frac{a}{2})$. By using the identity $T_{n,\alpha}^{\beta_n,\gamma_n}(1;x) = 1$, we get

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f;x) - f(x)\right| \leq \left|T_{n,\alpha}^{\beta_n,\gamma_n}(f - f_h;x)\right| + \left|T_{n,\alpha}^{\beta_n,\gamma_n}(f_h;x) - f_h(x)\right| + |f_h(x) - f(x)|$$

$$\leq 2\|f_h - f\| + \left|T_{n,\alpha}^{\beta_n,\gamma_n}(f_h;x) - f_h(x)\right|. \tag{9}$$

We have the following facts from [9] and [6]:

$$\|f_h - f\| \leq \frac{3}{4}\omega_2(f;h), \tag{10}$$

$$\|f_h''\| = \frac{3}{2h^2}\omega_2(f;h), \tag{11}$$

$$\|f_h'\| \leq \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\|. \tag{12}$$

By using Lemma 3, (11), (9) and (12), we get the following:

$$\begin{aligned}\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f_h;x) - f_h(x)\right| &\leq \sqrt{T_{n,\alpha}^{\beta_n,\gamma_n}((t-x)^2;x)}\,\|f_h'\| + \frac{1}{2}T_{n,\alpha}^{\beta_n,\gamma_n}((t-x)^2;x)\left\|f_h''\right\| \\ &\leq \left(\frac{2}{a}\|f_h\| + \frac{3a}{4h^2}\omega_2(f;h)\right)\sqrt{T_{n,\alpha}^{\beta_n,\gamma_n}((t-x)^2;x)} \\ &\quad + \frac{3}{4h^2}\omega_2(f;h)\,T_{n,\alpha}^{\beta_n,\gamma_n}((t-x)^2;x).\end{aligned}$$

Taking $h = h_n = \sqrt[4]{T_{n,\alpha}^{\beta_n,\gamma_n}((t-x)^2;x)}$, we get

$$\left|T_{n,\alpha}^{\beta_n,\gamma_n}(f_h;x) - f_h(x)\right| \leq \frac{2}{a}\|f\|\,h^2 + \frac{3\left(a + h^2\right)}{4}\omega_2(f;h). \tag{13}$$

The proof is completed by writing (10) and (13) in (9). □

3 Conclusion

In analytic number theory, Apostol-Genochi polynomials are widely studied polynomial sequences, but using this polynomial is new in approximation theory. Therefore, in this study, we have chosen to show its uniform continuity by defining an operator containing this polynomial sequence. Next, we determined the rate of approximation of this operator with the help of the modulus of continuity, the Lipschitz class, Peetre's K-functional, and the Steklov function.

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PMP Iterative Algorithm with Errors for Accretive Lipschitzian Operator and its Application

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Abstract: In this work, we performed the convergence and stability results of the PMP iterative algorithm with errors for the accretive Lipschitzian operator in Banach spaces. We also showed through a numerical example that the PMP algorithm with errors has a better convergence speed than some iterative algorithms with errors in the literature. Moreover, we investigated the iterative approximation of the solution for the variational inclusion problem in Banach spaces by using the PMP algorithm with errors.

Keywords: Accretive Lipschitzian operator, Convergence, Iterative algorithm, Stability, Variational inclusion.

1 Introduction

A number of researchers studied some basic fixed point theorems for different mapping classes, taking into account the margins of error arising from the operation of some fixed point algorithms. In 1995, Liu [1] worked on the Ishikawa and Mann iterative algorithms with errors for nonlinear strongly accretive mappings in Banach spaces. In 1998, Xu [2] revised definitions of Ishikawa and Mann iterative algorithms with errors. In 2001 Kim and Kim [3] worked on these iteration methods with errors for non-Lipschitzian mappings in Banach Spaces. In 2004, Cho [4] obtained several weak and strong convergence theorems for the three-step iterative algorithm with errors for asymptotically nonexpansive mappings. In 2016, Hussain et al. [5] obtained strong convergence and stability result of a three-step random iterative algorithm with errors for strongly pseudo-contractive Lipschitzian mappings on real Banach spaces. In 2020, Kumar et al. [6] analyzed the results of strong convergence and stability of the SP iterative algorithm with errors using the strongly accretive Lipschitzian operator on a Banach space. In 2020, Kumar and Hussain [7] also studied on the strong convergence and stability results of the SP iterative algorithm mixed errors for accretive Lipschitzian operators in a Banach space.
Variational inclusions are the generalization of variational inequalities and they have been widely studied by researchers (see [8–11]). One of the most interesting and important problems in the theory of variational inclusions in the development of an efficient and implementable iterative algorithm. Various kinds of iterative methods have been studied to find the approximate solutions for variational inclusions (see [6, 7, 12]).

Now, we give the PMP iteration method which introduced by Karakaya et al [13] as follows:

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)\, z_n + \alpha_n T z_n \\ z_n = Tx_n \ (n \in \mathbb{N}) \end{cases} \tag{1}$$

in which $(\alpha_n)_{n=1}^{\infty} \in [0,1]$ and T is a self-map a nonempty set of X.
In the light of the information given above, we can rewrite the PMP iterative algorithm with errors as under:

$$\begin{cases} x_{n+1} = Ty_n + u_n \\ y_n = (1-\alpha_n)\, z_n + \alpha_n T z_n + v_n \\ z_n = Tx_n + w_n \ (n \in \mathbb{N}) \end{cases} \tag{2}$$

in which $0 \leq \alpha_n \leq 1$ and u_n, v_n, w_n are sequences in X.
Let us now give some definitions and lemmas that are useful for obtaining our main results.

Definition 1. *Let C be a closed and convex subset of Banach space X. A mapping $T : C \subset X \to X$ is called:*

(i) $\mu-$Lipschitzian if for all $x, y \in C$, there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu \|x - y\|, \tag{3}$$

(ii) accretive if for all $r > 0$ and $x, y \in C$, we have

$$\|x-y\| \leq \|x-y+r\left[Tx-Ty\right]\|. \tag{4}$$

[7]

Definition 2. *Let $T : C \to C$ be a mapping. Define an iterative algorithm by*

$$x_{n+1} = f\left(T, y_n\right) \tag{5}$$

such that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in C. Set $\epsilon_n = \|y_{n+1} - f\left(T, y_n\right)\|$ for $n \geq 1$. The iterative algorithm (5) is said to be T-stable or stable w.r.t. T if the following condition is satisfied:

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ if and only if } \lim_{n\to\infty} y_n = p.$$

[14, 15].

Lemma 1. *Let $\{a_n\}_{n=0}^{\infty}$ be nonnegative real sequence satisfying the following inequality:*

$$a_{n+1} \leq \sigma a_n + b_n$$

$n \geq 1$ where $b_n \geq 0$, $\lim_{n\to\infty} b_n = 0$ and $0 \leq \sigma < 1$. Then $a_n \to 0$ [16].

Let $T, A : X \to X, g : X \to X^*$ be three mappings on a real reflexive Banach space X and $\varphi : X^* \to \mathbb{R} \cup \{\infty\}$ be a function with continuous subdifferential $\partial\varphi : X^* \to 2^{X^*}$ defined by $(\partial\varphi)x = \{x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle Tx - Ax - y, f - g(x)\rangle \geq \varphi(g(x) - \varphi(x))\}$. If for any given $y \in X$, there exists a $x \in X$ such that

$$g(x) \in D(\partial\varphi), \langle Tx - Ax - y, f - g(x)\rangle \geq \varphi(g(x) - \varphi(x)), \qquad \forall f \in X^* \tag{6}$$

holds, then, x is solution of a variational inclusion problem (6).

Lemma 2. *Let $\partial\varphi \circ g : X \to 2^X$ be a mapping on a real reflexive Banach space X. Then the followings are equivalent:*

i. $p \in X$ is a solution of variational inclusion problem (6);
ii. $p \in X$ is a fixed point of the mapping $R : X \to 2^X$ such that $Rx = y - (Tx - Ax + \partial\varphi(g(x))) + x$;
iii. $p \in X$ is a solution of the equation $y = Tx - Ax + \partial\varphi(g(x))$.

[7]

In this work, we performed the convergence and stability results of the PMP iterative algorithm with errors for the accretive Lipschitzian operator in Banach spaces. We also investigated the iterative approximation of the solution for the variational inclusion problem in Banach spaces by using the PMP algorithm with errors.

2 Main Results

Theorem 1. *Let T be an accretive Lipschitzian self operator with a Lipschitz constant L on a real Banach space X. Let $\{x_n\}_{n=0}^{\infty}$ be iterative sequence generated by the iterative algorithm (2) with the following restrictions:*

i. $0 < L^2 - L^2(\alpha_n - \alpha_n L - \alpha_n^2 L^2) < 1 - \alpha < 1$
ii. $\sum_{i=1}^{\infty} \|u_n\| < \infty$, $\sum_{i=1}^{\infty} \|v_n\| < \infty$, and $\sum_{i=1}^{\infty} \|w_n\| < \infty$

Then, for $x_0 \in X$ the iterative algorithm (2) converges strongly to a unique fixed point p of T.

Proof: T is Lipschitzian operator with Lipschitz constant L, such that T is an accretive and hence using (4), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|x_{n+1} - p + \alpha_n(Tx_{n+1} - Tp)\| \\
&\leq \|x_{n+1} - p\| + \alpha_n \|Tx_{n+1} - Tp\| \\
&\leq \|Ty_n - p\| + \alpha_n L \|x_{n+1} - p\| + \|u_n\| \\
&\leq L\|y_n - p\| + \alpha_n L \|x_{n+1} - p\| + \|u_n\| \\
&\leq L\|y_n - p\| + \alpha_n L \|Ty_n - p\| + \alpha_n L \|u_n\| + \|u_n\| \\
&\leq L(1 + \alpha_n L)\|y_n - p\| + (1 + \alpha_n L)\|u_n\|
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\|z_n - p\| &= \|Tx_n + w_n - p\| \\
&\leq \|Tx_n - p\| + \|w_n\| \\
&\leq L\|x_n - p\| + \|w_n\|
\end{aligned} \tag{8}$$

and

$$\begin{aligned}\|y_n - p\| &= \|(1-\alpha_n)z_n + \alpha_n T z_n + v_n\| \\ &\le (1-\alpha_n)\|z_n - p\| + \alpha_n \|Tz_n - Tp\| + \|v_n\| \\ &\le (1-\alpha_n)\|z_n - p\| + \alpha_n L \|z_n - p\| + \|v_n\| \\ &\le (1+\alpha_n(L-1))\|z_n - p\| + \|v_n\|\end{aligned} \tag{9}$$

Substituting (8) and (9) in (7), we have

$$\begin{aligned}\|x_{n+1} - p\| &\le L^2[1 - \{\alpha_n - \alpha_n L - \alpha_n^2 L^2\}]\|x_n - p\| \\ &+ L(1+\alpha_n L)(1+\alpha_n(L-1))\|w_n\| \\ &+ L(1+\alpha_n L)\|v_n\| + (1+\alpha_n L)\|u_n\| \\ &\le (1-\alpha)\|x_n - p\| + L(1+\alpha_n L)(1+\alpha_n(L-1))\|w_n\| \\ &+ L(1+\alpha_n L)\|v_n\| + (1+\alpha_n L)\|u_n\|\end{aligned} \tag{10}$$

Denote that

$$\begin{aligned}a_n &= \|x_n - p\| \\ \sigma &= \alpha \\ b_n &= L^2(1+\alpha_n L)\|w_n\| + L(1+\alpha_n L)\|v_n\| + (1+\alpha_n L)\|u_n\|\end{aligned}$$

It is now easy to check that (10) satisfies all the requirements of Lemma 1. Hence it follows by its conclusion that

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

To prove uniqueness of fixed point p, let q be an another fixed point of T. Since $-T$ is accretive, we get

$$\begin{aligned}\|p - q\| &= \|p - q - \alpha_n(Tq - Tp)\| \\ &\le (1-\alpha_n)\|p - q\|\end{aligned}$$

which is possible only when $p = q$. □

Example 1. *[7] Let $X = [0,3] \subset \mathbb{R}$. Let $T : X \to X$ be a mapping defined by $T(x) = 3 - x$ for all $x \in X$ with fixed point p $= 1.5$. It is easy to see that $-T$ is a Lipschitz accretive operator with the Lipschitz constant $L = 1$. Choose $\alpha = 0.008$, $\alpha_n = \frac{1}{8}$, $\beta_n = \gamma_n = \frac{1}{64}$ and $\|u_n\| = \frac{1}{8(n+1)}$, $\|v_n\| = \frac{1}{(n+2)^2}$, $\|w_n\| = \frac{1}{8(n+3^2)}$. Then all the conditions in Theorem 1 are satisfied. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iterative algorithm (2) converges strongly to p $= 1.5$. By taking the initial value as $x_0 = 0.99$, convergence comparison of different iterative algorithm with errors can be seen in Table 1:*

Iteration Steps	Mann Iteration with errors	Ishikawa Iteration with errors	SP with errors	PMP iteration with errors
1	0,9900	0,9900	0.9900	0.9900
2	1.4300	1.4141	1.5823	2.0251
3	1.6303	1.5802	1.7866	1.1788
⋮	⋮	⋮		
1441	1.5003	1.5004	1.5003	1.5001
1442	1.5003	1.5004	1.5003	1.5000
⋮	⋮	⋮		

Table 1 Convergence comparison of different iterative algorithm with errors with initial value $x_0 = 0.99$.

Theorem 2. *Let T and X be the same as in the Theorem 1. Suppose that $\{a_n\}_{n=1}^{\infty}$ be an iterative sequence generated by the following algorithm:*

$$\begin{cases} a_{n+1} = Tb_n + u_n, \\ \quad b_n = (1-\alpha_n)c_n + \alpha_n Tc_n + v_n, \\ \quad c_n = Ta_n + w_n, \end{cases} \tag{11}$$

Then, the iterative algorithm (2) with mixed errors is T-stable.

Proof: Suppose that $\{a_n\}_{n=1}^{\infty}$ is an arbitrary sequence in X and $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Then

$$\begin{aligned}\|a_{n+1} - p\| &\le \|a_{n+1} - Tb_n - u_n\| + \|Tb_n + u_n - p\| \\ &\le \|a_{n+1} - Tb_n - u_n\| + \|Tb_n - p\| + \|u_n\| \\ &\le \varepsilon_n + (1-\alpha)\|a_n - p\| \\ &+ L(1+\alpha_n L)(1+\alpha_n(L-1))\|w_n\| \\ &+ L(1+\alpha_n L)\|v_n\| + (1+\alpha_n L)\|u_n\|\end{aligned} \tag{12}$$

Denote that

$$\begin{aligned}a_n &= \|a_n - p\| \\ \sigma &= \alpha \\ b_n &= \varepsilon_n + L^2(1+\alpha_n L)\|w_n\| + L(1+\alpha_n L)\|v_n\| + (1+\alpha_n L)\|u_n\|\end{aligned}$$

It is now easy to check that (13) satisfies all the requirements of Lemma 1. Hence it follows by its conclusion that

$$\lim_{n\to\infty} \|a_n - p\| = 0.$$

Conversely, let $\lim_{n\to\infty} \|a_n - p\| = 0$, then

$$\begin{aligned}\varepsilon_n &= \|a_{n+1} - Tb_n - u_n\| = \|a_{n+1} - Tb_n - u_n - p + p\| \\ &\le \|a_{n+1} - p\| + \|Tb_n - p\| + \|u_n\| \\ &\le \|a_{n+1} - p\| + (1-\alpha)\|a_n - p\| \\ &+ L(1+\alpha_n L)(1+\alpha_n(L-1))\|w_n\| \\ &+ L(1+\alpha_n L)\|v_n\| + (1+\alpha_n L)\|u_n\|\end{aligned} \tag{13}$$

which implies that $\lim_{n\to\infty} \|\varepsilon_n - p\| = 0$. Therefore, the algorithm (2) is T-stable. □

Theorem 3. *Suppose that X is a real reflexive Banach space, $T, A : X \to X$, $g : X \to X^*$ are three nonexpansive mappings and $\varphi : X^* \to \mathbb{R} \cup \{\infty\}$ is a function with non-expansive sub differential $\partial\varphi$. Define an operator $R : X \to X$ by $Rx = f - (Tx - Ax + \partial\varphi(g(x))) + x$, in which $f \in X$ is any given point. Suppose that $\{x_n\}_{n=1}^{\infty}$ be iterative sequence generated by the following algorithm:*

$$\begin{cases} x_{n+1} = Ry_n + u_n, \\ \quad y_n = Rz_n + v_n, \\ \quad z_n = (1-\alpha_n)x_n + \alpha_n Rx_n + w_n, \end{cases} \tag{14}$$

in which $0 \le \alpha_n < 1$ and $\{u_n\}$, $\{v_n\}$ $\{w_n\}$ are sequences in E with following restrictions:

i. $0 < L^2 - L^2(\alpha_n - \alpha_n L - \alpha_n^2 L^2) < 1 - \alpha < 1$.
ii. $\sum\limits_{i=1}^{\infty} \|u_n\| < \infty$, $\sum\limits_{i=1}^{\infty} \|v_n\| < \infty$, *and* $\sum\limits_{i=1}^{\infty} \|w_n\| < \infty$

Then the iterative algorithm (2) converges to $p \in X$ and p is the unique solution of nonlinear variational inclusion problem (6)

Proof: As T, A, g, and $\partial\varphi$ are nonexpansive operators, so $(-A)$ and $\partial\varphi \circ g$ are nonexpansive operators. Hence, we get

$$\|x - y\| = \|x - y + r[(T - A + \partial\varphi g - I)x - (T - A + \partial\varphi g - I)y]\|. \tag{15}$$

Therefore, $T - A + \partial\varphi g - I : E \to E$ is an Lipschitzian accretive operator with a Lipschitz constant $L \ge 1$. Since $T - A + \partial\varphi \circ g - I$ is Lipschitzian accretive operator, so $T - A + \partial\varphi \circ g - I$ is m-accretive operator. Hence, for any $f \in X$, the equation $f = (T - A + \partial\varphi \circ g - I)x + x$ has a unique solution $p^* \in X$. Using Lemma 2, it is easy to see that $p^* \in X$ is a solution of nonlinear variational inclusion problem (6) and it is the fixed point of operator R . Again, since $T - A + \partial\varphi \circ g - I : X \to X$ Lipschitzian accretive operator with a Lipschitz constant $L \ge 1$, so $R : X \to X$ is Lipschitzian operator with Lipschitz constant $L^* = L + 1$, such that R is an accretive. Replacing T by R in the iterative algotrithm (2), L by L^* in condition (i) of Theorem 1 and following the procedure of the proof of Theorem 1, it is easy to see that the iterative algotrithm (2) converges to the unique solution $p^* \in X$ of nonlinear variational inclusion problem (6). □

3 Conclusion

In this work, we investigate some fixed point theorems such as convergence and stability by using PMP iterative algorithm with errors for the accretive Lipschitzian operator. We have given a numerical example to demonstrate the effectiveness of the PMP algorithm with errors in terms of convergence speed. Finally, we have shown that the solution to the variational inclusion problem can be reached using this iteration algorithm. The results obtained in this study can be interpreted as an improvement of the corresponding results in the literature.

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Iterative Approximation Technique for Functional Integral Equation

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Abstract: In this study, the iterative approximation technique is used in Banach spaces to reach the uniquely determined solution of a functional integral equation under suitable conditions. In addition, the data dependency of this solution for contraction mappings is examined and a numerical example supporting this result is given.

Keywords: Iterative approximation, Functional integral equation, Data dependence, Contraction mappings.

1 Introduction

Fixed point theory is one of the main tools in studying the solution of various equations such as the differential, integral, functional differential, and functional integral. (see: [1–4]). One of them is the existence, uniqueness, and convergence of successive approximations results for a functional integral equation by Ilea et al. [1]. Throughout this work, we consider the space of continuous functions with maximum norm, and we study in the following integral equation:

Definition 1 ([1])**.** *Let* $(\mathbb{B}, \|\cdot\|)$ *be a Banach space*

$$x(t) = \int_a^c H(t, s, Ax(s))\, ds + \int_a^t K(t, s, Bx(s))\, ds + f(t), \quad t \in [a, b], \tag{1}$$

in which a, b, c *are real numbers with* $c \in (a, b)$*,* $H \in C([a, b] \times [a, c] \times \mathbb{B}, \mathbb{B})$*,* $K \in C\left([a, b]^2 \times \mathbb{B}, \mathbb{B}\right)$*,* $f \in C[a, b]$*,* $A : C([a, c], \mathbb{B}) \to C([a, c], \mathbb{B})$ *and* $B : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$*.*

Define the integral operator $T : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ by

$$Tx(t) = \int_a^c H(t, s, Ax(s))\, ds + \int_a^t K(t, s, Bx(s))\, ds + f(t), \quad t \in [a, b], \tag{2}$$

Ilea et al. [1] obtained the following conditions for finding a solution to the integral equation (1).

Theorem 1 ([1])**.** *Let* $H \in C([a, b] \times [a, c] \times \mathbb{B}, \mathbb{B})$*,* $K \in C\left([a, b]^2 \times \mathbb{B}, \mathbb{B}\right)$*,* $f \in C[a, b]$*,* $A : C([a, c], \mathbb{B}) \to C([a, c], \mathbb{B})$ *and* $B : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ *Suppose that:*

(i) there exist $L_H > 0$ such that

$$|H(t, s, u) - H(t, s, v)| \le L_H |u - v|$$

for all $t \in [a, b]$, $s \in [a, c]$ and all $u, v \in \mathbb{B}$.
(ii) there exist $L_K > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \le L_K |u - v|$$

for all $t \in [a, b]$, $s \in [a, c]$ and all $u, v \in \mathbb{B}$.
(iii) there exist $L_A > 0$ such that

$$|Ax(t) - Ay(t)| \le L_A max_{[a,c]} |x(t) - y(t)|$$

for all $x, y \in C([a, c], \mathbb{B})$.

(iv) there exist $L_B > 0$ such that

$$|Bx(t) - By(t)| \leq L_B max_{[a,t]} |x(t) - y(t)|$$

for all $t \in [a, b]$.
(v) $L_H L_A(c - a) + L_K L_B(b - a) < 1$.

Then equation (2) has a unique solution and the following sequence

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2...$$

in which T is a Picard operator, converges to the solution $p \in C([a, c], \mathbb{B})$ for any initial point $x_0 \in C([a, c], \mathbb{B})$.

Now, the sequence obtained from the following iteration method defined by Sahu [5] is discussed with the idea that it can converge faster to the solution of the integral equation (1). Because the iterative sequence [5] has a better rate of convergence than all Picard [6], Mann [7], and Ishikawa [8] iterative sequences for the class of contraction mappings in the sense of the definition of Berinde [9].

Definition 2. *[5] Let X be a Banach space and let T be a selfmap of X. A normal S-iterative method is defined by*

$$\begin{cases} x_0 \in X \\ x_{n+1} = Ty_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n \end{cases} \tag{3}$$

in which $0 \leq \alpha_n < 1$.

Lemma 1. *[10] Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of non negative numbers for which one assumes there exists $n_0 \in \mathbb{N}$ (set of natural numbers), such that for all $n \geq n_0$*

$$\beta_{n+1} \leq (1 - \mu_n)\beta_n + \mu_n \gamma_n,$$

where $\mu_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. Then the following inequality holds:

$$0 \leq \lim_{n\to\infty} \sup \beta_n \leq \lim_{n\to\infty} \sup \gamma_n.$$

2 Main Results

If the Definition 2 is rewritten using the integral equation given in the Definition 1, we get:

Definition 3. *Let $\mathbb{B}$ be a Banach space and let T be a selfmap of $\mathbb{B}$. The iterative method is defined by*

$$\begin{cases} x_0 \in X \\ x_{n+1} = \int\limits_a^c H(t, s, Ay_n(s))\, ds + \int\limits_a^t K(t, s, By_n(s))\, ds + f(t), \\ y_n = (1 - \alpha_n)x_n + \alpha_n \left(\int\limits_a^c H(t, s, Ax_n(s))\, ds + \int\limits_a^t K(t, s, Bx_n(s))\, ds + f(t) \right) \end{cases} \tag{4}$$

in which $0 \leq \alpha_n < 1$, a, b, c are real numbers with $c \in (a, b)$, $H \in C([a, b] \times [a, c] \times \mathbb{B}, \mathbb{B})$, $K \in C\left([a, b]^2 \times \mathbb{B}, \mathbb{B}\right)$, $f \in C[a, b]$, $A : C([a, c], \mathbb{B}) \to C([a, c], \mathbb{B})$ and $B : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$.

Theorem 2. *Suppose that all the conditions in Theorem 1 are satisfied. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence in $[0, 1]$ satisfying $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Then equation (2) has a unique solution $p \in C([a, c], \mathbb{B})$ and the iterative algorithm (1) converges to p.*

Proof: Let $\{x_n\}_{n=0}^{\infty}$ be iterative sequence generated by iteration method (3) for the operator $T : C([a,b], \mathbb{B}) \to C([a,b], \mathbb{B})$ defined by (2). We will show that $x_n \to p$ as $n \to \infty$. Using all the conditions in Theorem 1, we get

$$\begin{aligned} |Tx_n(t) - Tp(t)| &= \Bigg| \left[\int_a^c H(t,s,Ax_n(s))\,ds + \int_a^t K(t,s,Bx_n(s))\,ds + f(t) \right] \\ &- \left[\int_a^c H(t,s,Ap(s))\,ds + \int_a^t K(t,s,Bp(s))\,ds + f(t) \right] \Bigg| \\ &\leq \int_a^c |H(t,s,Ax_n(s)) - H(t,s,Ap(s))|\,ds \\ &+ \int_a^t |K(t,s,Bx_n(s)) - K(t,s,Bp(s))|\,ds \\ &\leq \int_a^c L_H\,|Ax_n(s) - Ap(s)|\,ds + \int_a^t L_K\,|Bx_n(s) - Bp(s)|\,ds \\ &\leq \int_a^c L_H L_A max_{[a,c]}\,|x_n(s) - p(s)|\,ds + \int_a^t L_K L_B max_{[a,t]}\,|x_n(s) - p(s)|\,ds \\ &\leq (L_H L_A(c-a) + L_K L_B(b-a))\,\|x_n(s) - p(s)\| \end{aligned} \tag{5}$$

and

$$\begin{aligned} \|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n Tx_n - p\| \\ &\leq (1-\alpha_n)\,\|x_n - p\| + \alpha_n\,\|Tx_n - Tp\| \\ &\leq (1-\alpha_n)\,\|x_n - p\| + \alpha_n\,(L_H L_A(c-a) + L_K L_B(b-a))\,\|x_n - p\| \\ &\leq (1 - \alpha_n(1 - (L_H L_A(c-a) + L_K L_B(b-a))))\,\|x_n - p\| \end{aligned} \tag{6}$$

Similar to inequatlity (5)

$$|Ty_n(t) - Tp(t)| \leq (L_H L_A(c-a) + L_K L_B(b-a))\,\|y_n(s) - p(s)\| \tag{7}$$

Combining (6) and (7), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \theta\,\|y_n(s) - p(s)\| \\ &\leq \theta(1 - \alpha_n(1-\theta))\,\|x_n - p\| \end{aligned} \tag{8}$$

in which $\theta = (L_H L_A(c-a) + L_K L_B(b-a))$, by repeating this process n-times, we get

$$\begin{aligned} \|x_n - p\| &\leq \theta\,[1 - \alpha_{n-1}(1-\theta)]\,\|x_{n-1} - p\| \\ \|x_{n-1} - p\| &\leq \theta\,[1 - \alpha_{n-2}(1-\theta)]\,\|x_{n-2} - p\| \\ &\vdots \\ \|x_1 - p\| &\leq \theta\,[1 - \alpha_0(1-\theta)]\,\|x_0 - p\|. \end{aligned} \tag{9}$$

Then

$$\|x_{n+1} - p\| \leq \theta^{(n+1)} \sum_{i=0}^{n} [1 - \alpha_i(1-\theta)]\,\|x_0 - p\| \tag{10}$$

Hence by using this fact with (10), we obtain

$$\|x_{n+1} - p\| \leq \|x_0 - x^*\|\,\theta^{(n+1)} e^{-(1-\theta)\sum_{i=0}^{n} \alpha_i},$$

Hence it follows by its conclusion that

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

□

In order to examine the data dependence result of integral equation (2), we consider the following equation:

$$S(u(t)) = \int_a^c H_1(t,s,Au_n(s))\,ds + \int_a^t K_1(t,s,Bu_n(s))\,ds + f_1(t) \tag{11}$$

in which a, b, c are real numbers with $c \in (a,b)$, $H_1 \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $K_1 \in C\left([a,b]^2 \times \mathbb{B}, \mathbb{B}\right)$, $f_1 \in C[a,b]$, $A : C([a,c], \mathbb{B}) \to C([a,c], \mathbb{B})$ and $B : C([a,b], \mathbb{B}) \to C([a,b], \mathbb{B})$.

Now, we define the following iterative algorithms associated with S in (11),

$$\begin{cases} u_{n+1} = \int\limits_a^c H_1(t,s,Av_n(s))\,ds + \int\limits_a^t K_1(t,s,Bv_n(s))\,ds + f_1(t) \\ v_n = (1-\alpha_n)\,u_n + \alpha_n \left[\int\limits_a^c H_1(t,s,Au_n(s))\,ds + \int\limits_a^t K_1(t,s,Bu_n(s))\,ds + f_1(t)\right] \end{cases} \tag{12}$$

in which $0 \le \alpha_n < 1$, a, b, c are real numbers with $c \in (a,b)$, $H_1 \in C([a,b]\times[a,c]\times\mathbb{B},\mathbb{B})$, $K_1 \in C\left([a,b]^2 \times \mathbb{B},\mathbb{B}\right)$, $f_1 \in C[a,b]$, $A : C([a,c],\mathbb{B}) \to C([a,c],\mathbb{B})$ and $B : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$.

Theorem 3. *Consider the sequences $\{x_n\}_{n=0}$ and $\{u_n\}_{n=0}^{\infty}$ generated by (4) and (12), respectively, with the real sequence $\{\alpha_n\}_{n=0}^{\infty}$in $[0,1]$ satisfying $\frac{1}{2} \le \alpha_n$ for all $n \in \mathbb{N}$. Assume that all the conditions of Theorem 1 hold and p and q are solutions of equations (2) and (11), respectively and there exist non negative constants ε_1, ε_2, and ε_3 such that*
$|H(t,s,u) - H_1(t,s,u)| \le \varepsilon_1$, $|K(t,s,u) - K_1(t,s,u)| \le \varepsilon_2$ and $|f(t) - f_1(t)| \le \varepsilon_3$ for all $t, s \in [a,b]$,.
If the sequence $\{u_n\}_{n=0}^{\infty}$ converge to q as $n \to \infty$, then we have

$$\|p-q\| \le \frac{(2+\theta)\left[\varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2\right]}{1-\theta}. \tag{13}$$

Proof: Using (2), (4), (11), (12) and assumptions (i)-(v) in Theorem 1, we obtain

$$\begin{aligned} |x_{n+1}(t) - u_{n+1}(t)| &= |T(y_n)(t) - S(v_n)(t)| \\ &= \Bigg| \left[\int\limits_a^c H(t,s,Ay_n(s))\,ds + \int\limits_a^t K(t,s,By_n(s))\,ds + f(t)\right] \\ &\quad - \left[\int\limits_a^c H_1(t,s,Av_n(s))\,ds + \int\limits_a^t K_1(t,s,Bv_n(s))\,ds + f_1(t)\right]\Bigg| \\ &\le \int\limits_a^c |H(t,s,Ay_n(s)) - H(t,s,Av_n(s))|\,ds + |f(t) - f_1(t)| \\ &\quad + \int\limits_a^t |K(t,s,By_n(s)) - K(t,s,Bv_n(s))|\,ds \\ &\quad + \int\limits_a^c |H(t,s,Av_n(s)) - H_1(t,s,Av_n(s))|\,ds \\ &\quad + \int\limits_a^t |K(t,s,Bv_n(s)) - K_1(t,s,Bv_n(s))|\,ds \\ &\le (L_H L_A(c-a) + L_K L_B(b-a))\,\|y_n(t) - v_n(t)\| + \varepsilon_3 \\ &\quad + (c-a)\varepsilon_1 + (b-a)\varepsilon_2 \end{aligned} \tag{14}$$

and

$$\begin{aligned} \|y_n - v_n\| &\le (1-\alpha_n)\,\|x_n - u_n\| + \alpha_n \|Tx_n - Tu_n\| \\ &\le (1-\alpha_n)\,\|x_n - u_n\| + \alpha_n (L_H L_A(c-a) + L_K L_B(b-a))\,\|x_n - u_n\| \\ &\quad + \alpha_n\varepsilon_3 + \alpha_n(c-a)\varepsilon_1 + \alpha_n(b-a)\varepsilon_2 \\ &\le (1 - \alpha_n(1 - (L_H L_A(c-a) + L_K L_B(b-a))))\,\|x_n - u_n\| \\ &\quad + \alpha_n\varepsilon_3 + \alpha_n(c-a)\varepsilon_1 + \alpha_n(b-a)\varepsilon_2 \end{aligned} \tag{15}$$

and we have

$$\begin{aligned} \|x_{n+1}(t) - u_{n+1}(t)\| &\le \theta\,\|y_n(t) - v_n(t)\| + \varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2 \\ &\le \theta\,(1 - \alpha_n(1-\theta))\,\|x_n(t) - u_n(t)\| + (1+\theta\alpha_n)\left[\varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2\right] \\ &\le \theta\,(1 - \alpha_n(1-\theta))\,\|x_n(t) - u_n(t)\| + \alpha_n(2+\theta)\left[\varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2\right] \\ &\le \theta\,(1 - \alpha_n(1-\theta))\,\|x_n(t) - u_n(t)\| + \alpha_n(1-\theta)\frac{(2+\theta)\left[\varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2\right]}{1-\theta} \end{aligned} \tag{16}$$

Denote by

$$\begin{aligned}\beta_n &= \|x_n - u_n\|,\\ \mu_n &= \alpha_n(1-\delta) \in (0,1),\\ \gamma_n &= \frac{(2+\theta)\left[\varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2\right]}{1-\theta} \geq 0.\end{aligned}$$

The assumption $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$ implies $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Now it can be easily seen that (16) satisfies all the conditions of Lemma 1. Hence it follows by its conclusion that

$$0 \leq \lim\sup_{n\to\infty} \|x_n - u_n\| \leq \lim\sup_{n\to\infty} \frac{(2+\theta)\left[\varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2\right]}{1-\theta}.$$

By (i), we have that $\lim_{n\to\infty} x_n = p$. Using this fact and the assumption $u_n \to q$ as $n \to \infty$, we get

$$\|p-q\| \leq \frac{(2+\theta)\left[\varepsilon_3 + (c-a)\varepsilon_1 + (b-a)\varepsilon_2\right]}{1-\theta}.$$

□

Example 1. *Consider the following integral equation*

$$x(t) = \int_0^{\frac{1}{4}} \left[\frac{t^2+t+sinx(s)}{4} + \frac{3s}{8}\right] ds + \int_0^{\frac{1}{2}} \left[\frac{t^3-t+cosx(s)}{3} + \frac{2s}{5}\right] ds + \frac{5sint+1}{50},$$

in which $t \in [0,1]$, $H \in C([a,b]\times[a,c]\times\mathbb{B},\mathbb{B})$, $H(t,s,x(t)) = \frac{t^2+t+sinx(t)}{4} + \frac{3t}{8}$, $K \in C\left([a,b]^2\times\mathbb{B},\mathbb{B}\right)$, $K(t,s,x(t)) = \frac{t^3-t+cosx(t)}{3} + \frac{2t}{5}$, $f \in C[a,b]$, $f(t) = \frac{5sint+1}{50}$, $A : C([a,c],\mathbb{B}) \to C([a,c],\mathbb{B})$, $Ax(t) = \frac{3x(t)}{8}$ *and* $B : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$, $B = \frac{2x(t)}{5}$ *and its perturbed integral equation*

$$\tilde{x}(t) = \int_0^{\frac{1}{4}} \left[\frac{t^2+t+sinx(s)}{4} + \frac{5s}{8} - \frac{1}{10}\right] ds + \int_0^{\frac{1}{2}} \left[\frac{t^3-t+cosx(s)}{3} - \frac{3s}{5} + \frac{1}{5}\right] ds + \frac{sint}{10},$$

in which $t \in [0,1]$, $H_1(t,s,x(t)) = \frac{t^2+t+sinx(t)}{4} + \frac{5t}{8} - \frac{1}{10}$, $K_1 \in C\left([a,b]^2\times\mathbb{B},\mathbb{B}\right)$, $K_1(t,s,x(t)) = \frac{t^3-t+cosx(t)}{3} - \frac{3t}{5} + \frac{1}{5}$, $f_1 \in C[a,b]$, $f_1(t) = \frac{sint}{10}$, $A : C([a,c],\mathbb{B}) \to C([a,c],\mathbb{B})$, $Ax(t) = \frac{3x(t)}{8}$ *and* $B : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$, $B = \frac{2x(t)}{5}$. *Define the operator* $T : C[0,1] \to C[0,1]$ *by*

$$Tx(t) = \int_0^{\frac{1}{4}} \left[\frac{t^2+t+sinx(s)}{4} + \frac{3s}{8}\right] ds + \int_0^{\frac{1}{2}} \left[\frac{t^3-t+cosx(s)}{3} + \frac{2s}{5}\right] ds + \frac{5sint+1}{50},\ t \in [0,1],$$

One can easily show on the same lines as above that the mapping $\widetilde{T} : C[0,1] \to C[0,1]$ *defined by*

$$\tilde{T}\tilde{x}(t) = \int_0^{\frac{1}{4}} \left[\frac{t^2+t+sin\tilde{x}(s)}{4} + \frac{5s}{8} - \frac{1}{10}\right] ds + \int_0^{\frac{1}{2}} \left[\frac{t^3-t+cos\tilde{x}(s)}{3} - \frac{3s}{5} + \frac{1}{5}\right] ds + \frac{sint}{10}.$$

Since all the conditions of Theorem 1 *are satisfied by the integral equations* (1) *and* (11) *so by its conclusion, normal S-iterative method* (1) *converges to unique solution p and q, respectively in* $C[0,1]$.

Now we have the following estimates:

$$\begin{aligned}|H(t,s,u)-H(t,s,v)| &\leq L_H\,|u-v| \\ &\leq \frac{1}{4}\,|u-v| \\ |K(t,s,u)-K(t,s,v)| &\leq L_K\,|u-v| \\ &\leq \frac{1}{3}\,|u-v| \\ |Ax(t)-Ay(t)| &\leq L_A max_{[a,c]}\,|x(t)-y(t)| \\ &\leq \frac{3}{8} max_{[a,c]}\,|x(t)-y(t)| \\ |Bx(t)-By(t)| &\leq L_B max_{[a,t]}\,|x(t)-y(t)| \\ &\leq \frac{2}{5} max_{[a,t]}\,|x(t)-y(t)|\end{aligned}$$

and

$$\theta = (L_H L_A(c-a) + L_K L_B(b-a)) = \left(\frac{3}{64}+\frac{2}{15}\right) = 0.180208$$

in which $a=0$, $b=1$, $c=\frac{1}{2}$.

$$|H(t,s,u)-H_1(t,s,u)| \leq 0.4 = \varepsilon_1, \textit{ for all } t \in [0,1],$$

$$|K(t,s,u)-K_1(t,s,u)| \leq 0.8 = \varepsilon_2, \textit{ for all } t \in [0,1].$$

and

$$|f(t)-f_1(t)| = \left|\frac{5sint+1}{50} - \frac{sint}{10}\right| \leq \frac{1}{50} = 0.02 = \varepsilon_3, \textit{ for all } t \in [0,1].$$

In view of the above estimates, all the conditions of Theorem 3 *are satisfied and hence from (*13*), we have*

$$\|p-q\| \leq \frac{(2+\theta)\left[\varepsilon_3+(c-a)\varepsilon_1+(b-a)\varepsilon_2\right]}{1-\theta} = \frac{(2+0.180208)\left[0.02+0.5\cdot 0.4+0.8\right]}{1-0.180208}.$$

3 Conclusion

In this study, we have shown that the iteration method (3) converges to the solution of the functional integral equation. Finally, we have proven a data dependence result can be obtained for the solution of the integral equation. We have also provided a nontrivial numerical example to support the Data Dependence Theorem.

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Motions and Surfaces in Non-Euclidean Spaces

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Abstract: Motions play a significant part in differential geometry, as is well known. In the study of space kinematics or spatial mechanism, it is crucial to look into the geometry of the motion of a line or a point. This type of motion's geometry has numerous uses in geometric modeling, the creation of mechanical products, and the design of robotic motions. The motions of the points and curves are used to model many non-linear situations seen in both nature and science. For instance, the dynamics of vortices, interface motions, robotic movements, flame front propagation, image processing, supercoil DNAs, magnetic fluxes, membrane deformation, and protein dynamics. The differential equations describing the motion of curves as a family must be used in these nonlinear applications to describe the evolution of the curves.
In this study we search for the surfaces with constant Gauss curvatures, which are the orbit of a space curve in Lorentz 3-space and give some results.
Keywords: Gauss curvature, motions, Lorentz space, Galilean space

1 Introduction

Motions are important subjects of differential geometry. Many researchers have studied motions in different spaces and found interesting results. Beneki et al solved the problem of finding the helicoidal surfaces via helicoidal motions, with prescribed Gaussian or mean curvature given by smooth functions in three dimensional Minkowski space [3]. Lopez studied surfaces with constant Gauss curvature in Lorentz 3-space [4]. Jafari investigated some properties of homothetic motions in Euclidean 3-space [5]. Aksoyak and Yayl? defined a surface by means of homothetic motions in E_1^4 and gave some special subgroups of the Lie group P [1]. Mosa and Elzawy constructed helicoidal surfaces in Galilean 3-space and obtained the mean and gaussian curvature of these surfaces [6]. In this paper we study the homothetic motions in Lorentz 3-space and search for the surfaces with constant Gauss curvature.
Consider the Lorentz 3-Space E_1^3, that is, three dimensional real vector space $\mathbb{R}^3$ endowed with the metric $\langle , \rangle$ given by,

$$\langle , \rangle : E^3 \text{x} E^3 \longrightarrow E$$

$$(u,v) \longrightarrow \langle u,v \rangle = u_1v_1 + u_2v_2 - u_3v_3$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in E^3$.

Definition 1. *In three dimensional Lorentz space, one parameter homothetic motion of a body is generated by the transformation*

$$\begin{pmatrix} Y \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda A & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

where $A \in O_1(3)$, $A^t = \varepsilon A^{-1} \varepsilon$ and the signature matrix ε is the diagonal matrix $(\delta_{ij}\varepsilon_j)$ whose diagonal entries are $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = -1$. X and Y are real matrices of $3x1$ type and λ is the homothetic scale, A, λ and C are differentiable functions of C^∞ class of parameter t.
To avoid the case of affine transformation we assume that $\lambda = \lambda(t) \neq$constant. Also to avoid the cases of pure translation and pure rotation we assume that $\frac{d}{dt}(\lambda A) \neq 0$ and $\frac{d}{dt}(C) \neq 0$ [8].

Definition 2. *One parameter homothetic motion in E_1^3 can also be determined with an axis L and pitch $h \in \mathbb{R}$. Depending on the causal character of the axis there are three types of homothetic motions in E_1^3.*

1. If L is timelike, then $L=\langle(0,0,1)\rangle$ and the homothetic motion around this axis is

$$\phi_t(a,b,c)=\lambda(t)\begin{pmatrix}\cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+h\begin{pmatrix}0\\ 0\\ t\end{pmatrix}$$

2. If L is spacelike, then $L=\langle(1,0,0)\rangle$ and the homothetic motion around this axis is

$$\phi_t(a,b,c)=\lambda(t)\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh t & \sinh t\\ 0 & \sinh t & \cosh t\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+h\begin{pmatrix}t\\ 0\\ 0\end{pmatrix}$$

3. If L is lightlike, then $L=\langle(1,0,1)\rangle$ and the homothetic motion around this axis is

$$\phi_t(a,b,c)=\lambda(t)\begin{pmatrix}1-\frac{t^2}{2} & t & \frac{t^2}{2}\\ -t & 1 & t\\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2}\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+h\begin{pmatrix}\frac{t^3}{3}-t\\ t^2\\ \frac{t^3}{3}+t\end{pmatrix}$$

Here λ is a differentiable function of parameter t and h is a real scalar. In particular, if we take $\lambda(t)=1$ then we get helicoidal motion groups in Lorentz 3-space. Also if we take $\lambda(t)=1$ and $h=0$ then the new motion is a rotation in E_1^3.

2 Curvatures of a non degenerate surface in Lorentz 3-space

An immersion $x:M\longrightarrow E_1^3$ of a surface M is called spacelike if the tangent plane T_pM is spacelike, M is called timelike if the tangent plane T_pM is timelike for all $p\in M$. This is also equal to say that M is spacelike or timelike, respectively. In both cases the surface M is non-degenerate.

Let $\chi(M)$, be the class of tangent vector fields of M and ∇^0 be the Levi-Civita connection of E_1^3. Denote by ∇ the induced connection on M by immersion x, that is, $\nabla_XY=(\nabla_X^0Y)^\top$, where $\top$ denotes the tangent part of the vector field ∇_X^0Y. Then we have

$$\nabla_X^0Y=\nabla_XY+\sigma(X,Y) \tag{2.1}$$

called the Gauss formula. Here $\sigma(X,Y)$ is the normal part of the vector field ∇_X^0Y, that is $\sigma:\chi(M)\mathrm{x}\chi(M)\longrightarrow(\chi(M))^\perp$. Let Z be a normal vector field to x and $-\nabla_X^0Z$. $A_Z(X)=-(\nabla_X^0Z)^\top$ is the tangent component. From (2.1)

$$\langle A_Z(X),Y\rangle=\langle\sigma(X,Y),Z\rangle \tag{2.2}$$

The map $A_Z:\chi(M)\longrightarrow\chi(M)$ is called Weingarten endomorphism of Z. Because σ is symmetric, from (2.2) we have

$$\langle A_Z(X),Y\rangle=\langle X,A_Z(Y)\rangle \tag{2.3}$$

This means that A_Z is self-adjoint with respect to the metric $\langle,\rangle$ of M. Because our results are local, we only need local orientability, which is trivially satisfied. However a spacelike surface is globally orientable. Denote by N the gauss map on M. Define ε by $\langle N,N\rangle=\varepsilon$, where $\varepsilon=-1$ or 1, if the immersion is spacelike or timelike respectively. If we take $Z=N$ then $\langle\nabla_X^0N,N\rangle=0$ hence the normal part $(\nabla_XN)^\perp=0$ and we have the Weingarten formula

$$-\nabla_X^0N=A_N(X) \tag{2.4}$$

[7].

Definition 3. *The Weingarten endomorphism at $p\in M$ is defined by*
$A_p:T_pM\longrightarrow T_pM$, $A_p=A_{N(p)}$. If $v\in T_pM$ and $X\in\chi(M)$ is a tangent vector field, that is $X(P)=v$, then $A_p(v)=(A_N(X))_p$. Also from (2.4)

$$A_p(v)=-(dN)_p(v),\quad v\in T_pM$$

Here $(dN)_p$, is the differentiation of N at point p in E_1^3.

Also $X,Y\in\chi(M)$ and from (2.1) and (2.2) we have

$$\sigma(X,Y)=\varepsilon\langle\sigma(X,Y),N\rangle N=\varepsilon\langle A(X),Y\rangle N$$

and the equation (2.1) can be written as

$$\nabla_X^0Y=\nabla_XY+\varepsilon\langle A(X),Y\rangle N$$

[7].

Definition 4. *Let A be the Weingarten endomorphism of M in E_1^3. Then the Gauss curvature of M is defined as follows*

$$K = \varepsilon det(A)$$

Let $X : U \subset \mathbb{R}^2 \longrightarrow E_1^3$ be a parametrization of M with $X = X(u,v)$ and

$$X_u = \frac{\partial X(u,v)}{\partial u}, \qquad X_v = \frac{\partial X(u,v)}{\partial v}$$

Denote by $\{E,F,G\}$ and $\{e,f,g\}$ the coefficients of first and second fundamental form respectively. Then

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle$$

$$e = -\langle N_u, X_u \rangle, \quad f = -\langle N_u, X_v \rangle, \quad g = -\langle N_v, X_v \rangle$$

Also

$$K = \varepsilon \frac{eg - f^2}{EG - F^2}$$

Here

$$N = \frac{X_u \mathrm{x} X_v}{\sqrt{\varepsilon(EG - F^2)}}$$

$W = EG - F^2$ is positive if the immersion is spacelike and negative if the immersion is timelike [7].

Theorem 1. *Consider a non-degenerate surface which is generated by a graph of a polynomial under homothetic motion groups in E_1^3. If the surface have constant Gauss curvature then the degree of the polynomial is* 1 *or* 0*. That is the generating curve is a straight line and the surface is a ruled surface. Moreover*

(i) If the axis is timelike $L = \langle (0,0,1) \rangle$ the generating curve is $\gamma(s) = (s, 0, f(s))$, $f(s) = \sum_{n=0}^{m} a_n s^n$, $a_n \in \mathbb{R}$ then the parameterization of the surface generated by γ under homothetic motion group around this axis is

$$X(s,t) = (s\lambda(t)\cos t, s\lambda(t)\sin t, ht + \lambda(t) f(s)) \tag{3.1}$$

For this surface

a) If $K = 0$ then $f(s) = a_0 + a_1 s$ and $\lambda(t) = b_0 - \frac{h}{a_0} t$.

b) If K is a non-zero constant then $f(s) = a_0 \pm s$, $\lambda(t) = b_0 + b_1 t$ and we have $K = \frac{1}{(h + a_0 \lambda')^2}$

ii) If the axis is spacelike $L = \langle (1,0,0) \rangle$ the generating curve is $\gamma(s) = (f(s), s, 0)$, $f(s) = \sum_{n=0}^{m} a_n s^n, a_n \in \mathbb{R}$ and

$$X(s,t) = (ht + \lambda(t) f(s), s\lambda(t)\cosh t, s\lambda(t)\sinh t) \tag{3.2}$$

is the parameterization of the surface generated by γ under homothetic motion around spacelike axis.

a) If $K = 0$ then $f(s) = a_0$ or $f(s) = a_0 + a_1 s$ and $\lambda(t) = b_0 - \frac{h}{a_0} t$.

b) If $f(s) = \sum_{n=0}^{m} a_n s^n$ and $\lambda(t) = \sum_{k=0}^{l} b_k t^k$, $b_k \in \mathbb{R}$ then K can not be a non-zero constant.

iii) If the axis is lightlike $L = \langle (1,0,1) \rangle$ the generating curve is $\gamma(s) = (f(s), s, f(s))$, $f(s) = \sum_{n=0}^{m} a_n s^n$, $a_n \in \mathbb{R}$ then the parameterization of the surface generated by γ under homothetic motion group around this axis is

$$X(s,t) = (\lambda(t)(ts + f(s)) + h(\frac{t^3}{3} - t), s\lambda(t) + ht^2, \lambda(t)(ts + f(s)) + h(\frac{t^3}{3} + t)) \tag{3.3}$$

For this surface

a) If $f(s) = \sum_{n=0}^{m} a_n s^n$ and $\lambda(t) = \sum_{k=0}^{l} b_k t^k$, $b_k \in \mathbb{R}$ then K can not be zero.

b) If $f(s) = \sum_{n=0}^{m} a_n s^n$ and $\lambda(t) = \sum_{k=0}^{l} b_k t^k$, $b_k \in \mathbb{R}$ then K can not be a non-zero constant.

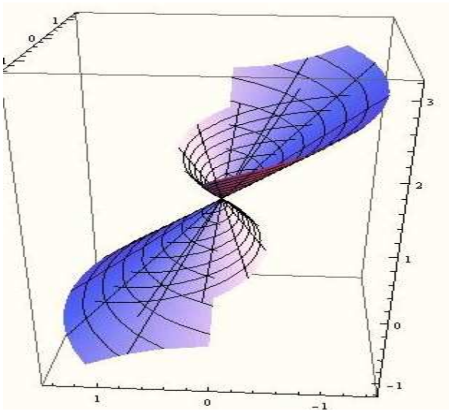

Fig. 1

Example 1. *The surface generated by $\gamma(s) = (s, 0, f(s))$ around timelike axis with homothetic motion where $f(s) = 1+2s$ and $\lambda(t) = 1-ht$, has zero Gauss curvature.*

3 Curvatures of a non degenerate surface in Galilean 3-space

For 3-dimensional Galilean space $\mathbb{G}_3$, the Galilean scalar product between two vectors $\xi = (\xi_1, \xi_2, \xi_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ is defined by

$$\langle \xi, \zeta \rangle_{G_3} = \begin{cases} \xi_1\zeta_1, & \text{if } \xi_1 \text{ or } \zeta_1 \text{ is not zero,} \\ \xi_2\zeta_2 + \xi_3\zeta_3, & \text{if } \xi_1 \text{ and } \zeta_1 \text{ are zero} \end{cases}$$

and the Galilean cross product is given as

$$(\xi \times \zeta)_{G_3} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix}, & \text{if } \xi_1 \text{ or } \zeta_1 \text{ is not zero,} \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix}, & \text{if } \xi_1 \text{ and } \zeta_1 \text{ are zero.} \end{cases}$$

where e_1, e_2, e_3 are Euclidean standard basis [2].

Definition 5. *Let $\alpha(u_1) = (u_1, 0, f(u_1))$ be a regular planar curve in Galilean 3-space. Then the surface parametrized by*

$$\phi(u_1, u_2) = (u_1 \cos u_2, u_1 \sin u_2, f(u_1) + bu_2)$$

is the helicoidal surface with the $oz-$axis, the pitch b and the generating curve α. Here f is a differentiable function [6].

Definition 6. *The mean curvature and the Gauss curvature of the helicoidal surface $\phi(u_1, u_2)$ is given by*

$$H = \frac{1}{2w^3}(-u_1^3 f''(u_1)\sin^2 u_2 + 2bu_1 \sin u_2 \cos u_2 - u_1^2 f'(u_1)\cos^2 u_2)$$

and

$$K = \frac{1}{w^4}(-u_1^3 f'(u_1) f''(u_1) - b^2)$$

respectively.
Here $w = \sqrt{b\cos u_2 + u_1 f'(u_1)\sin^2 u_2 + u_1^2}$
The mean curvature of the surface is obtain from

$$H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2w^2}$$

where $g_i = \phi_{u_i}, L_{11} = \frac{-u_1}{w} f''(u_1)$, $L_{22} = \frac{-u_1^2}{w} f'(u_1)$ and $L_{12} = \frac{b}{w}$. Then we have

$$H = \frac{1}{2w^3}(-u_1^3 f''(u_1)\sin^2 u_2 + 2bu_1 \sin u_2 \cos u_2 - u_1^2 f'(u_1)\cos^2 u_2)$$

Similar to the mean curvature, the Gauss curvature of the surface is obtain from

$$K = \frac{L_{11}L_{22} - L_{12}^2}{w^2}$$

With the necessary computations we have the Gauss curvature as

$$K = \frac{1}{w^4}(-u_1^3 f'(u_1) f''(u_1) - b^2)$$

[6].

4 Conclusion

It is commonly known that motions play a crucial role in differential geometry. In this study, we deal with a variety of motion groups in non-Euclidean spaces. In addition, we define surfaces with constant Gauss curvature in Lorentz 3-space and provide an illustration for these surfaces. We only briefly touch on certain fundamental ideas in the Galilean 3-space section, including definitions of helicoidal motions and helicoidal surface's curvatures. In additional research, we are looking for the criteria that must exist for surfaces to have constant Gauss curvature and mean curvature in Galilean 3-space.

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A New Separation Axiom in Ideal Topological Spaces

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Abstract: In this work, σ-R_0 ideal topological spaces is defined by using the concepts of σ-closure operator and local closure function. Moreover, the concept of a new kernel definition is introduce via local closure operator. The basic properties and characterizations of this new type of ideal topological spaces is obtained. The case that σ-topology obtained with the help of the local closure function is R_0-space is examined .

Keywords: Ideal topological spaces, Local closure function, R_0 topological spaces, Separation axioms.

1 Introduction and Preliminaries

Kuratowski defined the ideal and local function concepts ([6],[7]). He gave the basic properties of the local function. Later, Vaidyanathaswamy obtained new results on ideal topological spaces [11]. The concept of ideal was used not only in general topology but also in different branches of mathematics. Freud [4] generalized the Cantor-Bendixson Theorem. In [5], many results are given on the concepts of ideal and local function.

Shanin defined [10] the concept of R_0-space . Davis shown [2] every T_1-space is a R_0-space and gave the necessary inverse example. Dube introduced the concept of the kernel of a set in topological spaces [3]. He gave new characterizations of R_0-space using the kernel concept.

In recent years, new types of local functions have been defined ([1],[13],[14]). One of them is the local closure function. The basic properties of the local closure function were given by Al-Omari and Noiri [1]. They obtained two new topologies with the help of local closure function. Pavlovic gave theorems and examples showing the relationship between local function and local closure function [9]. In [8] also examined the case of the local closure function being idempotent.

In this study, we introduce the concept of σ-R_0-space and σKer_Γ by using the σ-topology produced with the help of the ideal and local closure function.

Definition 1. *[7] Let $X \neq \emptyset$ and $\mathcal{I} \subseteq \mathcal{P}(X)$. If the family $\mathcal{I}$ satifies the following conditions, it is called an ideal on X :*

1. $\emptyset \in \mathcal{I}$.
2. *If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.*
3. *If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.*

If (X, τ) is a topological space with the ideal $\mathcal{I}$, then the triplet $(X, \tau, \mathcal{I})$ is called an ideal topological space. We will denote the family of open neighborhoods of x with τ_x throughout this work.

Definition 2. *[1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. An operator $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by*

$$\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : Cl(U) \cap A \notin \mathcal{I} \text{ for every } U \in \tau_x\}$$

and is called the local closure function of A with respect to $\mathcal{I}$ and τ.

Theorem 3. *[1] Let $(X, \tau, \mathcal{I})$ be an ideal topological spaces.*

a) If $A \subseteq B$, then $\Gamma(A) \subseteq \Gamma(B)$.
b) If $A \in \mathcal{I}$, then $\Gamma(A) = \emptyset$.

Definition 4. *[1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. An operator $\Psi_\Gamma(A) : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as:*

$$\Psi_\Gamma(A) = \{x \in X : \text{there exists } U \in \tau_x \text{ such that } (Cl(U) \setminus A) \in \mathcal{I}\}$$

It is clear that $\Psi_\Gamma(A) = X \setminus \Gamma(X \setminus A)$.

Theorem 5. *[1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$.*

$$\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}$$

is a topology on X. Its elements is called σ-open.

Throughout this work, we will denote the family of closed sets according to this topology by $\mathcal{C}\sigma$, and we will denote the closure according to this topology by Cl_σ.

Definition 6. *[12] Let (X, τ) be a topological space. A subset A of X is called θ-open if each point of A has an open neighborhood U such that $Cl(U) \subseteq A$.*

The following diagram is given in [1].

$$\begin{array}{ccc} \theta\text{-open} & \rightarrow & \text{open} \\ \downarrow & & \\ \sigma\text{-open} & & \end{array}$$

Definition 7. *[10] Let (X, τ) be topological space. If $Cl(\{x\}) \subseteq U$ for every $U \in \tau$ and for each $x \in U$, then this space is said to be R_0-space.*

Theorem 8. *[2] If (X, τ) is T_1-space, then it is R_0-space.*

2 σ-R_0 Ideal Topological Spaces

Lemma 9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, for all $x \in X$,*

$$\Gamma(\{x\}) \subseteq Cl_\sigma(\{x\}).$$

Proof: Let $y \notin Cl_\sigma(\{x\})$. There exists $U \in \sigma_y$ such that $U \cap \{x\} = \emptyset$. Therefore $x \notin U$. Since $U \in \sigma$, $U \subseteq [X \setminus \Gamma(X \setminus U)] \subseteq X \setminus \Gamma(\{x\})$. Hence $\Gamma(\{x\}) \subseteq \Gamma(X \setminus U) \subseteq X \setminus U$. Since $y \notin X \setminus U$, $y \notin \Gamma(\{x\})$. Consequently, we obtain to the desired result. □

Definition 10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $Cl_\sigma(\{x\}) \setminus \Gamma(\mathcal{I}, \tau)(\{x\}) \subseteq U$ for every $U \in \sigma$ and for each $x \in U$, then this space is called σ-R_0-space.*

Theorem 11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If the topology (X, σ) obtained from this ideal space is R_0 topological space, then $(X, \tau, \mathcal{I})$ is σ-R_0-space.*

Proof: Let $x \in U \in \sigma$. Since (X, σ) is R_0-space, then $Cl_\sigma(\{x\}) \subseteq U$. Therefore $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq U$. □

Remark 1. *The converse of implication in Theorem* 11 *is not true in generally as shown in the next example.*

Example 12. *For $(\mathbb{R}, \tau = \{\mathbb{R}, \emptyset\}, \mathcal{I} = \{\emptyset, \{1\}\})$, we obtain the topology $\sigma = \{\mathbb{R}, \emptyset, \mathbb{R} \setminus \{1\}\}$. For $2 \in \mathbb{R} \setminus \{1\} \in \sigma$, $Cl_\sigma(\{2\}) = \mathbb{R}$. Therefore $(\mathbb{R}, \sigma)$ is not R_0-space. But $(\mathbb{R}, \tau, \mathcal{I})$ is σ-R_0-space. Because, for every $U \in \sigma$ and for every $x \in U$, $Cl_\sigma(\{x\}) \setminus \Gamma(\mathcal{I}, \tau)(\{x\}) \subseteq U$.*

One must focus on topologies on X. Related to this attention, we give the following examples.

Example 13. *In the previous example, although $(\mathbb{R}, \tau)$ is R_0-space, $(\mathbb{R}, \sigma)$ is not R_0-space.*

Example 14. *Consider the left topology $\tau_L = \{(-\infty, r) \subseteq \mathbb{R} : r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ on real numbers $\mathbb{R}$ with the ideal $\mathcal{I} = \mathcal{P}(\mathbb{R})$. It is obvious that is the topology $\sigma = \mathcal{P}(\mathbb{R})$. Therefore $(\mathbb{R}, \sigma)$ is R_0-space but $(\mathbb{R}, \tau_L)$ is not R_0-space.*

From Example 13 and Example 14, we obtain the following corollary.

Corollary 1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The fact that (X, τ) is R_0-space and (X, σ) is R_0-space are independent of each other.*

Theorem 15. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\{x\} \in \mathcal{I}$ for every $x \in X$.*

$$(X, \tau, \mathcal{I}) \text{ is } \sigma\text{-}R_0\text{-space} \Leftrightarrow (X, \sigma) \text{ is } R_0\text{-space}$$

Proof: Let $(X, \tau, \mathcal{I})$ be σ-R_0-space. For any $U \in \sigma$ and for any $x \in U$, $Cl_\sigma(\{x\}) \setminus \Gamma(\{x\}) \subseteq U$. Since $\{x\} \in \mathcal{I}$, $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] = Cl_\sigma(\{x\}) \setminus \emptyset = Cl_\sigma(\{x\}) \subseteq U$. Namely, (X, σ) is R_0. Conversely it is obtained by Theorem 11. □

From Theorem 15, we obtain the following corollary.

Corollary 2. *a) $(X, \tau, \mathcal{I}_f)$ is σ-R_0-space $\Leftrightarrow$ (X, σ) is R_0-space ($\mathcal{I}_f$ is the ideal of finite subsets of X).*

b) $(X, \tau, \mathcal{I}_c)$ is σ-R_0-space $\Leftrightarrow$ (X, σ) is R_0-space ($\mathcal{I}_c$ is the ideal of countable subsets of X).

Theorem 16. *The following statements are equivalent:*

a) The ideal topological space $(X, \tau, \mathcal{I})$ is σ-R_0-space.
b) For each $F \in \mathcal{C}\sigma$ and $x \notin F$, there exists $U \in \sigma$ such that $x \notin U$, $F \subseteq U \cup \Gamma(\{x\})$ and $U \cap \Gamma(\{x\}) = \emptyset$.
c) For each $F \in \mathcal{C}\sigma$ and $x \notin F$, $F \cap \Gamma(\{x\}) = F \cap Cl_\sigma(\{x\}) \subseteq \Gamma(\{x\})$.
d) For each $x, y \in X$, $Cl_\sigma(\{x\}) = Cl_\sigma(\{y\})$ or $[Cl_\sigma(\{x\}) \cap Cl_\sigma(\{y\})] = [Cl_\sigma(\{y\}) \cap \Gamma(\{x\})] \subseteq \Gamma(\{x\})$.

Proof:
a)⇒b) Let $F \in \mathcal{C}\sigma$ and $x \notin F$. Since $(X, \tau, \mathcal{I})$ is σ-R_0-space and $x \in X \setminus F \in \sigma$, $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq X \setminus F$. We have $F \subseteq [X \setminus Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] = [(X \setminus Cl_\sigma(\{x\})) \cup \Gamma(\{x\})]$. Then $x \notin U = [X \setminus Cl_\sigma(\{x\})] \in \sigma$ and $F \subseteq [(X \setminus Cl_\sigma(\{x\})) \cup \Gamma(\{x\})] = [U \cup \Gamma(\{x\})]$. Moreover $[U \cap \Gamma(\{x\})] \subseteq [U \cap Cl_\sigma(\{x\})] = ([X \setminus Cl_\sigma(\{x\})] \cap Cl_\sigma(\{x\})) = \emptyset$.
b)⇒c) Let $F \in \mathcal{C}\sigma$ and $x \notin F$. From b), there exists $U \in \sigma$ such that $x \notin U$, $F \subseteq U \cup \Gamma(\{x\})$ and $U \cap \Gamma(\{x\}) = \emptyset$. Since $x \notin U$, $U \cap \{x\} = \emptyset$. Therefore $[U \cap Cl_\sigma(\{x\})] = \emptyset$. Using Lemma 9,

$$\begin{aligned}
\Gamma(\{x\}) &= \Gamma(\{x\}) \cup (Cl_\sigma(\{x\}) \cap U) \\
&= [\Gamma(\{x\}) \cup Cl_\sigma(\{x\})] \cap [\Gamma(\{x\}) \cup U] \\
&\supseteq [Cl_\sigma(\{x\}) \cap F]
\end{aligned}$$

Moreover, from Lemma 9, $\Gamma(\{x\}) \subseteq Cl_\sigma(\{x\})$. Therefore $[\Gamma(\{x\}) \cap F] = [Cl_\sigma(\{x\}) \cap F] \subseteq \Gamma(\{x\})$.
c)⇒d) Let $Cl_\sigma(\{x\}) \neq Cl_\sigma(\{y\})$. There exists $z \in Cl_\sigma(\{x\})$ such that $z \notin Cl_\sigma(\{y\})$. Since $z \in Cl_\sigma(\{x\})$, $U \cap \{x\} \neq \emptyset$ for every $U \in \sigma_z$. That is, $x \in U$ for every $U \in \sigma_z$. Therefore, there exists $U \in \sigma_x$ such that $U \cap \{y\} = \emptyset$. That is, $x \notin Cl_\sigma(\{y\}) \in \mathcal{C}\sigma$. From c), $Cl_\sigma(\{y\}) \cap \Gamma(\{x\}) = \Gamma(\{y\}) \cap \Gamma(\{x\}) \subseteq \Gamma(\{x\})$.
d)⇒a) Let $U \in \sigma$ and $x \in U$. For any $y \in X \setminus U \in \mathcal{C}\sigma$, $Cl_\sigma(\{y\}) \subseteq X \setminus U$ and $\bigcup_{y \in X \setminus U} Cl_\sigma(\{y\}) = X \setminus U$. Since $[U \cap \{y\}] \subseteq [U \cap (X \setminus U)] = \emptyset$, $x \notin Cl_\sigma(\{y\})$. Therefore $Cl_\sigma(\{y\}) \neq Cl_\sigma(\{x\})$, for every $y \in X \setminus U$. From d), $[Cl_\sigma(\{x\}) \cap Cl_\sigma(\{y\})] = [Cl_\sigma(\{y\}) \cap \Gamma(\{x\})] \subseteq \Gamma(\{x\})$ for every $y \in X \setminus U$. Therefore $\bigcup_{y \in X \setminus U} [(Cl_\sigma(\{y\}) \cap Cl_\sigma(\{x\})] = [\bigcup_{y \in X \setminus U} (Cl_\sigma(\{y\}))] \cap Cl_\sigma(\{x\}) = [(X \setminus U) \cap Cl_\sigma(\{x\})] \subseteq \Gamma(\{x\})$. We have $(X \setminus U) \cap [(Cl_\sigma(\{x\}) \cap (X \setminus \Gamma(\{x\}))] \subseteq [\Gamma(\{x\}) \cap (X \setminus \Gamma(\{x\})] = \emptyset$. Therefore $[Cl_\sigma(\{x\}) \cap (X \setminus \Gamma(\{x\}))] \subseteq U$. Consequently, we have $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq U$. □

Corollary 3. *$(X, \tau, \mathcal{I})$ is σ-R_0-space if and only if $Cl_\sigma(\{x\}) \neq Cl_\sigma(\{y\})$ implies $[Cl_\sigma(\{x\}) \cap Cl_\sigma(\{y\})] = [Cl_\sigma(\{y\}) \cap \Gamma(\{x\})] \subseteq \Gamma(\{x\})$.*

Theorem 17. *The following statements are equivalent:*

a) $(X, \tau, \mathcal{I})$ is σ-R_0-space.
b) For any nonempty subset A and $G \in \sigma$ such that $A \cap G \neq \emptyset$, there exists $F \in \mathcal{C}\sigma$ such that $A \cap F \neq \emptyset$ and $F \setminus \Gamma(\{x\}) \subseteq G$, for every $x \in (A \cap G)$.

Proof: a)⇒b) Let A be nonempty subset of X and $G \in \sigma$ such that $(A \cap G) \neq \emptyset$. Since $(X, \tau, \mathcal{I})$ is σ-R_0-space, $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq G$ for any $x \in (A \cap G)$. Let $F = Cl_\sigma(\{x\}) \in \mathcal{C}\sigma$. Then $x \in (A \cap F)$ and $(F \setminus \Gamma(\{x\})) \subseteq G$.
b)⇒a) Let $G \in \sigma_x$ and $A = \{x\}$. Then $x \in (G \cap A) \neq \emptyset$. From b), there exists $F \in \mathcal{C}\sigma$ such that $(A \cap F) \neq \emptyset$ and $[F \setminus \Gamma(\{x\})] \subseteq G$. Since $x \in F$ and $F \in \mathcal{C}\sigma$, $Cl_\sigma(\{x\}) \subseteq F$. Therefore $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq [F \setminus \Gamma(\{x\})] \subseteq G$. □

Definition 18 ([3])**.** *Let (X, τ) be a topological space. The kernel of the subset A is defined as*

$$Ker(A) = \bigcap \{U \in \tau : A \subseteq U\}$$

.

To avoid any confusion, we will denote the kernel of a subset A with $\sigma Ker(A)$ according to the topology (X, σ). We now give a new definition.

Definition 19. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and the topology (X, σ) obtained from this ideal space. Then σ-kernel of the subset A is defined as*

$$\sigma Ker_\Gamma(A) = \bigcap \{U \cup \Gamma(A) : A \subseteq U \text{ and } U \in \sigma\}$$

.

Lemma 20. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then, $\sigma Ker_\Gamma(A) = \sigma Ker(A) \cup \Gamma(A)$.*

Proof: From the definition of $\sigma Ker_\Gamma(A)$, it is obtained. That is,

$$\begin{aligned}
\sigma Ker_\Gamma(A) &= \bigcap \{U \subseteq X : A \subseteq U \text{ and } U \in \sigma\} \cup \Gamma(A) \\
&= \sigma Ker(A) \cup \Gamma(A)
\end{aligned}$$

□

Lemma 21. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then,*

$$y \in \sigma Ker_\Gamma(\{x\}) \Leftrightarrow x \in Cl_\sigma(\{y\}) \text{ or } y \in \Gamma(\{x\}).$$

Proof: Let $y \in \sigma Ker_\Gamma(\{x\})$. From Lemma 20, $y \in \sigma Ker(\{x\})$ or $y \in \Gamma(\{x\})$. Therefore $x \in Cl_\sigma(\{y\})$ or $y \in \Gamma(\{x\})$.
Conversely, let $x \in Cl_\sigma(\{y\})$ or $y \in \Gamma(\{x\})$. Therefore $y \in \sigma Ker_\Gamma(\{x\})$ or $y \in \Gamma(\{x\})$. Consequently, $y \in \sigma Ker_\Gamma(\{x\})$. □

Lemma 22. *[3] Let (X, τ) be any topological space and $x, y \in X$. $Ker(\{x\}) \neq Ker(\{y\})$ if and only if $Cl(\{x\}) \neq Cl(\{y\})$.*

Lemma 23. *If $\sigma Ker_\Gamma(\{x\}) \neq \sigma Ker_\Gamma(\{y\})$, then at least one of the following is provided.*

a) $Cl_\sigma(\{x\}) \neq Cl_\sigma(\{y\})$ and $\sigma Ker(\{x\}) \neq \Gamma(\{y\})$.
b) $\Gamma(\{x\}) \neq \Gamma(\{y\})$ and $\Gamma(\{x\}) \neq \sigma Ker(\{y\})$.

Proof: Let $z \in \sigma Ker_\Gamma(\{x\})$ and $z \notin \sigma Ker_\Gamma(\{y\})$. Then $z \in \sigma Ker(\{x\})$ or $z \in \Gamma(\{x\})$. Moreover, $z \notin \sigma Ker(\{y\})$ and $z \notin \Gamma(\{y\})$. From Lemma 22,

a) If $z \in \sigma Ker(\{x\})$, then $Cl_\sigma(\{x\}) \neq Cl_\sigma(\{y\})$ and $\sigma Ker(\{x\}) \neq \Gamma(\{y\})$.
b) If $z \in \Gamma(\{x\})$, then $\Gamma(\{x\}) \neq \sigma Ker(\{y\})$ and $\Gamma(\{x\}) \neq \Gamma(\{y\})$.

□

Theorem 24. *The following statements are equivalent:*

a) $(X, \tau, \mathcal{I})$ is σ-R_0-space.
b) For each $x \in X$, $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq \sigma Ker_\Gamma(\{x\})$

Proof: b) $\Rightarrow$ a) Let $x \in G \in \sigma$.

$$\begin{aligned} y \in \sigma Ker_\Gamma(\{x\}) &\Leftrightarrow [y \in \sigma Ker(\{x\})] \text{ or } [y \in \Gamma(\{x\})] \\ &\Leftrightarrow x \in Cl_\sigma(\{y\}) \text{ or } y \in \Gamma(\{x\}) \\ &\Rightarrow y \in G \text{ or } y \in \Gamma(\{x\}) \\ &\Leftrightarrow y \in [G \cup \Gamma(\{x\})]. \end{aligned}$$

This shows that $\sigma Ker_\Gamma(\{x\}) \subseteq G \cup \Gamma(\{x\})$. From b), $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq [G \cup \Gamma(\{x\})]$ if and only if $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq G$.

a) $\Rightarrow$ b) Let any $x \in X$ and $y \in Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})$. Since $(X, \tau, \mathcal{I})$ is σ-R_0-space, $y \in Cl_\sigma(\{x\}) \setminus \Gamma(\{x\}) \subseteq G$ for every $G \in \sigma_x$. Namely, $y \in G$ for every $G \in \sigma_x$. Therefore $y \in Cl_\sigma(\{y\})$. From Lemma 21, $y \in \sigma Ker_\Gamma(\{x\})$. Consequently, we obtain $Cl_\sigma(\{x\}) \subseteq \sigma Ker_\Gamma(\{x\})$. □

3 Conclusion

We have given various characterizations for σ-R_0 ideal topological spaces. If we combine these obtained results, the following the corollary is obtained by Theorem 16, Corollary 3, Theorem 17 and Theorem 24.

Corollary 4. *The following statements are equivalent:*

a) $(X, \tau, \mathcal{I})$ is σ-R_0-space.
b) For each $F \in \mathcal{C}\sigma$ and $x \notin F$, there exists $U \in \sigma$ such that $x \notin U$, $F \subseteq [U \cup \Gamma(\{x\})]$ and $[U \cap \Gamma(\{x\})] = \emptyset$.
c) For each $F \in \mathcal{C}\sigma$ and $x \notin F$, $[F \cap \Gamma(\{x\})] = [F \cap Cl_\sigma(\{x\})] \subseteq \Gamma(\{x\})$.
d) For each $x, y \in X$, $Cl_\sigma(\{x\}) = Cl_\sigma(\{y\})$ or $[Cl_\sigma(\{x\}) \cap Cl_\sigma(\{y\})] = [Cl_\sigma(\{y\}) \cap \Gamma(\{x\})] \subseteq \Gamma(\{x\})$.
e) $Cl_\sigma(\{x\}) \neq Cl_\sigma(\{y\})$ implies $Cl_\sigma(\{x\}) \cap Cl_\sigma(\{y\}) = [Cl_\sigma(\{y\}) \cap \Gamma(\{x\})] \subseteq \Gamma(\{x\})$.
f) For any nonempty subset A and $G \in \sigma$ such that $A \cap G \neq \emptyset$, there exists $F \in \mathcal{C}\sigma$ such that $A \cap F \neq \emptyset$ and $[F \setminus \Gamma(\{x\})] \subseteq G$ for every $x \in (A \cap G)$.
g) For each $x \in X$, $[Cl_\sigma(\{x\}) \setminus \Gamma(\{x\})] \subseteq \sigma Ker_\Gamma(\{x\})$.

Moreover, in Theorem 15 and Corollary 2, characterizations of σ-R_0 space are obtained by using some special ideals.

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$\lambda-$Statistical Convergence in G-Metric Spaces

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Abstract: In this paper, we investigate $\lambda-$Statistical convergence in $G-$metric spaces. The $G-$metric function is based on the concept of distance between three points. Considering this new concept of distance, we examined the relationships between GS, GS_λ, GC_1 and GN_λ sequence spaces.

Keywords: G-metric spaces, lambda statistical convergence, statistical convergence.

1 Introduction and Backround

Let E be a subset of natural numbers $\mathbb{N}$ and

$$d(E) := \lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_E(j).$$

$d(E)$ is said to be natural density of E where χ_E is the characteristic function of E. With the help of this definition we can give the definition of statistical convergence.

Definition 1: ([6]) A number sequence (x_n) is statistically convergent to L provided that for every $\varepsilon > 0$, $d(\{k \le n : |x_k - L| \ge \varepsilon\}) = 0$. In this case we write $st - \lim x_k = L$ and usually the set of statistically convergent sequences is denoted by S.

Considering the definition of natural density, this definition can also be expressed as for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |k \le n : |x_k - L| \ge \varepsilon| = 0.$$

Zygmund first mentioned the concept of statistical convergence in her monograph in 1935 in Warsaw ([25]) and it was formally introduced by Fast ([6]) and Steinhaus ([24]), independently. Later on, Schoenberg gave some basic properties of statistical convergence and studied as a summability method ([23]). After the 1950s, studies on the concept of statistical convergence made rapid progress and many studies were conducted on this subject. The most well-known of these areas are number theory by Erdös and Tenenbaum ([5]), measure theory by Miller ([18]), trigonometric series by Zygmund ([25]), summability theory by Freedman and Sember ([7]). Fridy has an important study in which he studied the properties of statistical convergence ([8]) and Maio studied statistical convergence in topological spaces ([17]). This concept was also studied with ideals, weak convergence, modulus functions and $p-$Cesáro convergence ([3], [13], [14], [22]).

The idea to redefine statistical convergence with $\lambda = (\lambda_n)$ sequences where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1$ and $\lambda_1 = 1$ was thought by Mursaleen in 2000 ([19]). He named this new method by $\lambda-$statistical convergence and he denoted the set of all $\lambda-$statistical convergent sequences by S_λ. At the same time, he investigated the relation theorems between $[C, 1]-$summability and $[V, \lambda]-$summability where

$$[C, 1] = \left\{ x = (x_n) : \exists L \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}$$

and

$$[V, \lambda] = \left\{ x = (x_n) : \exists L \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |x_k - L| = 0 \right\}.$$

Throughout the paper, we use Λ for the set of all non-decreasing $\lambda = (\lambda_n)$ sequences satisfying the above conditions and $I_n = [n - \lambda_n + 1, n]$.

Definition 2: ([19]) Let $\lambda \in \Lambda$. The number sequence $x = (x_n)$ is $\lambda-$statistically convergent (or $S_\lambda-$convergent) to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |k \in I_n : |x_k - L| \ge \varepsilon| = 0$$

In this case we write $S_\lambda - \lim x_n = L$ and usually the set of $\lambda-$statistically convergent sequences is denoted by S_λ.

Now let's talk about $G-$metric spaces briefly. Today, due to very large and complex data sets, the definition of the distance function needs to be generalized. For this purpose, An ([2]), Dhage ([4]), Gahler ([12]), Ha ([15]), Khamsi ([16]) and Gaba ([9]) have already studied. They defined $2-$metric spaces, $D-$metric spaces and $G-$metric spaces. Among them, we will be particularly interested in $G-$metric spaces, which allows us to establish many topological properties.

Gahler claimed that a $2-$metric is a generalization of the usual notion of a metric, but different authors proved that there is no relation between these two functions. Further, there is no easy relationship between results obtained in the two metrics.

Definition 3: ([12]) Let X be a nonempty set. A function $d : X \times X \times X \to \mathbb{R}^+$ satisfying the following axioms:

$d1$) For every distinct points $x, y \in X$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

$d2$) If at least two of three points x, y, z are the same then $d(x, y, z) = 0$.

$d3$) $d(x, y, z) = d(x, z, y) = d(y, z, x) = ...$ (symmetry)

$d4$) $d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$ for all $x, y, z, t \in X$
(rectangle inequality)

is called a $2-$metric on X and (X, d) is called a $2-$metric space.

After Gahler's studies, Dhage defined $D-$metric spaces. Subsequently, these works have been the basis for over fourty papers by Dhage and other authors. However, several errors for fundamental topological properties in a $D-$metric space were found ([20], [21]).

Definition 4: ([4]) Let X be a nonempty set. A function $D : X \times X \times X \to \mathbb{R}^+$ satisfying the following axioms:

$D1$) $D(x, y, z) \geq 0$ for all $x, y, z \in X$.

$D2$) $D(x, y, z) \geq 0$ if and only if $x = y = z$.

$D3$) $D(x, y, z) = D(x, z, y) = D(y, z, x) = ...$ (symmetry in all three variables)

$D4$) $D(x, y, z) \leq D(x, y, t) + D(x, t, z) + D(t, y, z)$ for all $x, y, z, t \in X$ (rectangle inequality)

is called a $D-$metric on X and (X, D) is called a $D-$metric space.

All these developments led Mustafa and Sims to the idea of defining a more appropriate generalized metric space and they defined $G-$metric spaces. These properties are satisfied when $G(x, y, z)$ is the perimeter of a triangle with vertices at x, y and z in $\mathbb{R}^2$, further taking a in the interior of the triangle shows that $(G5)$ is best possible. $G-$metric function is a distance function that generalizes the concept of distance between 3 points. ([20])

Definition 5: ([20]) Let X be a nonempty set. The $G : X \times X \times X \to \mathbb{R}^+$ function that provides the following properties is called generalized metric or briefly $G-$metric on X.

$G1$) $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$

$G2$) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$

$G3$) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

$G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables)

$G5$) $G(x, y, z) \leq G(x, a, a)+ G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality)

The pair (X, G) is called by a $G-$metric space.

Example 1: Let $d(x, y, z)$ be the perimeter of the triangle with vertices at $x, y, z \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a $G-$metric space.

Example 2: Let $X = \{x, y\}$ and let $G(x, x, x) = G(y, y, y) = 0$, $G(x, x, y) = 1$, $G(x, y, y) = 2$ and extend G to all of $X \times X \times X$ by symmetry in the variables. Then G is a $G-$metric which is not symmetric.

Example 3: Let (X, d) be a metric space. The function

$$\psi(x, y, z) = \max \{d(x, y), d(y, z), d(x, z)\}$$

is a $G-$metric where $\psi : X \times X \times X \to \mathbb{R}^+$.

Of course, after these definitions, it became inevitable to study convergence types for sequences in these spaces. Abazari defined statistical convergence in $g-$metric spaces and studied some basic properties (In a $g-$metric space, the distance function defined between $n + 1$ points) ([1]) . In this section, we redefine some important definitions that Abazari studied in $g-$metric spaces for $G-$metric spaces.

Definition 6: Let $A \in \mathbb{N}^2$ and $A(n) = \{i_1, i_2 \leq n : (i_1, i_2) \in A\}$ then, $\rho_1(A) := \lim_{n\to\infty} \frac{2}{n^2} |A(n)|$ is called 2-dimensional asymptotic (or natural) density of the set A.

Definition 7: Let (x_i) be a sequence in a $G-$metric space (X, G). For every $\varepsilon > 0$, if

$$\lim_{n\to\infty} \frac{2}{n^2} |\{(i_1, i_2) \in \mathbb{N} : i_1, i_2 \leq n, \ G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}| = 0$$

then, (x_i) statistically converges to x in G. This situation is denoted by $GS - \lim x_i = x$ or $x_i \overset{GS}{\to} x$**.** The set of all statistically convergent sequences in a $G-$metric space is denoted by GS.

The following theorems were proved by abazari in $g-$metric spaces. Since our theorems are special cases of these theorems in $G-$metric spaces, we do not give proofs here.

Theorem 1. *In $G-$metric spaces, every convergent sequence is statistically convergent.*

Theorem 2. *Statistical limit in a $G-$metric space is unique.*

Theorem 3. *In $G-$metric spaces, every statistically convergent sequence has a convergent subsequence.*

2 Main Results

In this section the main definitions and results are introduced and discussed. First of all, we consider the definition of $\lambda-$ statistical convergence in $G-$metric spaces.

Definition 8: Let (X, G) be a $G-$metric space, (x_i) be a sequence in this space and $\lambda \in \Lambda$. The sequence (x_i) is said to be $\lambda-$statistically convergent to x in X provided that for all $\varepsilon > 0$,

$$\lim_{n} \frac{2}{\lambda_n^2} |\{i_1, i_2 \in I_n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}| = 0.$$

We write for this situation $GS_\lambda - \lim x_i = x$ or $x_i \overset{GS_\lambda}{\to} x$**.** The set of all this kind of sequences in X is denoted by GS_λ.

Definition 9: Let (X, G) be a $G-$metric space and (x_i) be a sequence in this space. The sequence (x_i) is said to be GC_1-statistically convergent to x provided that

$$\lim_{n} \frac{2}{n^2} \sum_{i_1, i_2=1}^{n} G(x, x_{i_1}, x_{i_2}) = 0.$$

We write for this situation $GC_1 - \lim x_i = x$ or $x_i \overset{GC_1}{\to} x$**.** The set of all this kind of sequences in X is denoted by GC_1.

Definition 10: Let (X, G) be a $G-$metric space, (x_i) be a sequence in this space and $\lambda \in \Lambda$. The sequence (x_i) is said to be $GN_\lambda-$statistically convergent to x provided that

$$\lim_{n} \frac{2}{\lambda_n^2} \sum_{i_1, i_2 \in I_n} G(x, x_{i_1}, x_{i_2}) = 0.$$

We write for this situation $GN_\lambda - \lim x_i = x$ or $x_i \overset{GN_\lambda}{\to} x$. The set of all this kind of sequences in X is denoted by GN_λ**.**

After all these definitions, we can give the following theorems that prove the relationship between GN_λ and GS_λ and the role of boundedness in this relationship.

Theorem 4. *Let (X, G) be a $G-$metric space and (x_i) be a sequence in this space.*

$$\textit{If } x_i \overset{GN_\lambda}{\to} x \textit{ then } x_i \overset{GS_\lambda}{\to} x.$$

Proof: Suppose that $x_i \overset{GN_\lambda}{\to} x$ and $\varepsilon > 0$ be given. Then,

$$\begin{aligned} \tfrac{2}{\lambda_n^2} \sum_{i_1, i_2 \in I_n} G(x, x_{i_1}, x_{i_2}) &\geq \tfrac{2}{\lambda_n^2} \sum_{\substack{i_1, i_2 \in I_n \\ G(x, x_{i_1}, x_{i_2}) \geq \varepsilon}} G(x, x_{i_1}, x_{i_2}) \\ &\geq \varepsilon \tfrac{2}{\lambda_n^2} |\{i_1, i_2 \in I_n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}| \end{aligned}$$

and if we take the limit of both sides,

$$\frac{1}{\varepsilon} \lim_{n\to\infty} \frac{2}{\lambda_n^2} \sum_{i_1, i_2 \in I_n} G(x, x_{i_1}, x_{i_2}) \geq \lim_{n\to\infty} \frac{2}{\lambda_n^2} |\{i_1, i_2 \in I_n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}|$$

and we have $x_i \overset{GS_\lambda}{\to} x$. □

Theorem 5. *Let (X,G) be a $G-$metric space and G be bounded function in X.*

$$\textit{If } x_i \stackrel{GS_\lambda}{\to} x \textit{ then } x_i \stackrel{GN_\lambda}{\to} x.$$

Proof: This time suppose that $x_i \stackrel{GS_\lambda}{\to} x$ and $\varepsilon > 0$ be given. From the boundedness of G there is a positive M such that $G(x, x_{i_1}, x_{i_2}) \leq M$ for all $x, x_{i_1}, x_{i_2} \in X$. Then,

$$\begin{aligned} \tfrac{2}{\lambda_n^2} \sum_{i_1,i_2 \in I_n} G(x, x_{i_1}, x_{i_2}) &= \tfrac{2}{\lambda_n^2} \sum_{\substack{i_1,i_2 \in I_n \\ G(x,x_{i_1},x_{i_2}) \geq \varepsilon}} G(x, x_{i_1}, x_{i_2}) + \tfrac{2}{\lambda_n^2} \sum_{\substack{i_1,i_2 \in I_n \\ G(x,x_{i_1},x_{i_2}) < \varepsilon}} G(x, x_{i_1}, x_{i_2}) \\ &\leq \tfrac{2}{\lambda_n^2} M \left| \{ i_1, i_2 \leq n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \right| + \varepsilon. \end{aligned}$$

Similarly, considering the limits of both side we have $x_i \stackrel{GN_\lambda}{\to} x$. □

In the following theorem, we explain the relationship between GS and GS_λ.

Theorem 6. *Let (X,G) be a $G-$metric space, (x_i) be a sequence in this space and $\lambda \in \Lambda$.*

$$\textit{If } \liminf_n \frac{\lambda_n^2}{n^2} > 0 \textit{ then } GS \subseteq GS_\lambda.$$

Proof: Suppose that $x_i \stackrel{GS}{\to} x$ and $\liminf_n \frac{\lambda_n^2}{n^2} > 0$. Then there exists $\eta > 0$ such that $\frac{\lambda_n^2}{n^2} \geq \eta$ for sufficiently large n. On the other hand, for given $\varepsilon > 0$ we know that in any case

$$\{ i_1, i_2 \leq n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \supset \{ i_1, i_2 \in I_n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \}.$$

Therefore,

$$\begin{aligned} \tfrac{2}{n^2} \left| \{ i_1, i_2 \leq n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \right| &\geq \tfrac{2}{n^2} \left| \{ i_1, i_2 \in I_n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \right| \\ &\geq \tfrac{\lambda_n^2}{n^2} \tfrac{2}{\lambda_n^2} \left| \{ i_1, i_2 \in I_n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \right|. \\ &\geq \eta \tfrac{2}{\lambda_n^2} \left| \{ i_1, i_2 \in I_n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \right| \end{aligned}$$

We know that the limit of the left side is zero. So, the limit of the right side should be zero and this gives the proof. □

Now, let's prove if $x_i \stackrel{GC_1}{\to} x$ then $x_i \stackrel{GS}{\to} x$ and also if G is bounded then the inverse of the theorem satisfies.

Theorem 7. *Let (X,G) be a $G-$metric space, (x_i) be a sequence in this space.*

$$\textit{If } x_i \stackrel{GC_1}{\to} x \textit{ then } x_i \stackrel{GS}{\to} x.$$

Proof: Let $\varepsilon > 0$ and $x_i \stackrel{GC_1}{\to} x$. Hence,

$$\begin{aligned} \tfrac{2}{n^2} \sum_{i_1,i_2=1}^{n} G(x, x_{i_1}, x_{i_2}) &\geq \tfrac{2}{n^2} \sum_{\substack{i_1,i_2=1 \\ G(x,x_{i_1},x_{i_2}) \geq \varepsilon}}^{n} G(x, x_{i_1}, x_{i_2}) \\ &\geq \varepsilon \tfrac{2}{n^2} \left| \{ i_1, i_2 \leq n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \right| \\ \tfrac{1}{M} \lim_{n\to\infty} \tfrac{2}{n^2} \sum_{i_1,i_2=1}^{n} G(x, x_{i_1}, x_{i_2}) &\geq \lim_{n\to\infty} \tfrac{2}{n^2} \left| \{ i_1, i_2 \leq n : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon \} \right| \end{aligned}$$

Considering that $x_i \stackrel{GC_1}{\to} x$ we have the result. □

Theorem 8. *Let (X,G) be a $G-$metric space, (x_i) be a sequence in this space and G is a bounded function.*

$$\textit{If } x_i \stackrel{GS}{\to} x \textit{ then } x_i \stackrel{GC_1}{\to} x.$$

Proof: This time suppose that $x_i \stackrel{GS}{\to} x$ and $\varepsilon > 0$ is given. From the boundedness of G there is a positive M such that □

$G(x, x_{i_1}, x_{i_2}) \leq M$ for all $x, x_{i_1}, x_{i_2} \in X$. Then,

$$\begin{aligned}
\frac{2}{n^2}\sum_{i_1,i_2=1}^{n} G(x,x_{i_1},x_{i_2}) &= \frac{2}{n^2}\sum_{\substack{i_1,i_2=1\\ G(x,x_{i_1},x_{i_2})\geq\varepsilon}}^{n} G(x,x_{i_1},x_{i_2}) \\
&+ \frac{2}{n^2}\sum_{\substack{i_1,i_2=1\\ G(x,x_{i_1},x_{i_2})<\varepsilon}}^{n} G(x,x_{i_1},x_{i_2}) \\
&\leq M\frac{2}{n^2}\left|\{i_1,i_2\leq n : G(x,x_{i_1},x_{i_2})\geq\varepsilon\}\right| + \varepsilon
\end{aligned}$$

Considering the limits of both side we have $x_i \overset{GC_1}{\to} x$.

3 Conclusion

Defining the $\lambda-$statistical convergence with the help of the G-metric, which defines the concept of distance between three points and handles it from different aspects, will bring an interesting study to the literature.

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Spectral Deferred Correction Time Integration Methods

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Abstract: In this study, the Spectral Deferred Correction (SDC) method developed to solve the initial value problems consisting of ordinary differential equations (ODEs) or the systems of ordinary differential equations with arbitrary order of accuracy has been examined. The essence of the spectral deferred correction method is based on an iterative correction procedure that increases the order of accuracy of a primitive time integration method such as forward or backward Euler methods. The stability properties of the method were examined through test problems that contain non-stiff cases. The stability regions of the method were compared with the same order Runge-Kutta (RK) methods. Our numerical finding indicates better stability behaviour of the SDC method versus the RK methods. In other words, we obtained fairly larger stability regions with the SDC method that is consistent with the literature [1, 2]. We have solved several test problems ranging from scalar to system of differential equations. We performed numerical studies to verify the order of convergence of the numerical scheme for explicit integrations. We managed to verify up to 11th order of accuracy in explicit cases. We also compared our code with MATLAB's built in ODE45 solver that is fifth order accurate. We found that the fifth order explicit SDC method stably runs with relatively larger time steps versus the ODE45 routine. Our numerical results also show that fifth order SDC method possesses smaller errors if we run both codes with the same size of time steps.

Keywords: Deferred correction methods, Explicit and Implicit time integrations, Spectral deferred correction methods, Stiff and non-stiff ODEs.

1 Introduction

Lots of real-life problems are modeled with differential equations. Differential equations form the basis of many physical theories. For this reason, the solution of differential equations or systems of differential equations is very important in terms of understanding and interpreting physical phenomenas. In most cases, it may not be possible to obtain the exact solution of the equations or systems modeling physical problems since these equations or systems are usually non-linear. In such cases, rather than an exact solution, a numerical solution is searched. Many numerical methods have been developed in the literature to obtain these solutions. Runge - Kutta (RK) and Euler methods are the most popular numerical methods [3]. In this study, another numerical approach that is referred to as the Spectral Deferred Correction (SDC) method is investigated. In general, the SDC methods are built on top of a primitive time integration method such as forward or backward Euler methods. The aim is to increase the order of accuracy of the Euler methods with an iterative correction procedure. The essence of SDC methods is based on finding the numerical solution of a new initial value problem created with the equation based on the error function. The first step of the SDC method is called the provisional step in which a set of the numerical solution is obtained by forward or backward Euler methods. The second step of the SDC method is based on an iterative correction procedure that improves the accuracy of the primitive Euler methods. To present the SDC methods consider the following general form of the ODE:

$$\phi'(t) = F(t, \phi(t)), \quad t \in [a, b] \tag{1}$$

$$\phi(a) = \phi_a. \tag{2}$$

The SDC method utilizes the so-called Picard integral equation that corresponds to the equations (1)-(2). Then, the residual function is defined by applying Picard iteration to this integral equation and the error function is defined by the difference between the exact solution and the approximate solution [1, 2]. More mathematical details of the derivation will be presented in Section 2.

In Section 3, stability region analysis of RK methods and SDC methods, are performed. The stability regions of the SDC methods were compared with the same order RK methods. Both RK and SDC methods were built on top of the first order convergent forward Euler methods depending on explicit discretization meaning that both numerical approaches are identical when the order of accuracy is equal to *one*. In addition to this, a meaningful comparison is obtained whenever the order of accuracy is *two* or *higher*. A head-on comparison between from second to fourth order RK and SDC methods was performed. In all cases, the stability properties of the SDC methods are better than the RK methods. In Section 4, several test problems are solved ranging from scalar to system of ordinary differential equations only for non-stiff cases. Using the obtained results, the theoretical order of the method and the numerical order are compared. Also, our code is compared with MATLAB's built-in ODE45 subroutine which is fifth order accurate. In our comparison, we have found that SDC method can easily compete MATLAB's ODE45 solver. We have also observed that the 5th order explicit SDC code can run with relatively larger time steps.

In summary, in this study, the stability, accuracy, and efficiency performance of spectral deferred correction methods for solving initial value problems were examined.

2 Spectral Deferred Correction Methods

The SDC procedure is described here but for details, we refer the reader to [1, 2, 4, 5, 6, 7, 8]. The SDC method is based on the Picard integral form of the equations (1)-(2). To obtain the Picard integral form, we integrate equations (1) and (2) from a to t with respect to t.

$$\phi(t) = \phi_a + \int_a^t F(\tau, \phi(\tau))\, d\tau. \tag{3}$$

For approximate solution $\widetilde{\phi}$, the aim of the SDC strategy is to create a new initial value problem for the error function $\delta(t) = \phi(t) - \widetilde{\phi}(t)$. Also, a residual function based on the integral equation can be written as:

$$E\left(t, \widetilde{\phi}(t)\right) = \phi_a + \int_a^t F(\tau, \widetilde{\phi}(\tau)) d\tau - \widetilde{\phi}(t). \tag{4}$$

Using $\phi(t) = \delta(t) + \widetilde{\phi}(t)$, equation (3) becomes

$$\delta(t) + \widetilde{\phi}(t) = \phi_a + \int_a^t F\left(\tau, \widetilde{\phi}(\tau) + \delta(\tau)\right) d\tau. \tag{5}$$

Then, equation (5) can be combined with (4) to give

$$\delta(t) = \int_a^t \left[F\left(\tau, \widetilde{\phi}(\tau) + \delta(\tau)\right) - F\left(\tau, \widetilde{\phi}(\tau)\right)\right] d\tau + E\left(t, \widetilde{\phi}(t)\right). \tag{6}$$

which is called the *correction equation*. Now, suppose that the interval $[t_n, t_{n+1}]$ is a sub-interval on the interval $[a, b]$. The SDC methods proceed by dividing the interval $[t_n, t_{n+1}]$ into p sub-intervals with points t_m for $m = 0, \ldots, p$ such that

$$t_n = t_0 < t_1 < \ldots < t_p = t_{n+1}. \tag{7}$$

Then an approximate solution $\phi^0(t_m)$ is computed for $m = 0, \ldots, p$ with the forward Euler method for non-stiff equations. Also, correction solutions $\delta^k(t_m)$ are computed with respect to equation (6). Then, to provide an increasingly accurate approximation $\phi^{k+1} = \phi^k + \delta^k$ solutions will be calculated. Here, the function $E\left(t, \widetilde{\phi}(t)\right)$ is calculated with Gaussian quadrature. Our choice of quadrature nodes is the Gauss–Lobatto nodes. Because the Gauss–Lobatto nodes include the endpoints, so that we do not have to do extrapolations of the final solution [4]-[6]. Using $\phi^k(t_m) = \phi_m^k$, $\delta^k(t_m) = \delta_m^k$ and $E(\phi_m^k, t_m) = E_m\left(\phi^k\right)$, a discretization for equation (6) can be written as

$$\delta_{m+1}^k = \delta_m^k + \Delta t_m \left[F(t_m, \phi_m^k + \delta_m^k) - F(t_m, \phi_m^k)\right] + E_{m+1}\left(\phi^k\right) - E_m\left(\phi^k\right). \tag{8}$$

for non-stiff cases. Assuming,

$$I_m^{m+1}\left(\phi^k\right) = \int_{t_m}^{t_{m+1}} F\left(\tau, \phi^k(\tau)\right) d\tau. \tag{9}$$

for residual function the following equation can be obtained

$$E_{m+1}\left(\phi^k\right) - E_m\left(\phi^k\right) = I_m^{m+1}\left(\phi^k\right) - \phi_{m+1}^k + \phi_m^k. \tag{10}$$

Using equation (10), $\phi_m^{k+1} = \phi_m^k + \delta_m^k$ and $\phi_{m+1}^{k+1} = \phi_{m+1}^k + \delta_{m+1}^k$, the equation (8) is rewritten as

$$\phi_{m+1}^{k+1} = \phi_m^{k+1} + \Delta t_m \left[F\left(t_m, \phi_m^{k+1}\right) - F\left(t_m, \phi_m^k\right)\right] + I_m^{m+1}\left(\phi^k\right). \tag{11}$$

This equation is a direct update to the correction solutions for the explicit cases. Each iteration of correction equations raises the order of accuracy by one [1, 2, 4, 5, 6, 7, 8].

3 The Stability Region Analysis of Runge - Kutta Methods and Spectral Deferred Correction Methods

In this section, the stability analysis of both RK methods and SDC methods will be performed. The stability region of a numerical method is defined by the following test problem [2]:

$$\phi'(t) = \lambda\phi, \quad \phi(0) = 1, \lambda < 0 \tag{12}$$

3.1 The Stability Region Analysis of Runge - Kutta Methods

The stability region for the RK methods will be examined with the test problem and second-order RK method. One of the second-order RK methods [9] is described by the following equations:

$$k_1 = F(t_m, \phi(t_m)), \tag{13}$$

$$k_2 = F(t_m + \Delta t_m, \phi(t_m) + \Delta t_m k_1), \tag{14}$$

$$\phi_{m+1} = \phi_m + \Delta t_m \frac{(k_1 + k_2)}{2}. \tag{15}$$

If the equalities $\phi'(t_m, \phi_m) = \lambda\phi_m$ and $\Delta t\lambda = z$ are substituted for the k values, the following equation can be obtained:

$$\phi_{m+1} = \left[1 + z + \frac{z^2}{2}\right]\phi_m \tag{16}$$

The equation $R(z) = 1 + z + \frac{z^2}{2}$ is called the stability function of the second order RK method. The stability region of the method is investigated by plotting the inequality $|1 + z + \frac{z^2}{2}| \leq 1$ on the complex plane [10].

3.2 The Stability Region Analysis of Spectral Deferred Correction Methods

To determine the stability regions of the SDC methods, provisional solutions and correction solutions will be calculated for each order. For the purpose of the meaningful comparison between RK and SDC methods, the stability behavior of the SDC methods is investigated by the second-order method. For explicit schemes, provisional solutions and correction solutions are calculated by the following equations respectively.

$$\phi_{m+1}^{[k]} = \phi_m^{[k]} + \lambda\Delta t_m \phi_m^{[k]}, \tag{17}$$

$$\phi_{m+1}^{[k+1]} = \phi_m^{[k+1]} + \lambda\Delta t_m \left[\phi_m^{[k+1]} - \phi_m^{[k]}\right] + I_m^{m+1}\left(\phi^{[k]}\right). \tag{18}$$

For a second-order method, one iteration of the correction equation is needed [4]-[8]. By using the second-order SDC method for solving the test problem the following equation can be obtained:

$$\phi_2^{[1]} = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{8}z^3\right)\phi_0^{[0]}. \tag{19}$$

The stability function is defined as $R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{8}z^3$ and the stability region of the method will be obtained by plotting the inequality $|1 + z + \frac{1}{2}z^2 + \frac{1}{8}z^3| \leq 1$ on the complex plane. The comparison between RK methods and SDC methods can be observed in the following figure.

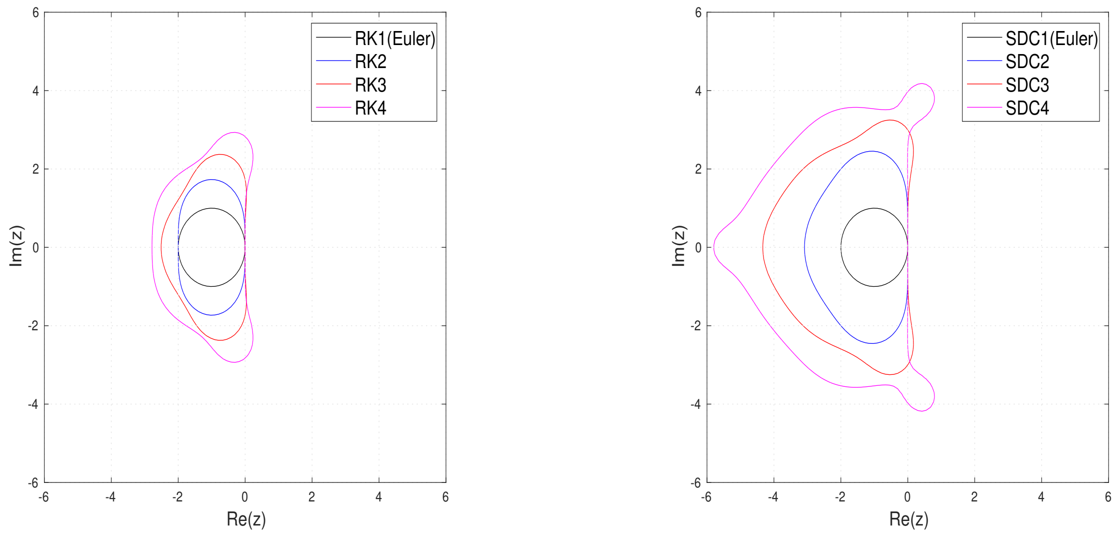

Fig. 1: The Stability Regions of RK and SDC Methods

According to the figure, the first-order stability regions in both methods are equal to each other and the same as the stability region obtained by the explicit Euler method. However, at least the stability regions of the second, third, and fourth-order SDC methods are larger than the RK methods. That means the SDC methods are more stable for the test problem according to the RK methods, that is consistent with the literature.

In the following part of this section, the stability regions of several explicit SDC methods will be given. The stability region analysis was performed using the MATLAB programming language.

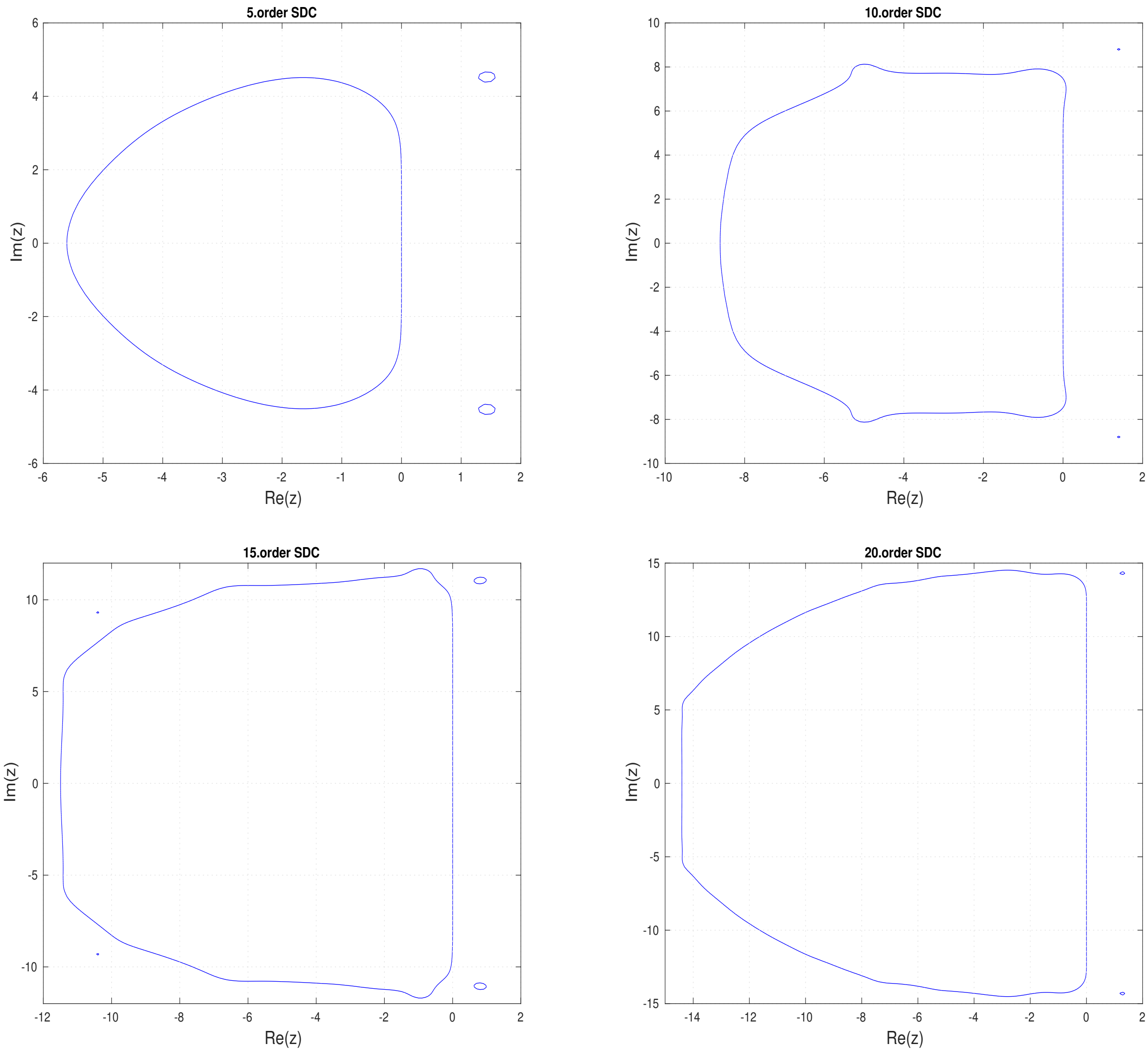

Fig. 2: The Stability Regions of SDC Methods

3.2.1 Stability properties of explicit schemes:

1. It is observed in Fig. 2 that the explicit schemes are stable for $Re(z) < 0$.
2. In all cases, the stability regions are getting larger as the order increases.

4 Numerical Results

In this section, several test problems are solved ranging from scalar to system of ordinary differential equations only for non-stiff cases. Numerical experiments are performed to verify the order of convergence of the proposed scheme for explicit integrations. Using the obtained results, the theoretical order of the method and the numerical order are compared. The code is set to quad precision to achieve meaningful error analysis. As a result, up to 11th order of accuracy is verified. Also, the code is compared with MATLAB's built-in ODE45 subroutine which is fifth order accurate. The solutions obtained with both ODE45 and 5th order SDC methods are superimposed on the same figures.

4.1 Problem 1

The first initial value problem is

$$\phi'(t) = -\lambda\phi(t), \;\; \phi(0) = 1, \; t \in [0,1]. \tag{20}$$

Assuming $\lambda = 1$,

$$\phi'(t) = -\phi(t), \;\; \phi(0) = 1, \; t \in [0,1], \tag{21}$$

the obtained result is investigated with the following table:

Table 1 The Numerical Results of Problem 1.

Theoretical Order	Absolute Error	Numerical Order
1	$1.92010E - 0002$	1.03145
2	$1.52825E - 0004$	1.99671
3	$2.35702E - 0006$	3.00460
4	$1.70700E - 0008$	3.98961
5	$9.70200E - 0011$	4.96877
6	$4.57009E - 0013$	5.95163
7	$1.84064E - 0015$	6.93544
8	$6.47903E - 0018$	7.92010
9	$2.02648E - 0020$	8.90562
10	$5.70431E - 0023$	9.89181

4.2 *Problem 2*

To make the analysis more comprehensive, the SDC methods will be used to solve another initial value problem:

$$\phi'(t) = -2\pi \sin(2\pi t) - \frac{1}{\epsilon}(\phi - \cos(2\pi t)), \phi(0) = 1, \ \ t \in [0, 1] \tag{22}$$

Assuming $\epsilon = 1$, the problem will become

$$\phi'(t) = -2\pi \sin(2\pi t) - (\phi - \cos(2\pi t)), \phi(0) = 1, \ \ t \in [0, 1] \tag{23}$$

The obtaining results are investigated in Table 2:

Table 2 The Numerical Results of Problem 2.

Theoretical Order	Absolute Error	Numerical Order
1	$1.18726E - 0002$	0.10719
2	$5.46855E - 0003$	2.00255
3	$1.64664E - 0006$	2.24394
4	$1.18673E - 0008$	3.22779
5	$6.93893E - 0011$	4.24305
6	$3.38679E - 0013$	5.26596
7	$1.40279E - 0015$	6.28124
8	$5.05402E - 0018$	7.29196
9	$1.61306E - 0020$	8.30005
10	$4.62082E - 0023$	9.30801

Remark 2. As can be seen from the results in Table 1 and Table 2, the numerical order of the SDC methods are capture the theoretical order. But, for Problem 2 the numerical order doesn't capture the theoretical order so well since each error term contains derivatives from the truncation error. When the derivatives of trigonometric functions are considered, the truncation error will begin to become more dominant after a while, since the derivatives will constantly repeat.

Table 3 The 5th Order Results of Problem 2.

Mesh Refinement	Absolute Error	Numerical Order
Δt	$6.93983E - 0011$	
$\Delta t/2$	$3.66437E - 0012$	4.24305
$\Delta t/4$	$1.39442E - 0013$	4.71452
$\Delta t/8$	$4.75979E - 0015$	4.87260
$\Delta t/16$	$1.55129E - 0016$	4.93933

Table 3 shows the fifth order solutions to Problem 2. The numerical order approaches the theoretical order as the time step gets smaller. That means the SDC methods give good results for this problem.

4.3 *Problem 3*

Now, a system of ordinary differential equations that is called the Van Der Pol problem will be solved by the explicit SDC method.

$$\phi_1'' - \mu(1 - \phi_1^2)\phi_1' + \phi_1 = 0, \quad \mu > 0, \quad \phi_1(0) = 2, \phi_2(0) = 0. \tag{24}$$

Assuming $\mu = 1$, this system is non-stiff. By making the substitution $\phi_1' = \phi_2$, the equation can be rewritten as a system of ODEs:

$$\phi_1' = \phi_2, \tag{25}$$

$$\phi_2' = \mu\left(1 - \phi_1^2\right)\phi_2 - \phi_1, \quad \mu > 0, \quad \phi_1(0) = 2, \phi_2(0) = 0. \tag{26}$$

Table 4 shows the fifth order results for Problem 3.

Table 4 The 5th Order Results of Problem 3.

Mesh Refinement	Error	Numerical Order
$\lvert\phi_{2,\Delta t} - \phi_{2,\Delta t/2}\rvert$	$3.90512E-0009$	
$\lvert\phi_{2,\Delta t/2} - \phi_{2,\Delta t/4}\rvert$	$1.31071E-0010$	4.89695
$\lvert\phi_{2,\Delta t/4} - \phi_{2,\Delta t/8}\rvert$	$4.20592E-0012$	4.96178
$\lvert\phi_{2,\Delta t/8} - \phi_{2,\Delta t/16}\rvert$	$1.32921E-0013$	4.98378
$\lvert\phi_{2,\Delta t/16} - \phi_{2,\Delta t/32}\rvert$	$4.17522E-0015$	4.99257

Remark 3. As can be seen from the last column of Table 4, the numerical order successfully captures the theoretical order. In addition, as the time step gets smaller, the numerical order becomes very close to the theoretical order. That means, the method also successfully solve the system of equations of ODEs.

Remark 4. Since the analytical solution of the Van Der Pol problem is unknown, numerical solutions are calculated for different Δt, and obtaining results are compared. Here, to obtain the numerical order ϕ_2 solutions are used.

4.4 Problem 4

Now, we solve the Euler equations for a rigid body without external forces. The rigid body problem is defined by the following equations and initial values:

$$\phi_1' = \phi_2\phi_3, \tag{27}$$

$$\phi_2' = -\phi_1\phi_3, \tag{28}$$

$$\phi_3' = -0.51\phi_1\phi_2, \quad \phi_1(0) = 0,\ \phi_2(0) = 1, \phi_3(0) = 1. \tag{29}$$

Table 5 The 5th Order Results of Problem 4.

Mesh Refinement	Error	Numerical Order
$\lvert\phi_{1,\Delta t} - \phi_{1,\Delta t/2}\rvert$	$3.97359E-0010$	
$\lvert\phi_{1,\Delta t/2} - \phi_{1,\Delta t/4}\rvert$	$1.24977E-0011$	4.99071
$\lvert\phi_{1,\Delta t/4} - \phi_{1,\Delta t/8}\rvert$	$3.92095E-0013$	4.99431
$\lvert\phi_{1,\Delta t/8} - \phi_{1,\Delta t/16}\rvert$	$1.22794E-0014$	4.99689
$\lvert\phi_{1,\Delta t/16} - \phi_{1,\Delta t/32}\rvert$	$3.84163E-0016$	4.99838

Remark 5. When the results given in Table 5 are investigated, the numerical order again captures the theoretical order. In other words, the method is also successful in solving this system.

4.5 Graphs for The Problems

Now, the solutions to the Van Der Pol and the rigid body problem will be examined on graphs. The solutions will be performed with MATLAB ODE45 subroutine and the 5th order SDC method. Considering the information in the literature, ODE45 is fourth or fifth order accurate [11]. In order to be consistent, comparisons will be performed with the 5th order SDC method.

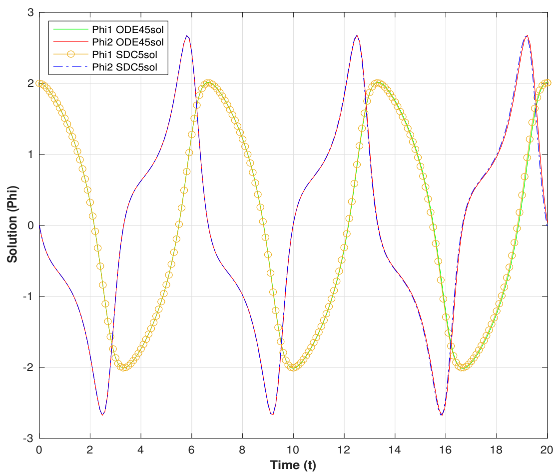

Fig. 3: The Solution of Van Der Pol Problem with ODE45 and SDC5

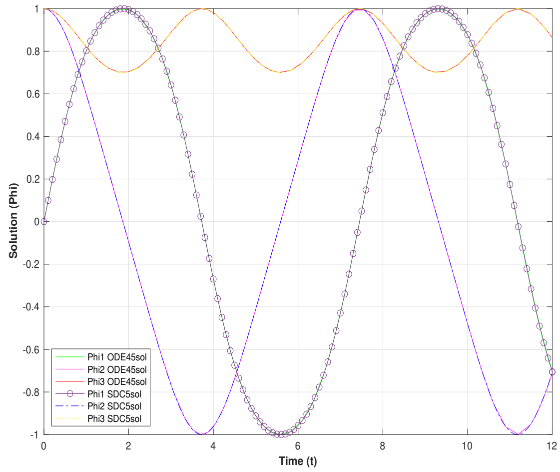

Fig. 4: The Solution of Rigid Body Problem with ODE45 and SDC5

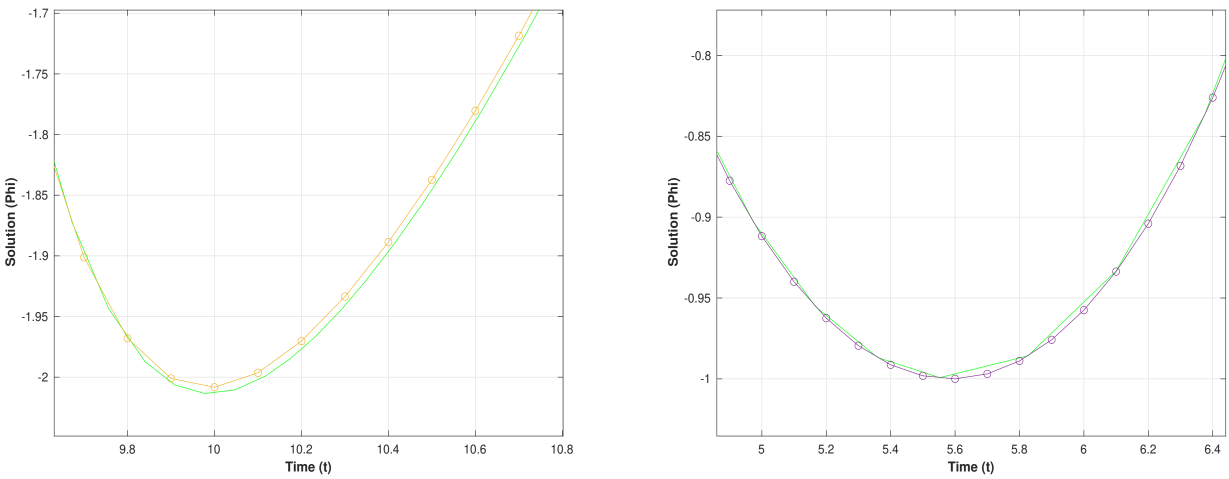

Fig. 5: Zoom of ϕ_1 Solutions to the Problems

Remark 6. Fig 3 and Fig 4 show that there is no visible difference between the solutions for both methods. This is because the 5th order error is very small. If the graphs are zoomed in at the inflection points the difference between the results can be seen in Fig 5. However, we can not decide which method is more successful since the exact solution is not known. Now, to understand which method can provide a better approximate solution, the solutions to Problem 1 and 2 will be performed.

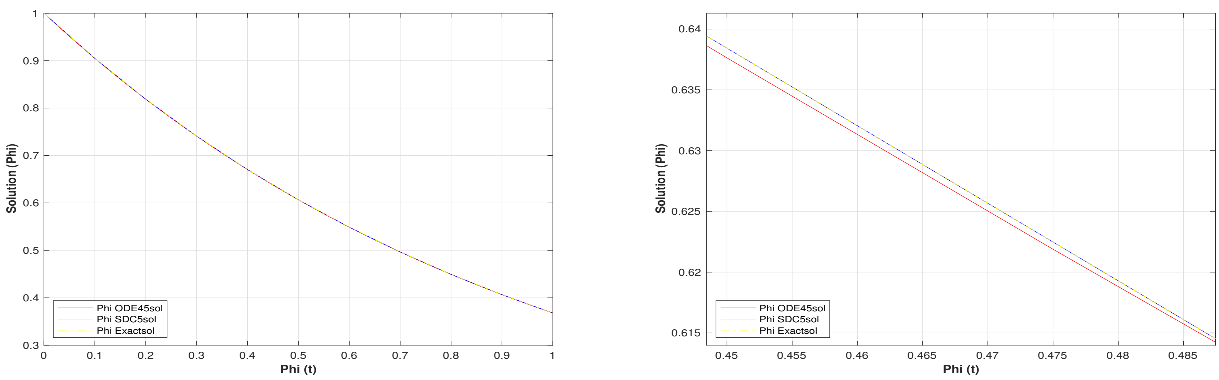

Fig. 6: The Solution of Problem 1 with ODE45 and SDC5

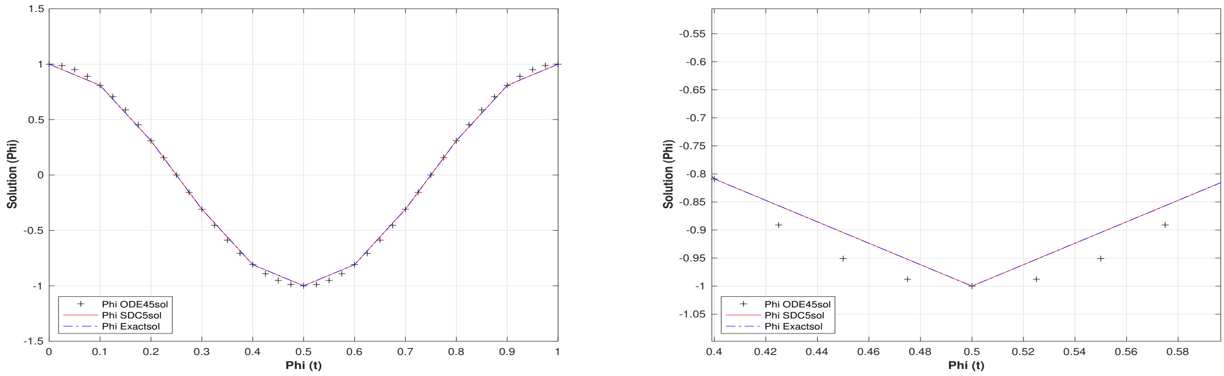

Fig. 7: The Solution of Problem 2 with ODE45 and SDC5

As can be seen from Fig 6 and Fig 7, both the ODE45 and 5th order SDC methods give good results. In addition, the solutions of both methods are very close to each other. However, the SDC method gives better approximate solutions.

5 Conclusion

In this study, the SDC methods for the solution of initial value problems coming from ordinary differential equations or systems are examined. The linear stability analysis of the method is performed. We have found that the stability regions of the presented numerical scheme are larger than the well-known RK methods. Several test problems have been solved ranging from non-stiff scalar to system of equations with explicit SDC methods. Our numerical findings indicate that SDC methods performed very well consistent with literature. In other words, we were able to to verify the theoretical accuracy of the numerical scheme with relatively larger time step compare to classical RK methods. This finding is also consistent with the SDC method having larger stability regions versus the RK methods.

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A Study on Kantorovich Type Operator Involving Adjoint Euler Polynomials

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Abstract: In this study, we give a new kind Kantorovich type operator with related to Euler type polynomials by using the its generating functions. For investigating approximation properties of our operator, we give moments and central moments functions. We obtain a uniformly convergence theorem for our operator with the help of moments and KorovkinâĂŹs theorem. We show rate of convergence of our operator by using modulus of continuity notation.

Keywords: Euler polynomials, Generating functions, Korovkin's theorem.

1 Introduction

Special polynomial families have many applications in many branches of mathematics such as analytical number theory, combinatoric, CAGD and so on. One of the special polynomials is Euler polynomials. Many researchers have introduced useful and important studies on Euler type polynomials. Ozden and Simsek constructed a new type generating function of (h, q)- Euler numbers and polynomials and derived identities and relations for these polynomials [1]. Simsek investigated a new generating function for q-Eulerian type polynomials and numbers at nonnegative real parameters and found some identities and results [2]. Kilar and Simsek derived some new identities and relations for the cosine-Euler and the sine-Euler polynomials by the aid of generating functions [3].

In approximation theory, many positive linear operators are defined with the help of special polynomials. Varma *et al* obtained a linear positive of generalization of SzÃąsz operators including the Brenke type polynomials as follows:

$$L_n(f,x)=\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right),$$

where A and B are analytic functions which a part of generating function of Brenke type polynomials [4].

YÄślmaz presented a linear positive operator involving the generating function of Apostol-Genocchi polynomials of order alpha and investigated its approximation properties as follows:

$$A_n^{(\alpha,\beta,m)}(f,x)=e^{-(n+\mu)x}\left(\frac{2}{\beta e+1}\right)^{-\alpha}\sum_{k=0}^{\infty}\frac{\mathcal{G}_k^{\alpha}((n+\mu)x,\beta)}{k!}f\left(\frac{k+m}{n+\mu}\right),$$

where $\mathcal{G}_k^{\alpha}(x,\beta)$ is called as Apostol-Genocchi polynomials [5].

Natalini and Ricci introduced to the adjunction property for Appell polynomials and applied to special Appell type polynomials family such as Appell-Euler polynomials. The adjoint-Euler polynomials are defined by the aid of generating function at the following equation:

$$\sum_{k=0}^{\infty}\tilde{\varepsilon}_k(x)\frac{t^k}{k!}=\frac{e^t+1}{2}e^{xt}. \tag{1}$$

Our operator is given by using generating function of adjoint-Euler polynomials for $t=1$ at the following equation:

$$A_n^*(f,x)=n\left(\frac{2}{e+1}\right)e^{-nx}\sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)dt \tag{2}$$

In this part, firstly, we give moments and central moments functions for constructing the Korovkin's theorem for $A_n^*(f,x)$.And then, we obtain convergence properties of $A_n^*(f,x)$ by the aid of Korovkin's theorem and modulus of continuity.

2 Main Results

Lemma1 For all $x \in [0, \infty)$, Eq.(2) satisfies at the following equalities:

$$A_n^*(1, x) = 1 \tag{3}$$

$$A_n^*(s, x) = x + \frac{3e+1}{2n(e+1)} \tag{4}$$

$$A_n^*(s^2, x) = x^2 + \left(\frac{2e}{e+1}\right)\frac{x}{n} + \left(\frac{7e+1}{3e+3}\right)\frac{1}{n^2}. \tag{5}$$

Proof: Using the generating fuction of Adjoint-Euler polynomials given by (1), we have

$$\sum_{k=0}^{\infty} \tilde{\varepsilon}_k(x)\frac{t^k}{k!} = \frac{e+1}{2}e^{xt}.$$

$$\sum_{k=0}^{\infty} \tilde{\varepsilon}_k(x)\frac{kt^k}{k!} = \frac{1}{2}e^{xt}\left[e^t + x(e^t+1)\right].$$

$$\sum_{k=0}^{\infty} \tilde{\varepsilon}_k(x)\frac{k(k-1)t^k}{k!} = \frac{1}{2}e^{tx}\left(e^t(x+1)^2 + x^2\right),$$

where $t = 1$ and $x \to nx$

From definition of $A_n^*(f, x)$, it is easy to see

For $f = 1$,

$$\begin{aligned} A_n^*(1,x) &= n\left(\frac{2}{e+1}\right)e^{-nx}\sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt \\ &= n\left(\frac{2}{e+1}\right)e^{-nx}\frac{e+1}{2}e^{nx}\left(\frac{k+1}{n} - \frac{k}{n}\right) \\ &= 1 \end{aligned}$$

For $f = t$,

$$\begin{aligned} A_n^*(1,x) &= n\left(\frac{2}{e+1}\right)e^{-nx}\sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} tdt \\ &= n\left(\frac{2}{e+1}\right)e^{-nx}\sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!}\left((\frac{k+1}{n})^2 - (\frac{k}{n})^2\right) \\ &= n\left(\frac{2}{e+1}\right)e^{-nx}\left(\sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!}(\frac{k+1}{n})^2 - \sum_{k=0}^{\infty}\frac{\tilde{\varepsilon}_k(nx)}{k!}(\frac{k}{n})^2\right) \\ &= x + \frac{3e+1}{2n(e+1)} \end{aligned}$$

We complete the proof by using the same method for $f = t^2$. □

Lemma2 For all $x \in [0, \infty)$, the operator satisfies at the following equalities:

$$A_n^*((s-x), x) = \frac{3e+1}{2n(e+1)}, \tag{6}$$

$$A_n^*((s-x)^2, x) = \frac{5e+1}{(e+1)n}x + \frac{7e+1}{(3e+31)n^2}. \tag{7}$$

Proof: By applying the linearity property of A_n^*, we have

$$\begin{aligned} A_n^*((s-x),x) &= A_n^*(s,x) - xA_n^*(1,x) \\ &= x + \frac{3e+1}{2n(e+1)} - x \\ &= \frac{3e+1}{2n(e+1)}. \end{aligned}$$

$$\begin{aligned} A_n^*((s-x)^2,x) &= A_n^*(s^2,x) - 2xA_n^*(s,x) + x^2A_n^*(1,x) \\ &= x^2 + \left(\frac{2e}{e+1}\right)\frac{x}{n} + \left(\frac{7e+1}{3e+3}\right)\frac{1}{n^2} - 2x\left(x + \frac{3e+1}{2n(e+1)}\right) + x^2 \\ &= \frac{5e+1}{(e+1)n}x + \frac{7e+1}{(3e+31)n^2}. \end{aligned}$$

□

Now we define a set of investigating convergence properties for $A_n^*(f,x)$. Let the set E is defined as follows:

$$E = \{f \mid x \in [0,\infty), \lim_{x\to-\infty} \frac{f(x)}{1+x^2} \; exist\}.$$

Theorem 1. *Let $f \in [0,\infty] = C[0,\infty] \cap E$.*

$$\lim_{n\to\infty} \|E_n^* f - f\| = 0. \tag{8}$$

Proof: By applying the well-known KorovkinâĂŹs first theorem and lemma 1, we obtain

$$\lim_{n\to\infty} E_n^*(t^i,x) = x^i, \tag{9}$$

for $i = 0,1,2$.

The operator $A_n^*(f,x)$ converges uniformly in each compact subset of $[0,\infty]$. The proof is completed by using the property (vii) of Theorem 4.1.4 in [6] . □

Let f be uniformly continuous function on $[0,\infty)$ and $\delta > 0$. The modulus of continuity $\omega(f,\delta)$ of the function f is defined as follows:

$$\omega(f,\delta) := sup|f(x) - f(y)| \tag{10}$$

where $x,y \in [0,\infty)$ and $|x-y| \le \delta$.

Then, for any $\delta > 0$ and each $x \in [0,\infty)$ the following relation holds:

$$|f(x) - f(y)| = \omega(f,\delta_n)\left(\frac{|x-y|}{\delta} + 1; x\right). \tag{11}$$

Theorem 2. *Let f is uniformly continuous function on $[0,1)$ and also belongs to set E. Then, we have*

$$\left|A_n^*(f;x) - f\right| \le 2\omega\left(f; \sqrt{A_n^*\left((s-x)^2;x\right)}\right), \tag{12}$$

where ω is the modulus of continuity of the function f.

Proof: It follows from Lemma 2 and monotonicity of $A_n^*(f;x)$ that

$$\left|A_n^*(f;x) - f(x)\right| \le A_n^*(|f(s) - f(x)|;x). \tag{13}$$

Using the definition of modulus of continuity, we obtain at the following inequality from $A_n^*((s-x)^2,x)$

$$\left|A_n^*(f;x) - f(x)\right| \le \omega(f,\delta)\left(1 + \frac{1}{\delta}A_n^*(|x-y|;x)\right). \tag{14}$$

Applying the Cauchy-Schwarz inequality to the right side of $A_n^*((s-x)^2,x)$, we get

$$\left|A_n^*(f;x) - f(x)\right| \le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{A_n^*\left((x-y)^2;x\right)}\right). \tag{15}$$

By choosing $\delta := \delta_n(x) = \sqrt{\frac{5e+1}{(e+1)n}x + \frac{7e+31}{(3e+31)n^2}}$ in (11), the proof is completed. □

3 Conclusion

In this study, the applications of Euler type polynomials, which have many applications in analytical number theory and combinatorics with the help of operators, are also examined in approximation theory. We gave a Kantorovich type operators by means of generating function of Euler type polynomials. We obtained moment and central moment functions. And also, we investigated convergence properties of such as uniform convergence and rate of convergence with the aid of Korovkin theorem and modulus of continuity.

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An optimal control strategy to prevent the spread of COVID-19

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Abstract: In this study, control strategies to prevent the spread of the COVID-19 virus created with the SEIR model are discussed. For this, a third control term, meaning mask, social distancing, and hygiene, is adapted to an available optimal control strategy consisting of the combination of vaccination and plasma transfusion (treatment). The aim is to reduce the number of exposed and infected individuals. Therefore, optimality systems are firstly introduced for optimal control of the spread of the virus. Afterward, optimal solutions are numerically obtained by applying the forward-backward fourth-order Runge-Kutta method. The numerical results drawn by MATLAB demonstrate that the triple control strategy could be even more effective in controlling the pandemic.

Keywords: COVID-19, Optimal control, SEIR Model, Fourth-order Runge-Kutta method,

1 Introduction

The World Health Organization announced that a new coronavirus (2019-nCoV), now known as COVID-19, was detected in Wuhan, China at the end of December 2019. Public health was under global threat over how many people were infected and susceptible to the disease. Therefore, the spread and dynamics of the coronavirus had to be predicted urgently. Since mathematical modeling is an efficient tool utilized to predict the dynamics of many infectious diseases in the literature, researchers began immediately to develop a mathematical model of the coronavirus. The developed model of the dynamics of disease spread have been based on various models (SIR, SEIR, etc.) [1–3].

On top of that, the researchers also suggested methods such as virus immunity, quarantine, smart health care, and vaccination to prevent the spread of the virus [5–8]. One of these methods is optimal control theory. Optimal control is the procedure of determining a control function which optimizes an objective functional that provides specific dynamic constraints [9]. In recent years, investigations have shown that optimal control is an effective way to control diseases such as tuberculosis, malaria, and ebola [10–12]. Couras et al. [13] have lately proposed an optimal control problem, whose dynamic is modeled with SEIR, to prevent the spread of the coronavirus. As a result, they have shown that optimal control functions representing vaccination and plasma transfusion can be effective in preventing the spread of the virus.

As is known, some variants of coronavirus were emerged which are more contagious than each other in the process of disease. To cope with these variants, some preventive measures such as increasing social distance and spending less time indoors have been taken. Until a vaccine or a treatment are found, one of the effective method to control the disease is the measures taken personally or socially. Therefore, the purpose of this work is to mathematically examine the impact of the measures on the spread of the coronavirus. To achieve the purpose, we adapt a third control term, which means wearing masks, social distance, and hygiene, to the optimal control strategy discussed by Couras et al. [13]. Thus, we aim to ensure that the rate of infected individuals is at the minimum while the preventing coronavirus cost is minimum.

1.1 *The model formulation for the Spread of the COVID-19*

The dynamic system proposed by Couras et al. is as follows:

$$\begin{cases} \frac{dS(t)}{dt} = -\beta S(t) I(t) - u_1(t) S(t), \\ \frac{dE(t)}{dt} = \beta S(t) I(t) - \gamma E(t), \\ \frac{dI(t)}{dt} = \gamma E(t) - u_2(t) R(t) I(t) - \mu I(t), \\ \frac{dR(t)}{dt} = \mu I(t) + u_2(t) R(t) I(t) + u_1(t) S(t), \end{cases} . \tag{1}$$

in which, β is the rate of exposed individuals, γ is the rate of infected individuals, and μ is the rate of natural recovery of infected individuals. Control function $u_1(t)$ vaccinates susceptible individuals, and the control function $u_2(t)$ recovers infected individuals by plasma transfusion.

2 A strategy to prevent the spread of the COVID-19

We improve System (1) by adapting a third control term as follows:

$$\begin{cases} \frac{dS(t)}{dt} = -(1-u_3(t))\,\beta S(t)\,I(t) - u_1(t)\,S(t), \\ \frac{dE(t)}{dt} = (1-u_3(t))\,\beta S(t)\,I(t) - \gamma E(t), \\ \frac{dI(t)}{dt} = \gamma E(t) - u_2(t)\,R(t)\,I(t) - \mu I(t), \\ \frac{dR(t)}{dt} = \mu I(t) + u_2(t)\,R(t)\,I(t) + u_1(t)\,S(t), \end{cases} \tag{2}$$

where $u_3(t)$ is the control term that represents precaution (mask, social distance, and hygiene). Let the control strategy be applied to System (1) during the time $[0,T]$, and the set of admissible control functions $u_1(t)$ vaccination, $u_2(t)$ plasma transfusion and $u_3(t)$ precaution is given as:

$$U_{ad} = \{(u_1,u_2,u_3)\,|\, 0 \le u_1 \le 0.5,\ 0 \le u_2,u_3 \le 0.3,\ 0 \le t \le T\}. \tag{3}$$

Adapting the control term $u_3(t)$, it is aimed to minimize both the rate of infected individuals and the cost of the precautions required. We suggest an optimal control strategy by balancing these two factors.

3 Optimal control problem

In this section, we will introduce the optimal control problem. The aim of maximizing the following objective functional

$$\underset{\max}{J(u_1,u_2,u_3)} = \int_0^T \left(-S(t) - E(t) - I(t) - u_1^2(t) - u_2^2(t) - u_3^2(t)\right) dt \tag{4}$$

is to find the control functions that will minimize the rate of susceptible, exposed, and infected individuals and the cost of vaccination, plasma transfusion, and precaution, respectively.

In order to obtain the optimal solution, we first find Lagrangian and Hamiltonian for optimal control problem (2) and (4). The Lagrangian function of the problem is

$$L(S,E,I,u_1,u_2,u_3) = -S(t) - E(t) - I(t) - u_1^2(t) - u_2^2(t) - u_3^2(t) \tag{5}$$

To solve the problem, we need to find the minimum value of Lagrangian. For this purpose, Hamiltonian formulation H^* at $(t,S,E,I,R,u_1,u_2,u_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is defined as:

$$\begin{aligned} \mathcal{H} =\ & \mathcal{L}(S,E,I,u_1,u_2,u_3) + \lambda_1(t)\,\dot{S}(t) + \lambda_2(t)\,\dot{E}(t) + \lambda_3(t)\,\dot{I}(t) + \lambda_4(t)\,\dot{R}(t), \\ \mathcal{H} =\ & -S(t) - E(t) - I(t) - u_1^2(t) - u_2^2(t) - u_3^2(t) \\ & +\lambda_1(t)\left(-(1-u_3(t))\,\beta S(t)\,I(t) - u_1(t)\,S(t)\right) \\ & +\lambda_2(t)\left((1-u_3(t))\,\beta S(t)\,I(t) - \gamma E(t)\right) \\ & +\lambda_3(t)\left(\gamma E(t) - u_2(t)\,R(t)\,I(t) - \mu I(t)\right) \\ & +\lambda_4(t)\left(\mu I(t) + u_2(t)\,R(t)\,I(t) + u_1(t)\,S(t)\right). \end{aligned} \tag{6}$$

Now, the Pontryagin Maximum Principle is used to obtain the necessary optimality conditions.

3.1 Optimality systems

Theorem 1. *Let $u_i^* \in U_{ad}$, $(i=1,2,3)$ be the optimal controls that maximize the objective functional (4) and (S^*,E^*,I^*,R^*) is the optimal state solution for System (2). Hence, there are costate variables $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ that ensure*

$$\begin{cases} \dot{\lambda}_1(t) = 1 + \lambda_1(t)\left((1-u_3(t))\,\beta I(t) + u_1(t)\right) - \lambda_2(t)\,(1-u_3(t))\,\beta I(t) - \lambda_4(t)\,u_1(t), \\ \dot{\lambda}_2(t) = 1 + \lambda_2(t)\,\gamma - \lambda_3(t)\,\gamma, \\ \dot{\lambda}_3(t) = 1 + \lambda_1(t)\,(1-u_3(t))\,\beta S(t) - \lambda_2(t)\,(1-u_3(t))\,\beta S(t) + \lambda_3(t)\,(u_2(t)\,R(t) + \mu) \\ \qquad - \lambda_4(t)\,(u_2(t)\,R(t) + \mu), \\ \dot{\lambda}_4(t) = \lambda_3(t)\,u_2(t)\,I(t) - \lambda_4(t)\,u_2(t)\,I(t), \end{cases}$$

with transversality conditions $\lambda_i(T) = 0$ $(i=1,2,3,4)$. Besides, the optimal controls u_i^* $(i=1,2,3)$ are revealed by means

$$\begin{cases} u_1(t) = \max\left\{\min\left\{\frac{(-\lambda_1(t)+\lambda_4(t))S(t)}{2}, 0.5\right\}, 0\right\}, \\ u_2(t) = \max\left\{\min\left\{\frac{(-\lambda_3(t)+\lambda_4(t))I(t)R(t)}{2}, 0.3\right\}, 0\right\}, \\ u_3(t) = \max\left\{\min\left\{\frac{(\lambda_1(t)-\lambda_2(t))\beta S(t)I(t)}{2}, 0.5\right\}, 0\right\} \end{cases}$$

Proof: For the existence of optimal control, the Lipschitz condition is satisfied for the state variables S, E, I, and R of System (2). Thus, the existence of control (u_1, u_2, u_3) is deduced [14, 15]. Then, so as to solve the optimal control problem, the optimality system needs first to be put forth. For this, by means of Hamiltonian formula given by equation (6), the necessary optimality conditions (the Euler-Lagrange equations) are

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial \lambda}(t, x(t), u(t), \lambda(t)) \text{ (state system)}, \\ \dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \lambda(t)) \text{ (costate system)}, \\ \frac{\partial H}{\partial u}(t, x(t), u(t), \lambda(t)) = 0 \text{ (control system)}. \end{cases}$$

We obtain the costate and control system by applying the necessary optimality conditions to Hamiltonian formulation as follows:

$$\begin{cases} \dot{\lambda}_1(t) = 1 + \lambda_1(t)\left((1 - u_3^*(t))\beta I^*(t) + u_1^*(t)\right) - \lambda_2(t)(1 - u_3^*(t))\beta I^*(t) \\ \qquad - \lambda_4(t) u_1^*(t), \\ \dot{\lambda}_2(t) = 1 + \lambda_2(t)\gamma - \lambda_3(t)\gamma, \\ \dot{\lambda}_3(t) = 1 + \lambda_1(t)(1 - u_3^*(t))\beta S^*(t) - \lambda_2(t)(1 - u_3^*(t))\beta S^*(t) \\ \qquad + \lambda_3(t)(u_2(t) R^*(t) + \mu) - \lambda_4(t)(u_2^*(t) R^*(t) + \mu), \\ \dot{\lambda}_4(t) = \lambda_3(t) u_2^*(t) I^*(t) - \lambda_4(t) u_2^*(t) I^*(t), \end{cases}$$

with transversality conditions $\lambda_i(T) = 0$ $(i = 1, 2, 3, 4)$,

$$\begin{cases} \frac{\partial H}{\partial u_1}\Big|_{u_1 = u_1^*(t)} = -2u_1^*(t) + (-\lambda_1(t) + \lambda_4(t)) S^*(t) = 0, \\ \frac{\partial H}{\partial u_2}\Big|_{u_2 = u_2^*(t)} = -2u_2^*(t) + (-\lambda_3(t) + \lambda_4(t)) I^*(t) R^*(t) = 0, \\ \frac{\partial H}{\partial u_3}\Big|_{u_3 = u_3^*(t)} = -2u_3^*(t) + (-\lambda_1(t) - \lambda_2(t)) \beta S^*(t) I^*(t) = 0, \end{cases}$$

Using the Pontryagin maximum principle, considering the upper and lower boundaries supplied in the control set, it acquires the optimal control values

$$\begin{cases} u_1^*(t) = \max\left\{\min\left\{\frac{(-\lambda_1(t) + \lambda_4(t)) S^*(t)}{2}, 0.5\right\}, 0\right\}, \\ u_2^*(t) = \max\left\{\min\left\{\frac{(-\lambda_3(t) + \lambda_4(t)) I^*(t) R^*(t)}{2}, 0.3\right\}, 0\right\}, \\ u_3^*(t) = \max\left\{\min\left\{\frac{(\lambda_1(t) - \lambda_2(t)) \beta S^*(t) I^*(t)}{2}, 0.3\right\}, 0\right\}. \end{cases}$$

□

4 Numerical Result

This section will present the non-linear state, costate, and control systems. For this, we solved the optimality system using the fourth-order Runge-Kutta numerical scheme combined with the forward-backward method. While the state system is solved forward, the costate system is solved backward. The control system is updated in a loop.

Using the parameter values $\beta = 0.3$, $\gamma = 0.1887$, $\mu = 0.1$, given by Couras et al. [13], initial conditions $S(0) = 0.88$, $E(0) = 0.07$, $I(0) = 0.05$, and $R(0) = 0$, and transversality conditions $\lambda_i(T) = 0$ $(i = 1, 2, 3, 4)$ with final time value chosen as time T = 20, numerical simulations are obtained with the help of MATLAB.

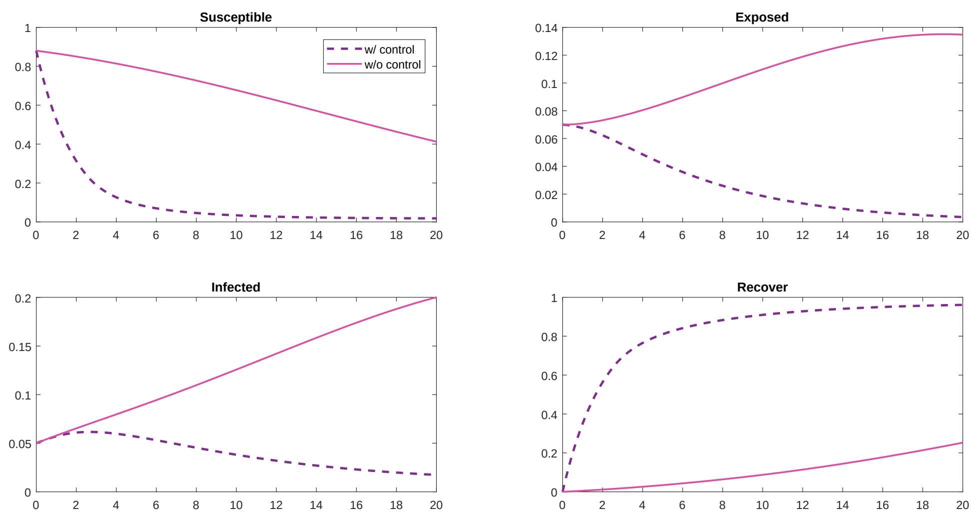

Fig. 1: The effect of precaution and vaccination of susceptible individuals and also implementing plasma transfusions to infected individuals

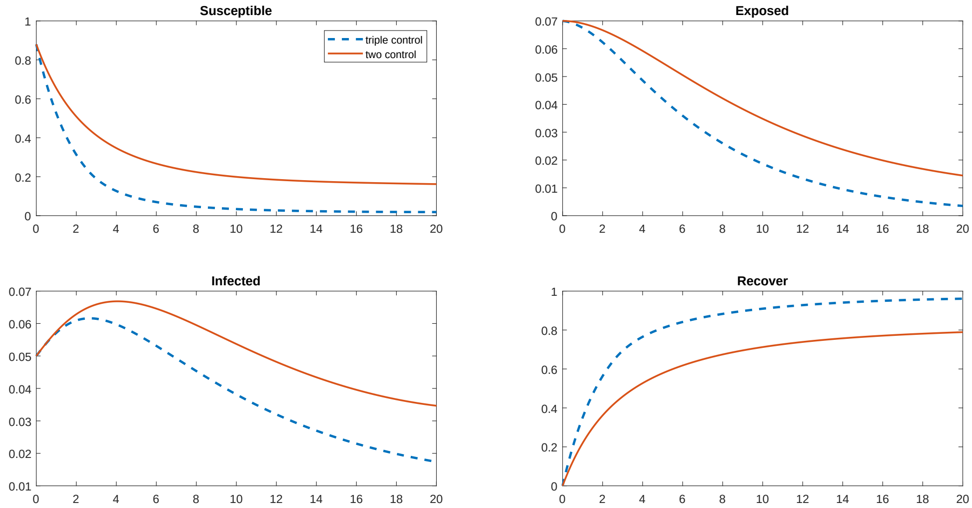

Fig. 2: Comparison of triple and double control strategies

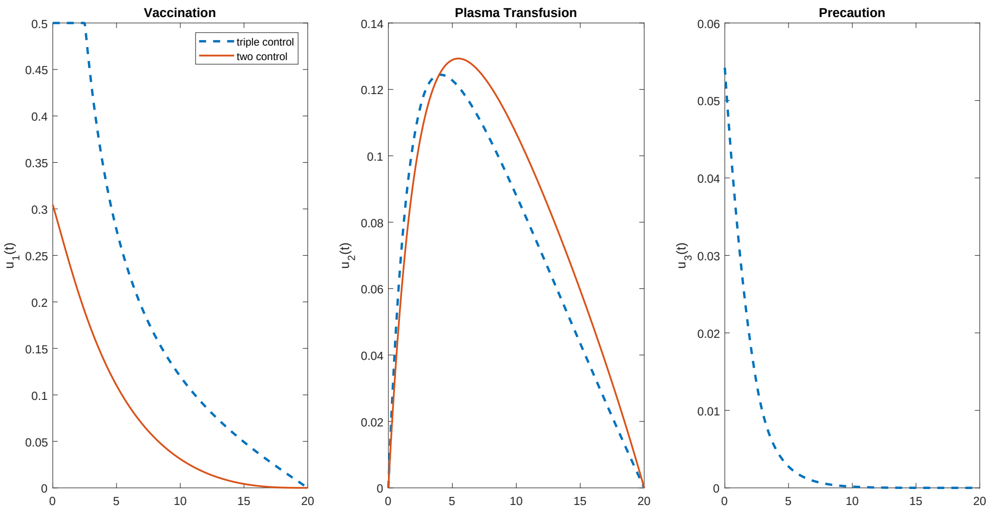

Fig. 3: The situation of vaccination, plasma transfusion, and prevention controls according to strategies

Figure 1 reveals that the applied strategy reduces the rate of susceptible, exposed, and infected individuals and increases the rate of individuals becoming immune as compared with and without control cases. Figure 2 illustrates the comparative of control strategies, and the triple control strategy appears to yield more effective results for individuals in the population. In other words, the rate of infected and exposed individuals is lower. Vaccination was applied at a higher rate and for a longer period of time in order to make society immune to the disease. As a result, susceptible individuals have become immune to the disease without losing their health. Figure 3 contrasts the control functions of the two strategies. As the rate of individuals infected with the precaution decreases, the plasma transfusion used in their treatment implement earlier and at a lower rate. Besides, in the triple control strategy, vaccination continued for a longer period of time than in the two control strategy. Although this seems like a disadvantage in terms of cost and effort, with the help of balancing, individuals have developed immunity without losing their health.

5 Conclusion

By the strategy we have proposed, the rate of exposed and infected individuals diminishes which indicates the significance of precautions in preventing the spread of the virus. As a result, plasma transfusion therapy is performed earlier and at a lower rate during the pandemic. Furthermore, since the rate of infected individuals is less, vaccination for community immunity continues for a long period of time. In addition, the rate of susceptible individuals is less and the rate of individuals recovery is high. It is better for public health for individuals to acquire immunity through vaccination rather than by being infected. In conclusion, numerical simulations indicate that the triple control strategy can be an even more robust method of controlling the pandemic.

Acknowledgement

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Cat^1-group-groupoids and double group-groupoids

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Abstract: Cat^1-groups are introduced as an algebraic model for homotopy 2-types and it is proved that there is a categorical equivalence between cat^1-groups and crossed modules over groups which are also called as 2-dimensional groups. Crossed modules, and hence cat^1-groups, play a crucial role in many areas of mathematics. After the introducing of cat^1-groups, cat^1-objects are investigated in other algebraic categories such as rings, Lie, Leibniz and commutative algebras etc. In this study we describe 2-dimensional version of cat^1-groups namely cat^1-group-groupoids and obtain fruitful properties about this kind of algebraic objects. Moreover, an equivalence is noted between the category of cat^1-group-groupoids and that of double group-groupoids.

Keywords: Cat^1-object, double group-groupoid, group-groupoid.

1 Introduction

In this section we recall the definitions and properties of cat^1–groups, of group-groupoids and of double group-groupoids.

1.1 Cat^1–groups

Definition 1. *A cat^1–group (or 1-cat-group) consists of a group G and two endomorphisms $s, t\colon G \to G$ such that*

1. $st = t$, $ts = s$ *and*
2. $[\ker s, \ker t] = 0$.

See [2] and [4] for further studies on cat^1–groups.

Remark 1. *One can see that for a cat^1–group $ss = s$, $tt = t$ and $\operatorname{Im} s = \operatorname{Im} t$. Therefore we will denote the image of G under s (or under t) by R_G. A cat^1–group will be denoted by (G, R_G, s, t) or briefly by (G, s, t) when no confusion arise.*

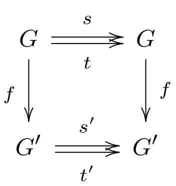
Example 1. *Any group G can be regarded as a cat^1–group where $s = 1_G = t$. Here $R_G = G$.*

Example 2. *Let A be an abelian group. Then with the direct product $A \times A$ of A with itself $(A \times A, s, t)$ becomes a cat^1–group where s and t are given by $s(a, a_1) = (a, a)$ and $t(a, a_1) = (a_1, a_1)$ for any $a, a_1 \in A$. Here note that $R_{A\times A} = A \times A$ and $\ker s = \{0\} \times A \cong A \cong A \times \{0\} = \ker t$.*

Example 3. *Let (A, B, α) becomes a crossed module. Then the semi-direct product $A \rtimes B$ has a structure of a cat^1–group where $s(a, b) = (0, b)$ and $t(a, b) = (0, \alpha(a) + b)$ for any $(a, b) \in A \rtimes B$.*

Example 4. *Let X be a topological group and $\pi(X)$ be the set of all homotopy classes of paths in X. It is a well known fact that $\pi(X)$ has a group structure induced from that of X. $\pi(X)$ also has a cat^1–group structure with $s([\alpha]) = [1_{\alpha(0)}]$ and $t([\alpha]) = [1_{\alpha(1)}]$ for any $[\alpha] \in \pi(X)$.*

Let (G, s, t) and (G', s', t') be two cat^1–groups. Then a group homomorphism $f\colon G \to G'$ is called a morphism of cat^1–groups if $fs = s'f$ and $ft = t'f$.



The category of all cat^1–groups with morphisms defined above is denoted by $\mathsf{Cat}^1-\mathsf{Gr}$.

1.2 Group-groupoids

Now we recall the definition of an internal category within a category with pullbacks as given by Datuashvili [3].

Definition 2. *Let $\mathbb{C}$ be a category with pullbacks. An internal category $\mathbb{D}$ within $\mathbb{C}$ is a sextuple $(D_0, D_1, d_0, d_1, \varepsilon, m)$ where D_0, D_1 are objects in $\mathbb{C}$, and $d_0, d_1 \colon D_1 \to D_0$, $\varepsilon \colon D_1 \to D_0$ and $m \colon D_{1\,d_1}\times_{d_0} D_1 \to D_1$ are morphisms in $\mathbb{C}$ such that*

1. $d_0\varepsilon = d_1\varepsilon = 1_{G_0}$,
2. $d_0 m = d_0\pi_2$, $d_1 m = d_1\pi_1$,
3. $m(m \times 1_{D_1}) = m(1_{D_1} \times m)$,
4. $m(\varepsilon d_0, 1_{D_1}) = m(1_{D_1}, \varepsilon d_1) = 1_{D_1}$.

Morphisms between internal categories within a category $\mathbb{C}$ are actually functors whose components are morphisms in the category $\mathbb{C}$. These morphisms are called by internal functors. For a given category $\mathbb{C}$, all internal categories within $\mathbb{C}$ and all internal functors form a category and this category is denoted by $\mathsf{Cat}(\mathbb{C})$. A category or an internal category $\mathbb{C} = (C_0, C_1, d_0, d_1, \varepsilon, m)$ will be denoted briefly by $\mathbb{C} = (C_0, C_1)$.

An internal category within the category Gr of groups becomes a groupoid, i.e. a category whose morphisms are isomorphisms. Hence an internal category within the category Gr of groups is called a group-groupoid. Let $\mathbb{G} = (G_0, G_1)$ be a group-groupoid. Then the class of objects G_0 and the class of morphisms G_1 are groups, and d_0, d_1, ε, m are group homomorphisms such that the conditions given in Definition 2 are satisfied. Here note that, m being a group homomorphism is equivalent to the following equation:

$$(b \circ a) + (d \circ c) = (b + d) \circ (a + c)$$

for any $a, b, c, d \in G_1$ whenever both sides are meaningful.

Group-groupoid morphisms are actually functors whose components are group-homomorphisms. The category of all group-groupoids and morphisms between them is denoted by $\mathsf{Cat}(\mathsf{Gr})$ or briefly by GpGd.

We sketch the proof of following theorem due to Loday [5] since we need some details later.

Theorem 1 ([5])**.** *The category $\mathsf{Cat}^1-\mathsf{Gr}$ of cat^1–groups and the category GpGd of group-groupoids are naturally equivalent.*

Proof: Let (G, R_G, s, t) be a cat^1–group. Then the corresponding group-groupoid to (G, R_G, s, t) is $(R_G, G, d_0, d_1, \varepsilon, m)$ where $d_0 = s$, $d_1 = t$, $\varepsilon = e$ the embedding of R_G into G, and $m(g, g_1) = g - t(g) + g_1$ for any g, g_1 such that $t(g) = s(g_1)$.

Conversely, let $\mathbb{G} = (G_0, G_1)$ be a group-groupoid. Then the corresponding cat^1–group to $\mathbb{G}$ is (G_1, G_0, s, t) where $s = \varepsilon d_0$ and $t = \varepsilon d_1$. □

Definition 3. *Let $\mathbb{G} = (G_0, G_1)$ be a group-groupoid and let $\mathbb{H} = (H_0, H_1)$ be a subgroupoid of $\mathbb{G}$. Then we say that $\mathbb{H}$ is a subgroup-groupoid of $\mathbb{G}$ if H_1 is a subgroup of G_1 and that $\mathbb{H}$ is a normal subgroup-groupoid (or an ideal) of $\mathbb{G}$ if H_1 is a normal subgroup of G_1.*

If $\mathbb{H}$ is a subgroup-groupoid (an ideal) of $\mathbb{G}$ then we denote this by $\mathbb{H} \leq \mathbb{G}$ ($\mathbb{H} \lhd \mathbb{G}$).

Example 5. *Let $\mathbb{G}$ and $\mathbb{H}$ be two group-groupoids and $f = (f_0, f_1) \colon \mathbb{G} \to \mathbb{H}$ be a group-groupoid morphism. Then $\ker f = (\ker f_0, \ker f_1)$ is an ideal of $\mathbb{G}$ and $\operatorname{Im} f = (\operatorname{Im} f_0, \operatorname{Im} f_1)$ is a subgroup-groupoid of $\mathbb{H}$.*

Example 6. *A groupoid containing only one object and only one morphism can be regarded as a group-groupoid. This group-groupoid is denoted by $\mathbb{1} = (\{*\}, \{1_*\})$ where $*$ is the single object and 1_* is the identity morphism on $*$. Moreover, for any group-groupoid $\mathbb{G}$ the subgroupoid $\mathbb{0} = \left(\{0_{G_0}\}, \left\{0_{G_1} = 1_{(0_{G_0})}\right\}\right)$ is an ideal of $\mathbb{G}$.*

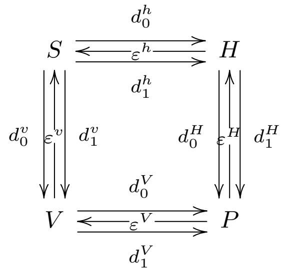
Definition 4. *Let $\mathbb{G}$ be a group-groupoid and let $\mathbb{H}, \mathbb{K} \lhd \mathbb{G}$. Then the group-groupoid $([H_0, K_0], [H_1, K_1])$ is called the commutator subgroup-groupoid of $\mathbb{G}$ generated by $\mathbb{H}$ and $\mathbb{K}$. This commutator subgroup-groupoid is in fact an ideal of $\mathbb{G}$ and denoted by $[\mathbb{H}, \mathbb{K}]$. In particular, $[\mathbb{G}, \mathbb{G}]$ is called the derived subgroup of G and denoted by $[\mathbb{G}, \mathbb{G}] = \mathbb{G}'$ for the sake of brevity.*

In [1] Brown and Spencer proved that there is a categorical equivalence between crossed modules over groups and group-groupoids. Hence the category of cat^1–groups and the category of crossed modules are naturally equivalent.

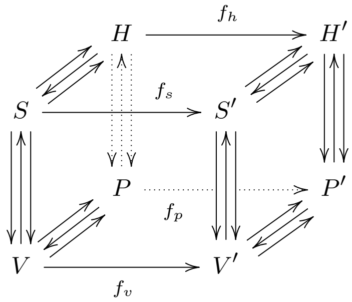
1.3 Double group-groupoids

Definition 5. *An internal category $\mathcal{G}$ in the category GpGd of group-groupoids is called a double group-groupoid. Therefore, a double group-groupoid $\mathcal{G} = (P, H, V, S)$ consist of four compatible group-groupoids (H, S), (V, S), (P, V) and (P, H).*

$\mathcal{G}$:



Let $\mathcal{G}$ and $\mathcal{G}'$ be two double group-groupoids. A morphism form $\mathcal{G}$ to $\mathcal{G}'$ is a double groupoid morphism $\mathcal{F} = (f_p, f_h, f_v, f_s) \colon \mathcal{G} \to \mathcal{G}'$ such that $f_s \colon S \to S'$, $f_h \colon H \to H'$, $f_v \colon V \to V'$ and $f_p \colon P \to P'$ are group homomorphisms. Such a morphism of double group-groupoids may be denoted by a diagram as follows:



All double group-groupoids and morphisms between them given above form a category which is denoted by DbGpGd. Temel et al. [6] defined the notion of crossed module over group-groupoids and proved that the category of double group-groupoids and the category of crossed modules over groups are naturally equivalent.

2 Cat^1-objects in the category of group-groupoids

Definition 6. *A cat^1–group–groupoid (or 1–cat–group–groupoid) $(\mathbb{G}, s, t)$ consists of a group-groupoid $\mathbb{G}$ and two endomorphisms*

$$s = (s_0, s_1), t = (t_0, t_1) \colon \mathbb{G} \to \mathbb{G}$$

of $\mathbb{G}$ such that

1. $st = t$, $ts = s$ *and*
2. $[\ker s, \ker t] = \mathbb{0}$.

Remark 2. *Here note from the condition* 1 *of the Definition* 6 *that $s_i t_i = t_i$, $t_i s_i = s_i$ for $i \in \{0, 1\}$ and from the condition* 2 *of the Definition* 6 *that*

$$[(\ker s_0, \ker s_1), (\ker t_0, \ker t_1)] = ([\ker s_0, \ker t_0], [\ker s_1, \ker t_1]) = (\{0_{G_0}\}, \{0_{G_1}\}),$$

that is, $[\ker s_0, \ker t_0] = \{0_{G_0}\}$ *and* $[\ker s_1, \ker t_1] = \{0_{G_1}\}$.

We can give the following corollary as a consequence of Remark 2.

Corollary 1. *Let $(\mathbb{G}, s, t)$ be a cat^1–group–groupoid. Then (G_0, s_0, t_0) and (G_1, s_1, t_1) have structures of cat^1–groups.*

Lemma 1. *If $(\mathbb{G}, s, t)$ is a cat^1–group–groupoid then*

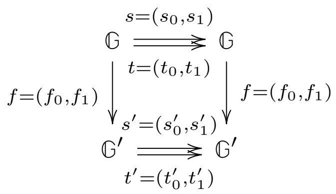
1. $d_0 s_1 = s_0 d_0$, $d_0 t_1 = t_0 d_0$,
2. $d_1 s_1 = s_0 d_1$, $d_1 t_1 = t_0 d_1$,
3. $\varepsilon s_0 = s_1 \varepsilon$, $\varepsilon t_0 = t_1 \varepsilon$,
4. $m(s_1 \times s_1) = s_1 m$, $m(t_1 \times t_1) = t_1 m$,

that is, the groupoid structural maps d_0, d_1, ε and m have structures of cat^1–group morphisms.

Proof: This is a direct result of $s = (s_0, s_1)$ and $t = (t_0, t_1)$ being group-groupoid morphisms. □

The result given in Lemma 1 tells us that a cat^1–group–groupoid can actually be considered as an internal category in the category of cat^1–groups.

Morphisms between cat^1–group–groupoids are group-groupoid morphisms which are compatible with the group-groupoid morphisms $s = (s_0, t_0)$ and $t = (t_0, t_1)$. Explicitly; let $(\mathbb{G}, s, t)$ and $(\mathbb{G}', s', t')$ be two cat^1–group–groupoids and let $f = (f_0, f_1) \colon \mathbb{G} \to \mathbb{G}'$ be a group-groupoid morphism. Then we say that f is a morphism of cat^1–group–groupoids if $fs = s'f$ and $ft = t'f$.



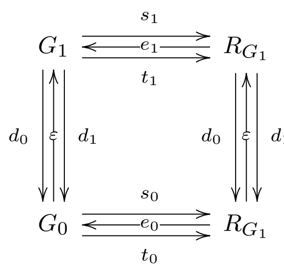
It is clear that all cat[1]–group–groupoids and morphisms between them as defined above have a structure of a category. This category is denoted by $\mathsf{Cat}^1 - \mathsf{GpGd}$.

3 Equivalence of categories

In this section, first of all, we prove the category of cat[1]–group–groupoids and the category of double group-groupoids are naturally equivalent. Using this equivalence we will be able to transfer special objects, special morphisms, examples and results from one to another. Further, we prove another categorical equivalence between cat[1]–group–groupoids and cat[2]–groups.

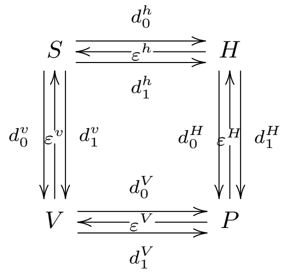
Theorem 2. *There is a categorical equivalence between cat[1]–group–groupoids and double group-groupoids.*

Proof: Let $(\mathbb{G}, R_{\mathbb{G}}, s, t)$ be a cat[1]–group–groupoid. Here we already know that $\mathbb{G} = (G_0, G_1)$ and $R_{\mathbb{G}} = (R_{G_0}, R_{G_1})$ are group-groupoids since $R_{\mathbb{G}} = \operatorname{Im} s$ (and hence $R_{\mathbb{G}} = \operatorname{Im} t$). It can easily be seen from Corollary 1 and Theorem 1 that (R_{G_1}, G_1) and (R_{G_0}, G_0) are also group-groupoids. Other details can be shown by direct calculations. Then $(G_1, G_0, R_{G_1}, R_{G_0})$ which is diagrammatically shown below is a double group-groupoid.



Conversely, let $\mathcal{G} = (P, H, V, S)$ be a double group-groupoid.

$\mathcal{G}$:



Then $((V, S), s, t)$ becomes a cat[1]–group-groupoid where $s_1 = \varepsilon^h d_0^h$, $t_1 = \varepsilon^h d_1^h$, $s_0 = \varepsilon^V d_0^V$ and $t_0 = \varepsilon^V d_1^V$. Checking of the conditions of the Definition 6 is easy. So we omit the proof. □

As a consequence of Theorem 2 and the equivalence obtained by Temel et al. [6] we can give the following corollary.

Corollary 2. *The category of cat[1]–group-groupoids and the category of crossed modules over group-groupoids are naturally equivalent.*

Now we recall another algebraic object which will be shown that these kind of objects are categorically equivalent to cat[1]–group-groupoids.

Definition 7. *Let G be a group and let s_0, s_1, t_0 and t_1 be four group endomorphisms of G, i.e. $s_0, s_1, t_0, t_1 \colon G \to G$ are group homomorphisms. The (G, s_0, s_1, t_0, t_1) is called a cat[2]–group (2–cat–group) if*

1. $s_i t_i = t_i$ *and* $t_i s_i = s_i$ *for* $i \in \{0, 1\}$,
2. $s_i s_j = s_j s_i$, $t_i t_j = t_j t_i$ *and* $s_i t_j = t_j s_i$ *for* $i \neq j$ *and*
3. $[\ker s_i, \ker t_i] = 0$ *for* $i \in \{0, 1\}$.

In other words a cat[2]–group consist of two independent cat[1]–group structures (G, s_0, t_0) and (G, s_1, t_1). Let (G, s_0, s_1, t_0, t_1) and $(G', s'_0, s'_1, t'_0, t'_1)$ be two cat[2]–groups and let $f \colon G \to G'$ be a group homomorphism such that $f s_i = s'_i f$ and $f t_i = t'_i f$ for $i \in \{0, 1\}$. Then we say that f is a morphism of cat[2]–groups.

$$
\begin{array}{ccccc}
G & \overset{s_0}{\underset{t_0}{\Longleftarrow}} & G & \overset{s_1}{\underset{t_1}{\Longrightarrow}} & G \\
{\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
G' & \overset{s'_0}{\underset{t'_0}{\Longleftarrow}} & G' & \overset{s'_1}{\underset{t'_1}{\Longrightarrow}} & G'
\end{array}
$$

The category of all cat^2–groups with morphisms reminded above is denoted by $\mathsf{Cat}^2 - \mathsf{Gr}$.

Theorem 3. *The category* $\mathsf{Cat}^2 - \mathsf{Gr}$ *of cat*^{2}*–groups is naturally equivalent to the category* $\mathsf{Cat}^1 - \mathsf{GpGd}$ *of cat*^{1}*–group–groupoids.*

Proof: Let (G, s_0, s_1, t_0, t_1) be a cat^2–group. Then we know from Theorem 1 that $\mathbb{G} = (\mathrm{Im}\, s_0, G, s_0, t_0, e_0, m_0)$ is a group-groupoid. Thus it can be seen that $(\mathbb{G}, s = (s_1, s_1), t = (t_1, t_1))$ becomes a cat^1–group–groupoid.

Conversely, let $(\mathbb{G}, s, t)$ be a cat^1–group–groupoid. In this case, (G, s_0, s_1, t_0, t_1) becomes a cat^2–group where $s_0 = \varepsilon d_0$ and $t_0 = \varepsilon d_1$. Other details are straightforward. □

4 Conclusion

It is a well known fact that cat^n–groups are algebraic models for (connected) homotopy (n+1)–types. In particular cat^1–groups are algebraic models for (connected) homotopy 2–types while cat^2–groups are algebraic models for (connected) homotopy 3–types. In this study we introduce a new algebraic concept namely cat^1–group–groupoid which defined as cat^1-object in the category of group-groupoids. Then we proved two categorical equivalences: One of them is between the category $\mathsf{Cat}^1 - \mathsf{GpGd}$ of cat^1–group–groupoids and the category DbGpGd (Theorem 2). The other one is between the category $\mathsf{Cat}^1 - \mathsf{GpGd}$ of cat^1–group–groupoids and the category $\mathsf{Cat}^2 - \mathsf{Gr}$ of cat^2–groups (Theorem 3. From the categorical equivalence given in Theorem 3 now it is possible to say that cat^1–group–groupoids are also models (connected) homotopy 3–types.

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Abstract: This study analyzes the optimal control of a fractional SEIR model representing the spread of computer viruses in an internet network under the influence of singular and non-singular fractional operators. The basic model is considered in terms of the Caputo fractional derivative. However, the model is also discussed with the Caputo-Fabrizio and Atangana-Baleanu derivatives for comparison purposes. The main aim is to save the network from the devastating effect of the computer virus at a minimum cost. Therefore, anti-virus software is considered as a control variable and adapted to the model. Optimality conditions are calculated by the Hamiltonian formalism. Subsequently, the Adams-type predictor-corrector and the forward-backward sweep algorithms are combined and applied to the system to obtain the numerical solution of the optimal system. Consequently, numerical simulations are held by the MATLAB software, and hence the effectiveness of the anti-virus program according to different fractional operators is revealed.

Keywords: Anti-virus software, Computer virus propagation, Optimal control, Caputo, Caputo-Fabrizio, Atangana-Baleanu, Adams-type predictor-corrector method.

1 Introduction

A computer virus is a man-made destructive computer program or code that is installed on a computer system without the user's knowledge and causes undesirable changes to the information stored on the computer, without the person's authorization [1]. Computer viruses have been named viruses because they reflect some characteristics of biological viruses. In other words, a computer virus is transmitted from computer to computer in the same way that a biological virus is transmitted from person to person. Thus, the spread of a biological virus in a living population is similar to the spread of a computer virus in a computer population [2].

Recently, rapid innovations in science, technology, and commerce have increased the use of computers, the internet, and various software. The rapidly increasing developments in computer technology have become an indispensable part of real life. But on the other hand, these developments have unfortunately led to an increase in the threat of malicious software such as computer viruses [3, 4]. Therefore, computer and software engineers and applied mathematicians are increasingly interested in developing measures to prevent this threat, which could cause a global crisis.

Mathematical models have been used as a tool to understand the transmission dynamics of epidemics and to predict their course [5, 6]. As with biological viruses, there is a fundamental need to develop mathematical models to clarify the behavior of computer viruses and control their spread. Because mathematical models allow to make predictions about their spread in a short time. Thus, many economic losses that computer viruses may cause at the global level can be prevented easily. Because of the close similarity between the dynamics of epidemiological diseases and the behavior of computer viruses, the spreading models of computer viruses are developed using the theory of mathematical epidemiology [7]-[9]. The basic ones of these models are: SIS [10], SIR [11, 12], SIRS [13, 14], SEIR [15], SEIS [16], SIC [17], VEISV [18] and SLBS [19].

Integer-order differential equation systems have been frequently used in modeling epidemics. However, the inherited and memory structure of viral diseases makes modeling them with integer-order derivatives insufficient. Fractional derivatives can overcome this shortcoming quite successfully, thanks to their non-local definition. These are basically classified as singular such as Riemann-Liouville (RL) and Caputo [20], [21] and non-singular fractional derivatives such as Caputo-Fabrizio [22] (CF) and Atangana-Baleanu (AB) [23], depending on their kernel functions. Deciding which derivative is more effective in modeling is related to which law the modeled phenomenon acts in accordance with [24]. While it is realistic to model the behavior of a power law-abiding phenomenon with RL or Caputo derivatives, modeling of exponential law-abiding processes with CF and AB derivatives is more useful.

The heterogeneity in the structure of a computer network (i.e. the complexity of the connections in the network) determines the behavior of virus propagation in the network. Since computer networks are multi-connected in today's information technology, computer viruses also behave like this and spread super fast instead of showing normal propagation. Therefore, it naturally leads to model their propagation with singular (low heterogeneity) or regular (high heterogeneity) fractional derivatives, depending on the structure of the network. Despite its importance, there are still limited studies on solutions of fractional order systems, stability analyzes, and control strategies for the propagation of computer viruses [25]-[30].

The first step of the present study has been determining possible optimal control strategies for the model under consideration and the stability analysis for the controlled system. This part has been studied recently [31]. In the current work, the optimal control problem of computer virus propagation is comparatively investigated according to the effect of singular and regular fractional derivatives. To obtain the numerical solutions of optimal system, we apply Adams-type predictor-corrector method with forward-backward sweep algorithm. In this hybrid method, it is important that the numerical integration coefficients vary according to the type of fractional derivative. This is taken into account when creating the algorithm.

The work can be summarized as follows: In Section 2, we give the definitions and basic properties of the fractional operators: Caputo, CF and ABC. In Section 3, we adapt the anti-virus control strategy to the system. In Section 4, we pose the optimal control problem and obtain the optimal system. In Section 5, the numerical algorithm used to obtain the solution of the optimal system is given in detail. Also, there are numerical simulations and discussions. In conclusion, the contribution of the study is presented in Section 6.

2 Preliminaries

The prescribed model is actually described with Caputo fractional derivative [26]. However, the main purpose of the study is to compare the behaviors of the model for the Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives. For this purpose, necessary definitions and some properties of these operators is given briefly as follows:

Definition 1. *[20] Assume that $f(\cdot)$ belongs to $AC[a,b]$. Hence, the Caputo fractional derivatives (CFDs) are, respectively,*

$$ {}_a^C D_t^{\alpha} f(t) = {}_a I_t^{n-\alpha} D f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)\, d\tau, \tag{1}$$

$$ {}_t^C D_b^{\alpha} f(t) = {}_t I_b^{n-\alpha} (-D) f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\xi-t)^{n-\alpha-1} f^{(n)}(\xi)\, d\tau, \tag{2}$$

where $\alpha \in (n-1, n]$ and $\Gamma(\alpha)$ is the Euler's gamma function.

Definition 2. *[22] Assume that $f \in H^1(a,b)$, $b > a$ and $\alpha \in [0,1]$. Then, the α-order left and right CF fractional derivatives of the f, respectively, are described as*

$$ {}^{CF}{}_a D_t^{\alpha} f(t) = \frac{M(\alpha)}{\alpha-1} \int_a^t \frac{df(\tau)}{d\tau} \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau, \tag{3}$$

$$ {}^{CF}{}_t D_b^{\alpha} f(t) = -\frac{M(\alpha)}{\alpha-1} \int_t^b \frac{df(\tau)}{d\tau} \exp\left[-\frac{\alpha}{1-\alpha}(\tau-t)\right] d\tau, \tag{4}$$

where $M(\alpha)$ is the normalization function such that $M(0) = M(1) = 1$.

Definition 3. *[23] Assume that $f \in H^1(a,b)$, $b > a$ and $\alpha \in [0,1]$. Then, the α-order left and right ABC fractional derivatives of the f, respectively, are described as*

$$ {}^{ABC}{}_a D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \frac{df(\tau)}{d\tau} E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] d\tau, \tag{5}$$

$$ {}^{ABC}{}_t D_b^{\alpha} f(t) = \frac{-B(\alpha)}{1-\alpha} \int_t^b \frac{df(\tau)}{d\tau} E_\alpha\left[-\frac{\alpha}{1-\alpha}(\tau-t)^{\alpha}\right] d\tau, \tag{6}$$

where E_α is the Mittag-Leffler function and $B(\alpha)$ is the normalization function such that $B(0) = B(1) = 1$.

Lemma 1. *[32] For $\alpha \in (0,1]$, the following property for CFDs is satisfied:*

$$ {}_t^C D_{t_f}^{\alpha} f(t) = {}_{t_0}^C D_t^{\alpha} f\left(t_f - t\right), \ \ t \in [t_0, t_f]. \tag{7}$$

3 Description of Virus Propagation Model with Optimal Control Strategy

In a network of computers, there are two possibilities: If the computer is connected to the internet, this connection is called internal, otherwise, it is called external. To model virus propagation among the internal computers, this population is considered in four compartments:

$S(t)$: denotes the susceptible computers on the network that are uninfected and newly connected to the network,
$E(t)$: represents the exposed computers that are newly infected or threatened by viruses,
$I(t)$: indicates the infected computers, that is, those hosting the virus,
$R(t)$: is the recovered computers that are virus-free and immune to viruses.

The discussed model was first proposed with the integer-order derivative by Peng et al. [33]. Subsequently, Bonyah et al. [26] generalized this model by considering the capability of the Caputo fractional derivative for a power-law network:

$$\begin{cases} {}_0^C D_t^\alpha S(t) = \Lambda - \beta_1 S(t) I(t) - \beta_2 S(t) E(t) - pS(t) - \mu S(t), \\ {}_0^C D_t^\alpha E(t) = \beta_1 S(t) I(t) + \beta_2 S(t) E(t) - kE(t) - \sigma E(t) - \mu E(t), \\ {}_0^C D_t^\alpha I(t) = \sigma E(t) - dI(t) - \mu I(t), \\ {}_0^C D_t^\alpha R(t) = pS(t) + kE(t) + dI(t), \end{cases} \tag{8}$$

where ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative, $\alpha \in (0,1]$, $N = S + E + I + R$, $(S,E,I,R) \in \mathbb{R}_+^4$, and $\Lambda = (1-p)N$. The parameters of the system are as follows:

Λ: the rate of connection of susceptible computers to the internet network, that is, the rate of recruitment,
p : the treatment rate of susceptible computers with the natural anti-virus ability of network,
β_1: the incidence rate of infected and susceptible computers,
β_2: the incidence rate of susceptible and exposed computers,
μ : the breaking-out rate of the computers due to mechanical reasons,
k : the treatment rate of exposed computers with the natural anti-virus ability of the network,
σ : the rate of exposed computers that may not be recovered by an anti-virus program and become corrupted,
d : the natural treatment rate of infected compartment.

Computers can become unusable due to viruses, and they can also be out of use for mechanical reasons. Therefore, different from the studies [26] and [33], we consider the mechanical death rate μ and then adapt the term $\mu R(t)$ to the system (8). In addition, unlike these studies, we also consider the R compartment while controlling the system optimally.

Our main motivation is to examine the optimal control for the model (8) by using an anti-virus software as the control function. Installing an anti-virus program on the network is inevitable, as a network's inherent anti-virus capability is often not sufficient. The virus protection program can be installed on computers in the network for different purposes. For this, we will give some scenario comparisons with simulations.

In our controlled model, we suppose that we have installed the anti-virus on susceptible and infected computers and others that interact with them. Let $u(t)$ denotes the cost of installing an anti-virus program. The Lebesque measurable set of admissible control functions is defined as

$$U_{ad} = \left\{ u(t) \in L^2 \; \left[0, t_f\right] : u(t) \in [0,1], \forall t \in \left[0, t_f\right] \right\}.$$

Now, we rearrange the model (8) under unit consistency as follows:

$$\begin{cases} {}_0^C D_t^\alpha S(t) = \Lambda^\alpha - (1-u(t))\beta_1^\alpha S(t) I(t) - (1-u(t))\beta_2^\alpha S(t) E(t) - (p + \mu^\alpha) S(t), \\ {}_0^C D_t^\alpha E(t) = (1-u(t))\beta_1^\alpha S(t) I(t) + (1-u(t))\beta_2^\alpha S(t) E(t) - (k^\alpha + \sigma^\alpha + \mu^\alpha) E(t) - u(t)E(t), \\ {}_0^C D_t^\alpha I(t) = \sigma^\alpha E(t) - (d^\alpha + \mu^\alpha) I(t) - u(t) I(t), \\ {}_0^C D_t^\alpha R(t) = pS(t) + k^\alpha E(t) + d^\alpha I(t) - \mu^\alpha R(t) + u(t)E(t) + u(t)I(t) \end{cases} \tag{9}$$

with the initial value $(S(0), E(0), I(0), R(0))$. Here, we motivate from the control strategy considered in [34].

4 Determination of Necessary Optimality Conditions

Main aim is to

(1) reduce both of the infected (I) and exposed (E) computers,
(2) minimize the cost of installing the anti-virus program.

For this purpose, the objective functional is defined as follows:

$$J(u) = \int_0^{t_f} \left[E(t) + I(t) + \frac{1}{2}\epsilon u^2(t) \right] dt, \tag{10}$$

subjected to the system (9). In which, ϵ is a weight coefficient. We should determine the optimality conditions. Let us consider the Hamiltonian function $\mathcal{H}$ as

$$\begin{aligned}
\mathcal{H}(t,S,E,I,R,\lambda_i,u) &= E(t)+I(t)+\frac{1}{2}\epsilon u^2(t) \\
&\quad +\lambda_1(t){}_0^C D_t^\alpha S(t)+\lambda_2(t)\,{}_0^C D_t^\alpha E(t)+\lambda_3(t)\,{}_0^C D_t^\alpha I(t)+\lambda_4(t)\,{}_0^C D_t^\alpha R(t) \\
&= E(t)+I(t)+\frac{1}{2}\epsilon u^2(t) \\
&\quad +\lambda_1\left\{\Lambda^\alpha-(1-u(t))\left[\beta_1^\alpha S(t)I(t)+\beta_2^\alpha S(t)E(t)\right]\right. \\
&\quad \left.-\lambda_1\left(p+\mu^\alpha\right)S(t)\right\} \\
&\quad +\lambda_2\left\{(1-u(t))\left[\beta_1^\alpha S(t)I(t)+\beta_2^\alpha S(t)E(t)\right]\right. \\
&\quad \left.-\lambda_2\left(k^\alpha+\sigma^\alpha+\mu^\alpha+u(t)\right)E(t)\right\} \\
&\quad +\lambda_3\left\{\sigma^\alpha E(t)-\left(d^\alpha+\mu^\alpha\right)I(t)-u(t)I(t)\right\} \\
&\quad +\lambda_4\left[pS(t)+\left(k^\alpha+u(t)\right)E(t)+\left(d^\alpha+u(t)\right)I(t)-\mu^\alpha R(t)\right]
\end{aligned} \tag{11}$$

with the transversality conditions $\lambda_i(t_f)=0,(i=1,2,3,4)$.

The optimality conditions for the system (9) obtained using the Pontryagin Maximum principle are given by

Theorem 1. *Assume that (S^*,E^*,I^*,R^*) be optimal solutions of the system (9), and u^* be optimal control minimizing $J(u)$. Hence, there exist co-states functions $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ satisfying*

$$\begin{cases}
{}_t^C D_{t_f}^\alpha \lambda_1(t)=-\lambda_1(t)\left((1-u(t))\beta_1^\alpha I(t)+(1-u(t))\beta_2^\alpha E(t)+p+\mu^\alpha\right) \\
+\lambda_2(t)\left((1-u(t))\beta_1^\alpha I(t)+(1-u(t))\beta_2^\alpha E(t)\right)+\lambda_4(t)\,p, \\
{}_t^C D_{t_f}^\alpha \lambda_2(t)=-\lambda_1(t)(1-u(t))\beta_2^\alpha S(t)+\lambda_2(t)\left((1-u(t))\beta_2^\alpha S(t)-k^\alpha-\sigma^\alpha-\mu^\alpha-u(t)\right) \\
+\lambda_3(t)\,\sigma^\alpha+\lambda_4(t)\left(k^\alpha+u(t)\right)+1, \\
{}_t^C D_{t_f}^\alpha \lambda_3(t)=-\lambda_1(t)(1-u(t))\beta_1^\alpha S(t)+\lambda_2(t)(1-u(t))\beta_1^\alpha S(t) \\
-\lambda_3(t)\left(d^\alpha+\mu^\alpha+u(t)\right)+\lambda_4(t)\left(d^\alpha+u(t)\right)+1, \\
{}_t^C D_{t_f}^\alpha \lambda_4(t)=-\lambda_4(t)\,\mu^\alpha
\end{cases} \tag{12}$$

with the transversality conditions

$$\lambda_i(t_f)=0, i=1,2,3,4,5. \tag{13}$$

Additionally, the optimal control function u^* is obtained as follows:

$$u^*(t)=\max\left\{\min\left\{\begin{array}{c}\frac{(\lambda_2(t)-\lambda_1(t))(\beta_1^\alpha S^*(t)I^*(t)+\beta_2^\alpha S^*(t)E^*(t))}{\epsilon} \\ +\frac{(\lambda_2(t)-\lambda_4(t))E^*(t)+(\lambda_3(t)-\lambda_4(t))I^*(t)}{\epsilon}\end{array},1\right\},0\right\}. \tag{14}$$

Proof: Consider the fractional Euler-Lagrange equations [35]:

$${}_0^C D_t^\alpha x(t)=\frac{\partial\mathcal{H}}{\partial\lambda}(t,x(t),u(t),\lambda(t)), \tag{15}$$

$${}_t^C D_{t_f}^\alpha\,\lambda(t)=\frac{\partial\mathcal{H}}{\partial x}(t,x(t),u(t),\lambda(t)), \tag{16}$$

$$\frac{\partial\mathcal{H}}{\partial u}(t,x(t),u(t),\lambda(t))=0, \tag{17}$$

which are denoting the optimality system, with the initial state and the final co-state values:

$$x(0)=x_0,\ \ \lambda\left(t_f\right)=0. \tag{18}$$

By substituting Hamiltonian function (11) into the Eqs. $(15)-(17)$, we derive the optimality system for co-state functions as follows:

$$\begin{cases}
{}_t^C D_{t_f}^\alpha \lambda_1(t)=-\lambda_1(t)\left((1-u(t))\beta_1^\alpha I(t)+(1-u(t))\beta_2^\alpha E(t)+p+\mu^\alpha\right) \\
+\lambda_2(t)\left((1-u)\beta_1^\alpha I(t)+(1-u(t))\beta_2^\alpha E(t)\right)+\lambda_4(t)\,p, \\
{}_t^C D_{t_f}^\alpha \lambda_2(t)=-\lambda_1(t)(1-u(t))\beta_2^\alpha S(t)+\lambda_2(t)\left((1-u(t))\beta_2^\alpha S(t)-k^\alpha-\sigma^\alpha-\mu^\alpha-u(t)\right) \\
+\lambda_3(t)\,\sigma^\alpha+\lambda_4(t)\left(k^\alpha+u\right)+1, \\
{}_t^C D_{t_f}^\alpha \lambda_3(t)=-\lambda_1(t)(1-u(t))\beta_1^\alpha S(t)+\lambda_2(t)(1-u(t))\beta_1^\alpha S(t) \\
-\lambda_3(t)\left(d^\alpha+\mu^\alpha+u(t)\right)+\lambda_4(t)\left(d^\alpha+u(t)\right)+1, \\
{}_t^C D_{t_f}^\alpha \lambda_4(t)=-\lambda_1(t)\,\mu^\alpha
\end{cases}$$

with the transversality conditions:

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0.$$

From Eq.(17), we obtain the optimal control function as below:

$$u^*(t) = \begin{cases} 0, & u(t) \le 0, \\ \frac{(\lambda_2(t)-\lambda_1(t))(\beta_1^\alpha S(t)I(t)+\beta_2^\alpha S(t)E(t))+(\lambda_2(t)-\lambda_4(t))E(t)+(\lambda_3(t)-\lambda_4(t))I(t)}{\epsilon}, & 0 < u(t) < 1, \\ 1, & u(t) > 1, \end{cases}$$

or equivalently

$$u^*(t) = \max\left\{ \min\left\{ \begin{array}{c} \frac{(\lambda_2(t)-\lambda_1(t))(\beta_1^\alpha S(t)I(t)+\beta_2^\alpha S(t)E(t))}{\epsilon} \\ +\frac{(\lambda_2(t)-\lambda_4(t))E(t)+(\lambda_3(t)-\lambda_4(t))I(t)}{\epsilon} \end{array}, 1 \right\}, 0 \right\}.$$

Consequently, the optimal system consists of Eqs. (9), (12), and (17). □

5 Numerical Scheme

There are a limited number of numerical algorithms to solve the optimal control problems for fractional-order systems. In the present study, we apply the Adams-type predictor-corrector method by combining it with the forward-backward sweep algorithm. Note that while introducing the numerical algorithm, we convert the right-handed derivatives of the co-state system to the left-handed ones using the property (7). The steps of numerical algorithm is given in detail as follows [32, 36]:

Step 1: Putting the initial values $S(0)$; $E(0)$; $I(0)$; $R(0)$ and system coefficients;

Step 2: Divide the interval $[0, t_f]$ into N sub-intervals of equal length and set $h = \frac{t_f}{N}$, $t_k = kh$, $k = 0, 1, ..., N$;

Step 3: The discrete form of control function is given by

$$\begin{aligned} u(t_k) = & \frac{(\lambda_2(t_k)-\lambda_1(t_k))(\beta_1^\alpha S(t_k)I(t_k)+\beta_2^\alpha S(t_k)E(t_k))}{\epsilon} \\ & + \frac{(\lambda_2(t_k)-\lambda_4(t_k))E(t_k)+(\lambda_3(t_k)-\lambda_4(t_k))I(t_k)}{\epsilon}, k = 0, 1, ..., M. \end{aligned} \tag{19}$$

The initial condition for u is calculated using $(S(0), E(0), I(0), R(0))$ and $\lambda_i(t_f) = 0$, $i = 1, 2, 3, 4, 5$. Other values of u for $k = 1, 2, ..., M-1$ are calculated with the following loop.

Step 4: Solve the system (9) forward-in-time with the initial conditions and the value of u. The critical point is to replace Eq. (9) by the following equivalent fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau, x(\tau), u(\tau))\, d\tau. \tag{20}$$

Then, applying the Adams-type predictor-corrector method is as follows:

$$\begin{cases} S(t_{k+1}) = S(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \Big[g_1(t_{k+1}, S^p(t_{k+1}), E^p(t_{k+1}), I^p(t_{k+1}), R^p(t_{k+1}), u(t_{k+1})) \\ \quad + \sum\limits_{j=0}^{k} a_{j,k+1} g_1(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\Big], \\ S^p(t_{k+1}) = S(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \left[\sum\limits_{j=0}^{k} b_{j,k+1} g_1(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\right], \\ E(t_{k+1}) = E(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \Big[g_2(t_{k+1}, S^p(t_{k+1}), E^p(t_{k+1}), I^p(t_{k+1}), R^p(t_{k+1}), u(t_{k+1})) \\ \quad + \sum\limits_{j=0}^{k} a_{j,k+1} g_2(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\Big], \\ E^p(t_{k+1}) = E(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \left[\sum\limits_{j=0}^{k} b_{j,k+1} g_2(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\right], \\ I(t_{k+1}) = I(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \Big[g_3(t_{k+1}, S^p(t_{k+1}), E^p(t_{k+1}), I^p(t_{k+1}), R^p(t_{k+1}), u(t_{k+1})) \\ \quad + \sum\limits_{j=0}^{k} a_{j,k+1} g_3(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\Big], \\ I^p(t_{k+1}) = I(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \left[\sum\limits_{j=0}^{k} b_{j,k+1} g_3(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\right], \\ R(t_{k+1}) = R(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \Big[g_4(t_{k+1}, S^p(t_{k+1}), E^p(t_{k+1}), I^p(t_{k+1}), R^p(t_{k+1}), u(t_{k+1})) \\ \quad + \sum\limits_{j=0}^{k} a_{j,k+1} g_4(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\Big], \\ R^p(t_{k+1}) = R(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \left[\sum\limits_{j=0}^{k} b_{j,k+1} g_4(t_k, S(t_k), E(t_k), I(t_k), R(t_k), u(t_{k+1}))\right], \end{cases} \tag{21}$$

where $k=0,1,...,M-1$, and $j=1,...,k$. In addition, the coefficients $a_{j,k+1}$ and $b_{j,k+1}$ in (21) are as follows:

$$a_{j,k+1}=\begin{cases} k^{\alpha+1}-(k-\alpha)(k+1)^{\alpha}, & j=0, \\ (k-j+2)^{\alpha+1}+(k-j)^{\alpha+1}-2(k-j+1)^{\alpha+1}, & 1\leq j\leq k, \end{cases} \tag{22}$$

$$b_{j,k+1}=(k+1-j)^{\alpha}-(k-j)^{\alpha}. \tag{23}$$

Step 5: Solve the system (12) forward-in-time with terminal conditions and the values of u and x. The Eq. (12) with the condition $\lambda\left(t_f\right)=0$ is equivalent to the following fractional integral equation:

$$\begin{aligned} \lambda\left(t_f-t\right) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \\ &\times\frac{\partial H}{\partial x}\left(t_f-\tau, x\left(t_f-\tau\right), u\left(t_f-\tau\right), \lambda\left(t_f-t\right)\right)d\tau. \end{aligned} \tag{24}$$

Then, applying the Adams-type predictor-corrector under consideration with the property (7) is as follows:

$$\begin{cases} \lambda_1(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left[\frac{\partial\mathcal{H}}{\partial S}\left(t_{M-k-1},S\left(t_{M-k-1}\right),E\left(t_{M-k-1}\right),I\left(t_{M-k-1}\right),R\left(t_{M-k-1}\right),u\left(t_{M-k-1}\right),\lambda_i^p\left(t_{M-k-1}\right)\right)\right. \\ \left.+\sum_{j=0}^{k}a_{j,k+1}\frac{\partial\mathcal{H}}{\partial S}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \\ \lambda_1^p(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{j=0}^{k}b_{j,k+1}\frac{\partial\mathcal{H}}{\partial S}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \\ \lambda_2(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left[\frac{\partial\mathcal{H}}{\partial E}\left(t_{M-k-1},S\left(t_{M-k-1}\right),E\left(t_{M-k-1}\right),I\left(t_{M-k-1}\right),R\left(t_{M-k-1}\right),u\left(t_{M-k-1}\right),\lambda_i^p\left(t_{M-k-1}\right)\right)\right. \\ \left.+\sum_{j=0}^{k}a_{j,k+1}\frac{\partial\mathcal{H}}{\partial E}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \\ \lambda_2^p(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{j=0}^{k}b_{j,k+1}\frac{\partial\mathcal{H}}{\partial E}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \\ \lambda_3(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left[\frac{\partial\mathcal{H}}{\partial I}\left(t_{M-k-1},S\left(t_{M-k-1}\right),E\left(t_{M-k-1}\right),I\left(t_{M-k-1}\right),R\left(t_{M-k-1}\right),u\left(t_{M-k-1}\right),\lambda_i^p\left(t_{M-k-1}\right)\right)\right. \\ \left.+\sum_{j=0}^{k}a_{j,k+1}\frac{\partial\mathcal{H}}{\partial I}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \\ \lambda_3^p(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{j=0}^{k}b_{j,k+1}\frac{\partial\mathcal{H}}{\partial I}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \\ \lambda_4(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left[\frac{\partial\mathcal{H}}{\partial R}\left(t_{M-k-1},S\left(t_{M-k-1}\right),E\left(t_{M-k-1}\right),I\left(t_{M-k-1}\right),R\left(t_{M-k-1}\right),u\left(t_{M-k-1}\right),\lambda_i^p\left(t_{M-k-1}\right)\right)\right. \\ \left.+\sum_{j=0}^{k}a_{j,k+1}\frac{\partial\mathcal{H}}{\partial R}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \\ \lambda_4^p(t_{M-k-1})=\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{j=0}^{k}b_{j,k+1}\frac{\partial\mathcal{H}}{\partial R}\left(t_{M-j},S\left(t_{M-j}\right),E\left(t_{M-j}\right),I\left(t_{M-j}\right),R\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-j}\right)\right)\right], \end{cases} \tag{25}$$

where $k=0,1,...,M-1$, and $j=1,...,k$. Also, the coefficients $a_{j,k+1}$ and $b_{j,k+1}$ in the system (25) are those given in Step 4.

Step 6: Iterative state and co-state values calculated in the Steps 4 and 5 are used to update the values of the control function calculated in Step 3.

Step 7: Determine tolerance values are determined for states and co-states functions. If the difference between consecutive values is negligibly close for the prescribed error data, the calculation is terminated and the results obtained correspond to the optimal solutions. Otherwise, it returns to Step 3.

The system (9) discussed in the study is in terms of Caputo fractional derivative. But "I wonder what would be the solution of the optimal control problem of the system with the fractional derivatives of Caputo-Fabrizio or Atangana-Baleanu?" The answer to this question is also sought in this thesis. For this, the algorithm whose steps are given above should be made suitable for the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives.

5.1 Algorithm for Caputo-Fabrizio Fractional Derivative

The first three steps and the last two steps of the algorithm are equally valid for Caputo-Fabrizio fractional derivative. The integral equation in step 4 and the coefficients a and b in step 5 will change. With these changes, the 4th and 5th steps of the algorithm for the Caputo-Fabrizio fractional derivative can be expressed as follows [36]:

Step 4: Solve the system (9) forward-in-time with initial conditions and the value of u. Then the equivalent fractional integral equation is obtained as follows:

$$x(t) = x_0 + (1-\alpha)\, g(t, x(t), u(t)) + \alpha \int_0^t g(\tau, x(\tau), u(\tau))\, d\tau. \tag{26}$$

Then, apply the Adams-type predictor-corrector method as follows:

$$\begin{cases} x(t_{k+1}) = x_0 + \frac{\alpha h}{2} g(t_{k+1}, x^p(t_{k+1}), u(t_{k+1})) + \sum\limits_{j=0}^{k} a_{j,k+1} . g(t_k, x(t_k), u(t_k)), \\ x^p(t_{k+1}) = x_0 + h \left[\sum\limits_{j=0}^{k} b_{j,k+1} . g(t_k, x(t_k), u(t_k)) \right], \end{cases} \tag{27}$$

where $k = 0, 1, ..., M-1$ and $j = 0, 1, ..., k$; In addition, the coefficients $a_{j,k+1}$ and $b_{j,k+1}$ in the system (27) are as follows:

$$a_{j,k+1} = \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}, & j = 0, \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}, & 1 \le j \le k, \\ 1, & j = k+1, \end{cases} \tag{28}$$

$$b_{j,k+1} = \begin{cases} 1 + \frac{1-\alpha}{h}, & j = k, \\ 1, & 0 \le j \le k-1. \end{cases} \tag{29}$$

Step 5: Solve the system (12) forward-in-time with terminal conditions and the values of u and x. The system (12) with the condition $\lambda(t_f) = 0$ is equivalent to the following fractional integral equation:

$$\begin{aligned} \lambda(t_f - t) &= (1-\alpha)\, g(t_f - \tau, x(t_f - \tau), u(t_f - \tau)) \\ &\quad + \alpha \int_0^t \frac{\partial H}{\partial x} (t_f - \tau, x(t_f - \tau), u(t_f - \tau), \lambda(t_f - t))\, d\tau. \end{aligned} \tag{30}$$

Then, applying the Adams-type predictor-corrector gives

$$\begin{cases} \lambda(t_{M-k-1}) = \frac{\alpha h}{2} \left[\frac{\partial H}{\partial x} (t_{M-k-1}, x(t_{M-k-1}), u(t_{M-k-1}), \lambda^p(t_{M-k-1})) \right. \\ \qquad \left. + \sum\limits_{j=0}^{k} a_{j,k+1} . \frac{\partial H}{\partial x} (t_{M-j}, x(t_{M-j}), u(t_{M-j}), \lambda_i(t_{M-k-1})) \right], \\ \lambda^p(t_{M-k-1}) = h \left[\sum\limits_{j=0}^{k} b_{j,k+1} . \frac{\partial H}{\partial x} (t_{M-j}, x(t_{M-j}), u(t_{M-j}), \lambda_i(t_{M-k-1})) \right], \end{cases} \tag{31}$$

where $k = 0, 1, ..., M-1$, and $j = 1, ..., k$. Also, the coefficients $a_{j,k+1}$ and $b_{j,k+1}$ are those given in Step 4.

5.2 *Algorithm for Atangana-Baleanu Fractional Derivative*

The first three steps and the last two steps of the algorithm are equally valid for Atangana-Baleanu fractional derivative. The integral equation in step 4 and the coefficients a and b in step 5 will change. With these changes, the 4th and 5th steps of the algorithm for the Atangana-Baleanu fractional derivative can be expressed as follows [36]:

Step 4: Solve the system (9) forward-in-time with initial conditions and the value of u. Then, rewrite the system (9) in the following equivalent form:

$$x(t) = x_0 + \frac{(1-\alpha)}{M(\alpha)} g(t, x(t), u(t)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau, x(\tau), u(\tau))\, d\tau. \tag{32}$$

Then, applying the Adams-type predictor-corrector method gives

$$\begin{cases} x(t_{k+1}) = x_0 + \frac{h^{\alpha}}{(\alpha+1)M(\alpha)\Gamma(\alpha)} \left[g(t_{k+1}, x^p(t_{k+1}), u(t_{k+1})) + \sum\limits_{j=0}^{k} a_{j,k+1} . g(t_k, x(t_k), u(t_k)) \right], \\ x^p(t_{k+1}) = x_0 + \frac{h^{\alpha}}{M(\alpha)\Gamma(\alpha)} \left[\sum\limits_{j=0}^{k} b_{j,k+1} g(t_k, x(t_k), u(t_k)) \right], \end{cases} \tag{33}$$

where $k = 0, 1, ..., M-1$ and $j = 0, 1, ..., k$; In addition, the coefficients $a_{j,k+1}$ and $b_{j,k+1}$ in system (33) are as follows:

$$a_{j,k+1} = \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}, & j = 0, \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}, & 1 \le j \le k, \\ 1, & j = k+1, \end{cases} \tag{34}$$

$$b_{j,k+1} = \begin{cases} 1 + \frac{(1-\alpha)\Gamma(\alpha)}{h^{\alpha}}, & j = k, \\ (k-j+1)^{\alpha} - (k-j)^{\alpha}, & 0 \le j \le k-1. \end{cases} \tag{35}$$

Step 5: Solve the system (12) forward-in-time with the terminal conditions and the values of u and x. The system (12) with the condition $\lambda\left(t_f\right)=0$ can be rewritten as follows:

$$\begin{aligned}\lambda\left(t_f-t\right) &= \frac{(1-\alpha)}{M(\alpha)}g\left(t_f-\tau,x\left(t_f-\tau\right),u\left(t_f-\tau\right)\right)\\ &\quad+\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t\frac{\partial H}{\partial x}\left(t_f-\tau,x\left(t_f-\tau\right),u\left(t_f-\tau\right),\lambda\left(t_f-t\right)\right)d\tau\end{aligned} \tag{36}$$

Then, applying the Adams-type predictor-corrector leads to

$$\begin{cases}\lambda\left(t_{M-k-1}\right)=\frac{h^\alpha}{(\alpha+1)M(\alpha)\Gamma(\alpha)}\left[\frac{\partial H}{\partial x}\left(t_{M-k-1},x\left(t_{M-k-1}\right),u\left(t_{M-k-1}\right),\lambda^p\left(t_{M-k-1}\right)\right)\right.\\ \qquad\left.+\sum_{j=0}^{k}a_{j,k+1}\cdot\frac{\partial H}{\partial x}\left(t_{M-j},x\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-k-1}\right)\right)\right],\\ \lambda^p\left(t_{M-k-1}\right)=\frac{h^\alpha}{M(\alpha)\Gamma(\alpha)}\left[\sum_{j=0}^{k}b_{j,k+1}\cdot\frac{\partial H}{\partial x}\left(t_{M-j},x\left(t_{M-j}\right),u\left(t_{M-j}\right),\lambda_i\left(t_{M-k-1}\right)\right)\right],\end{cases} \tag{37}$$

where $k=0,1,...,M-1$, and $j=1,...,k$. Also, the coefficients $a_{j,k+1}$ and $b_{j,k+1}$ are those given in Step 4.

5.3 Comparative Results

The numerical simulations are carried out using the parameters given in [26]. The purpose of this is to be able to compare controlled and uncontrolled systems. The weight coefficient is taken as $\epsilon=0.009$, the initial conditions are $S(0)=10, E(0)=1, I(0)=1$, and $R(0)=0$. Also, the system parameters are $p=0.7$, $\mu=0.02$, $\sigma=0.09$, $d=0.04$, $k=0.04$, $\omega=0.5$, $\beta_1=0.002$, and $\beta_2=0.003$. All the simulation results are held by MATLAB software.

Firstly, Figure 1 shows the controlled and uncontrolled behaviors of system (9). In this case, the order of the derivative is arbitrarily chosen as α=0.85. The anti-virus program with the control strategy indicates the situation when it is installed on infected, exposed and connected computers. As intended, the number of computers that are both infected and exposed at time t = 100 is almost 0. Therefore, the number of recovered computers has increased. There is a slight variation in the number of susceptible computers. This is because it uses a small initial population of values. Although not given between results, this difference was observed to be significant for larger populations. A similar situation applies to the recovered computer population. Figure 2 illustrates the dependence of optimal solutions for Caputo fractional derivative on α. The number of infected and exposed computers will take longer to reach 0 as α values are decreasing, i.e. the smaller α values, the longer the anti-virus program should run to reduce the effect of the virus on the network. Decreasing values of α less than 1 indicate the cases in which computers on the network are weaker against a virus attack. As a natural consequence of this, if paying attention to the behavior of the control in Figure 3, the lower the α value, the longer the effectiveness of the anti-virus program. In Figure 3, as the α value decreases from 1, the effectiveness of the control gets longer. In other words, if the anti-virus program works longer, we achieve the intended result. In Figure 4, for the value of α = 0.85, the most desirable case for the number of all compartments is the Caputo fractional derivative, followed by ABC derivative and the CF derivative last. Because the model in the study represents the spread of computer virus in the computer population connected to a computer network and this shows a moderately heterogeneous network structure. This gives the most desired result that the network behaves in accordance with the power law and therefore modeling with the Caputo fractional derivative with the power function kernel. Computer virus models that represent the situation where there is more than one internet network and devices such as mobile phones and USBs are connected to infect computers are more prone to exhibit exponential behavior, which helps in the selection of the fractional derivative. Because the kernel functions of CF and ABC fractional derivatives are of exponential type, CF and ABC fractional derivatives can be chosen for such models, that is, computer models with a fully heterogeneous network structure. While the value of α approaches 1 for the Caputo fractional derivative, the intended results for all compartments have been obtained. In Figure 5, For the value of α = 0.85, the behavior of the control is the same for Caputo, CF and ABC. In other words, the anti-virus program is effective for all fractional derivatives used at the same time.

6 Conclusion

This study presents the effects of Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives on an anti-virus spreading model equipped with the anti-virus control strategy. The aim of the proposed optimal control problem has to eliminated the damage caused by the virus in the network at minimum cost. The basis model has been first discussed by Bonyah et al. [26] in sense of stability analysis but without control. In the present work, Adams-type predictor-corrector method and forward-backward sweep algorithm are combined to obtain numerical solutions of the optimal system. According to the numerical results, the behaviors of the proposed model are close to each other for the Caputo and ABC derivatives. For the model equipped with CF derivative, it is observed that it takes longer to reduce infected computers with anti-virus software. Thus, reconsidering the present model with the CF derivative represents a situation where virus spread is more aggressive. In fact, the main purpose of comparison studies on fractional derivatives should not be which derivative is better. The important manner is which derivative represents the process to be modeled better. What is critical in the spread of a computer virus is the structure of the network, i.e. the heterogeneity degree of network. In other words, if the heterogeneity of the network behaves according to the power law, virus propagation in this network behaves similarly, and therefore modeling with the Caputo derivative is more realistic. On the other hand, as the complexity of connectivity in the network increases, it makes more sense to model virus propagation in this network with CF or ABC derivatives. Future studies are planned to focus on the different functional relationships of incidence rates, which significantly determine the content of the virus spread problem. It is aimed to investigate the effect of different incidence rates on the proposed optimal control problem.

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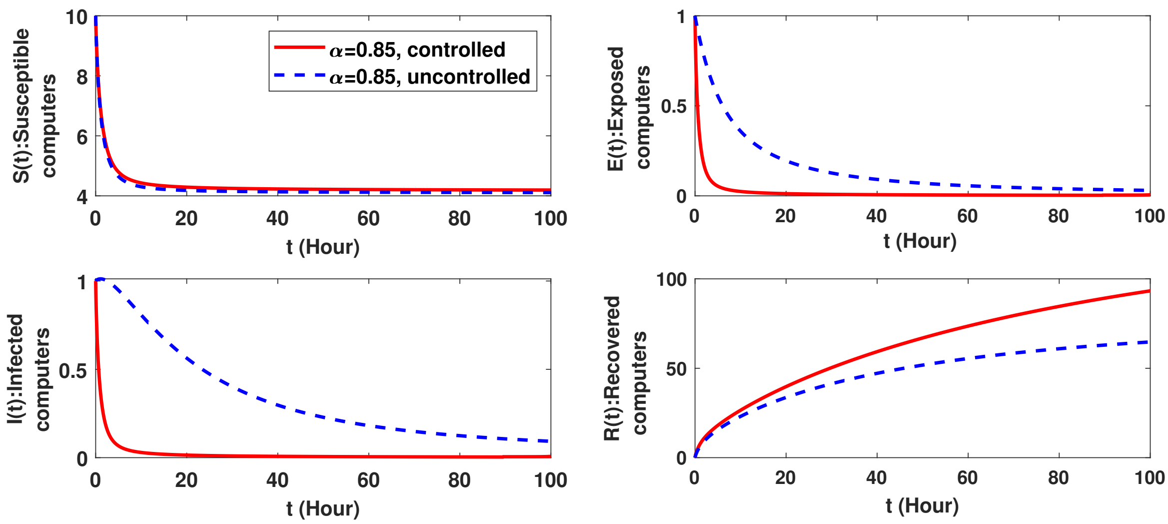

Fig. 1: System behaviours with and without control for Caputo fractional derivative: $\alpha = 0.85$ [31].

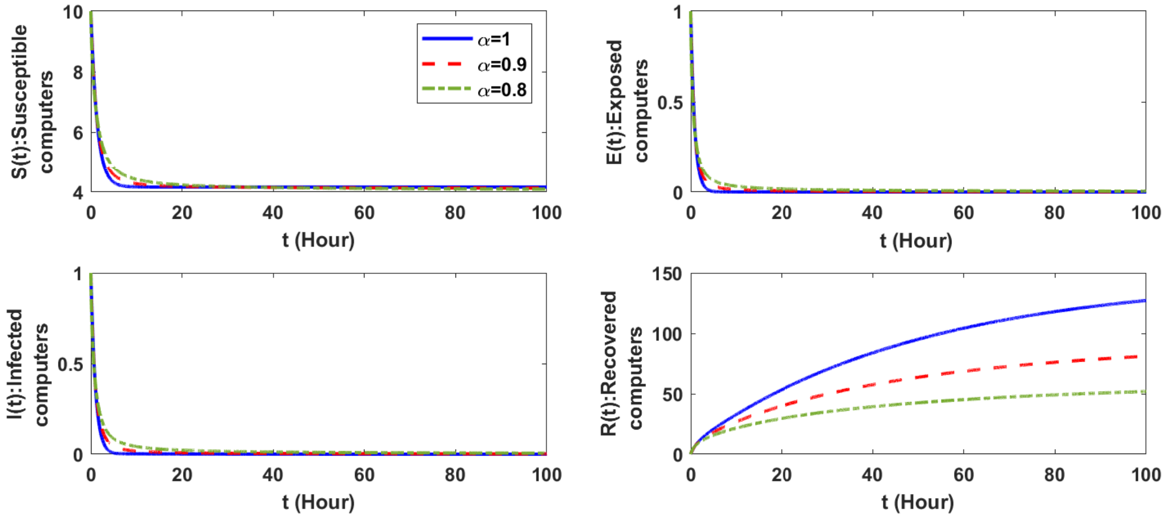

Fig. 2: The dependence of optimal solutions for Caputo fractional derivative on α [31].

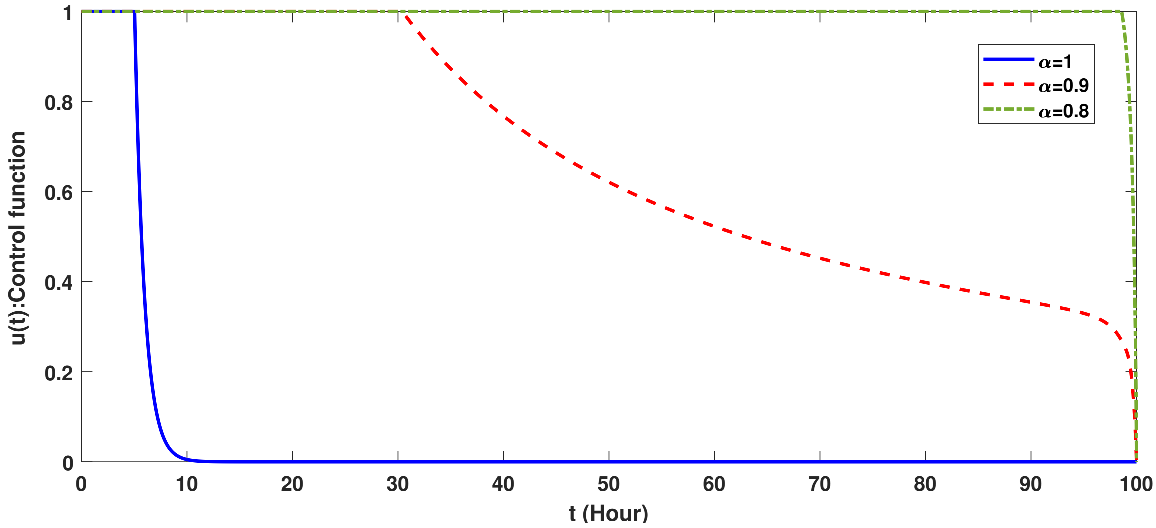

Fig. 3: The dependence of control function $u\left(t\right)$ for Caputo fractional derivative on α [31].

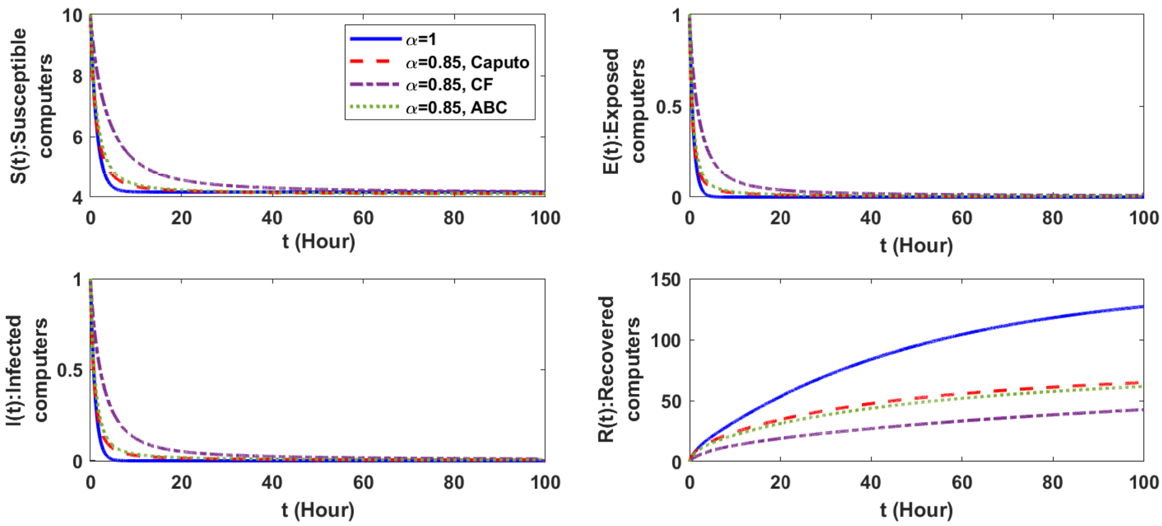

Fig. 4: Comparison of Caputo, CF and ABC fractional derivatives for $\alpha = 0.85$.

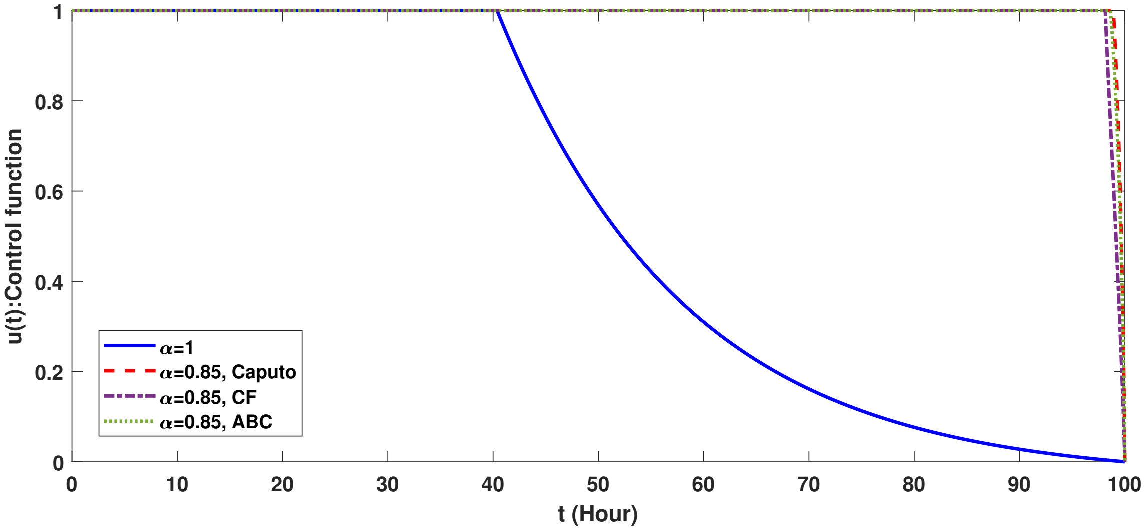

Fig. 5: Comparison of control function of CF and ABC fractional derivatives for $\alpha = 0.85$.

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Nonexistence of solutions for the quasilinear wave equation with variable coefficients

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Abstract: In this work, we consider the quasilinear wave equation with variable coefficients. Under suitable conditions on variable coeffcients, we prove the nonexistence of solutions.

Keywords: Nonexistence, Qusilinear wave equation, Variable coefficients.)

1 Introduction

In this work, we are concerned with the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) + \xi_1(t)|u_t|^{p-2}u_t = \xi_2(t)|u|^{q-2}u, & x\in\Omega, t>0, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x\in\Omega, \\ u(x,t) = \Delta u(x,t) = 0, & x\in\partial\Omega, t>0, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n $(n \in N)$, with a smooth boundary $\partial\Omega$, $m \geq 2$, $p \geq 2$, $q > 2$, $\xi_1(t)$ is a non-negative function of t and $\xi_2(t)$ is a positive functions of t. The quantity $|u_t|^{p-2}u_t$ is a damping term which assures global existence, and $|u|^{q-2}u$ is the source term which contributes to nonxistence of global solutions. $\xi_1(t)$ and $\xi_2(t)$ can be regarded as two control buttons which can dominate the polarity between damping term and source term.

Pişkin and Fidan [1] investigated

$$u_{tt} - \Delta u - \Delta u_t + \mu_1(t)|u_t|^{p-2}u_t = \mu_2(t)|u|^{q-2}u,$$

with boundary and initial conditions, and proved a nonexistence of solutions.

Messaoudi, [2] studied the following problem

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u. \tag{2}$$

He studied decay of solutions of the problem (2). Then the problem (2) was studied by Wu and Xue [3] and Pişkin [4]. For more depth, here are some papers that focused on the study of div term (see [5–10])

Zheng et al. [11] considered the fourth order equation

$$u_{tt} + \Delta^2 u + k_1(t)|u_t|^{m-2}u_t = k_2(t)|u|^{p-2}u$$

in a bounded domain. They proved the nonexistence of solutions.

In this work, we established the nonexistence of solutions. To our best konowledge, the nonexistence of solutions of the quasilinear wave equation with variable coefficients have not yet studied. By using the same techniques as in [11].

This work is organized as follows: In the next part, we present some lemmas, notations and local existence theorem. In part 3, the nonexistence of solutions are given.

2 Preliminaries

Throughout this work $\|u\|_p = \|u\|_{L^p(\Omega)}$ and $\|u\|_2 = \|u\|$ denote the usual $L^p(\Omega)$ norm and $L^2(\Omega)$ norm, respectively. Also, $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ is a Hilbert spaces (see [12, 13], for details).

Lemma 1. *[14]. Suppose that*

$$\begin{cases} m \le q < \infty, & n \le m, \\ m < q < \frac{nm}{n-m}, & n > m. \end{cases}$$

Then, there exist a positive constant $C > 1$, depending on Ω only, such that

$$\|u\|_q^s \le C \left(\|\nabla u\|_m^m + \|u\|_q^q \right) \tag{3}$$

for any $u \in W_0^{2,m}(\Omega)$ and $m \le s \le q$.

Lemma 2. *Assume that $\xi_1(t)$ is a nonnegative function of t, $\xi_2(t)$ is a positive functions of t and $\xi_2'(t) \ge 0$. Let $u(t)$ be a solution of problem (1) then the energy functional $E(t)$ is non-increasing, namely $E'(t) \le 0$.*

Proof: Multiplying the equation (1) with u_t and integrating with respect to x over the domain Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{m}\|\nabla u\|_m^m + \frac{1}{2}\|\Delta u\|^2 - \frac{\xi_2(t)}{q}\|u\|_q^q \right) = -\xi_1(t)\|u_t\|_p^p - \frac{\xi_2'(t)}{q}\|u\|_q^q. \tag{4}$$

By the equality (4), we have

$$E'(t) = -\xi_1(t)\|u_t\|_p^p - \frac{\xi_2'(t)}{q}\|u\|_q^q \le 0,$$

and $E(t) \le E(0)$ .Here

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{m}\|\nabla u\|_m^m + \frac{1}{2}\|\Delta u\|^2 - \frac{\xi_2(t)}{q}\|u\|_q^q \tag{5}$$

and

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{m}\|\nabla u_0\|_m^m + \frac{1}{2}\|\Delta u_0\|^2 - \frac{\xi_2(0)}{q}\|u_0\|_q^q.$$

□

In order to obtain our main results, we set

$$H(t) = -E(t) \tag{6}$$

In the following remark, C denotes a generic constant that varies from line to line.

Remark 1. *Suppose that*

$$\begin{cases} m \le q < \infty, & n \le m, \\ m < q < \frac{nm}{n-m}, & n > m. \end{cases}$$

and energy functional $E(t) < 0$. Then, there exist a positive constant C, depending only on Ω, such that

$$\|u\|_q^s \le C\left(H(t) + \|u_t\|^2 + \left(\frac{\xi_2(t)}{q} + 1 \right)\|u\|_q^q \right) \tag{7}$$

for any $u \in W_0^{2,m}(\Omega)$ and $m \le s \le q$.

Next, we state the local existence theorem that can be established by combining arguments of [15–17].

Theorem 1. *(Local existence). Assume that*

$$\begin{cases} m \le q < \infty, & n \le m, \\ m < q < \frac{nm}{n-m}, & n > m. \end{cases}$$

Then, for any given $u_0 \in W_0^{2,m}(\Omega)$ and $u_1 \in L^2(\Omega)$, the problem (1) has a local solution satisfying

$$u \in C\left([0,T] : W_0^{2,m}(\Omega), u_t \in C\left([0,T]; L^2(\Omega) \right) \cap L^p(\Omega, [0,T]) \right)$$

for some $T > 0$.

3 Nonexistence

In this part, we will consider the blow up of solutions for the problem (1).

Theorem 2. *Let the assumptions of Lemma 2 hold. And suppose that $\xi_1(t)$ is a nonnegative function of t, $\xi_2(t)$ is a positive functions of t, $\xi_2'(t) \geq 0$ and*

$$\lim_{t\to\infty} \xi_1(t)\, \xi_2(t)^{\alpha(p-1)}$$

exists, where

$$0 < \alpha \leq \min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}.$$

Then the solution of Eq. (1) blows up in finite time T^* and

$$T^* \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)}$$

if $q > p$ and the initial energy function

$$E(0) < 0,$$

where

$$L(0) = [H(0)]^{1-\alpha} + \epsilon\int_\Omega u_0 u_1 dx > 0.$$

Proof: From (4)-(6), we obtain

$$\frac{d}{dt}H(t) = \xi_1(t)\|u_t\|_p^p + \frac{\xi_2'(t)}{q}\|u\|_q^q \geq 0 \tag{8}$$

for almost, every $t \in [0, T)$. Also,

$$0 < H(0) \leq H(t) \leq \frac{\xi_2(t)}{q}\|u\|_q^q,\ t \in [0, T). \tag{9}$$

Define

$$L(t) = H^{1-\alpha}(t) + \epsilon\int_\Omega u u_t dx \tag{10}$$

where $\epsilon > 0$ is small to be chosen later, and

$$0 < \alpha \leq \min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}. \tag{11}$$

Differentiating (10) with respect to t and combining the first equation of (1), we get

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon\int_\Omega\left(uu_{tt} + u_t^2\right)dx \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) \\ &\quad +\epsilon\int_\Omega\left(u\mathrm{div}\left(|\nabla u|^{m-2}\nabla u\right) - u\Delta^2 u - \xi_1(t)|u_t|^{p-1}u + \xi_2(t)u^q + u_t^2\right)dx \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon\|u_t\|^2 - \epsilon\|\nabla u\|_m^m - \|\Delta u\|^2 \\ &\quad -\epsilon\xi_2(t)\|u\|_q^q - \epsilon\xi_1(t)\int_\Omega|u_t|^{p-1}u dx. \end{aligned} \tag{12}$$

Due to the Hölder's and Young's inequalities, we have

$$\begin{aligned} \left|\xi_1(t)\int_\Omega|u_t|^{p-1}u dx\right| &\leq \xi_1(t)\int_\Omega|u_t|^{p-1}u dx \\ &\leq \left(\int_\Omega\xi_1(t)|u_t|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega\xi_1(t)|u|^p dx\right)^{\frac{1}{p}} \\ &\leq \frac{p-1}{p}\xi_1(t)\delta^{-\frac{p}{p-1}}\|u_t\|_p^p + \frac{\delta^p}{p}\xi_1(t)\|u\|_p^p \end{aligned} \tag{13}$$

where δ is positive constant to be determined later. According to the conditions $\xi_1(t) \geq 0$, $\xi_2'(t) \geq 0$ and (8), we have

$$H'(t) \geq \xi_1(t)\|u_t\|_p^p. \tag{14}$$

Combining (5), (6), (12), (13) and (14), we have

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha)H^{-\alpha}(t) - \frac{p-1}{p}\epsilon\delta^{-\frac{p}{p-1}}\right]H'(t) \\ &\quad +\epsilon\left(qH(t) - \frac{\delta^p}{p}\xi_1(t)\|u_t\|_p^p - \left(\frac{q}{2}-1\right)\|\Delta u\|^2\right) \\ &\quad +\epsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \epsilon\left(\frac{q}{m}-1\right)\|\nabla u\|_m^m. \end{aligned} \tag{15}$$

Since the integral is taken over the variable x, it is reasonable to take δ depending on variable t. From (9), we have

$$0 < H^{-\alpha}(t) \leq H^{-\alpha}(0),$$

for every $t > 0$. Hence $H^{-\alpha}(t)$ is a positive function and bounded. Thus, by taking $\delta^{-\frac{p}{p-1}} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (15), we have

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\epsilon k\right] H^{-\alpha}(t) H'(t) \\ & + \epsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \epsilon\left(\frac{q}{m}-1\right)\|\nabla u\|_m^m \\ & + \epsilon\left[qH(t) - \frac{k^{1-p}}{p}\xi_1(t) H^{\alpha(p-1)}(t)\|u\|_p^p - \left(\frac{q}{2}-1\right)\|\Delta u\|^2\right] \end{aligned} \tag{16}$$

By using the (5), (6), (9) and the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$ $(q > p)$, we arrive at $\|u\|_p^p \leq C\|u\|_q^p$ and

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\epsilon k\right] H^{-\alpha}(t) H'(t) \\ & + \epsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \epsilon\left(\frac{q}{m}-1\right)\|\nabla u\|_m^m \\ & + \epsilon\left[qH(t) - \frac{Ck^{1-p}}{p}\left(\frac{\xi_2(t)}{q}\right)^{\alpha(p-1)}\|u\|_q^{p+q\alpha(p-1)} - \left(\frac{q}{2}-1\right)\|\Delta u\|^2\right]. \end{aligned} \tag{17}$$

From (11), we get $2 \leq s = p + q\alpha(p-1) \leq q$. Combining (5), (6), Remark 1 and (17), we get

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\epsilon k\right] H^{-\alpha}(t) H'(t) + \epsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \epsilon\left(\frac{q}{m}-1\right)\|\nabla u\|_m^m \\ & + \epsilon\left[qH(t) - C_1 k^{1-p}\xi_2(t)^{\alpha(p-1)}\xi_1(t)\left(H(t) + \|u_t\|_2^2 + \|\Delta u\|^2 + \frac{\xi_2(t)}{q} + 1\right)\|u\|_q^q\right] \\ & - \epsilon\left(\frac{q}{2}-1\right)\left(-H(t) - \frac{1}{2}\|u_t\|^2 - \frac{1}{m}\|\nabla u\|_m^m + \frac{\xi_2(t)}{q}\|u\|_q^q\right) \end{aligned} \tag{18}$$

$$\begin{aligned} \geq & \left[(1-\alpha) - \frac{p-1}{p}\epsilon k\right] H^{-\alpha}(t) H'(t) + \epsilon\left(\frac{q+2}{2} - C_1 k^{1-p}\xi_2(t)^{\alpha(p-1)}\xi_1(t)\right) H(t) \\ & + \epsilon\left[\frac{q+6}{4} - C_1 k^{1-p}\xi_2(t)^{\alpha(p-1)}\xi_1(t)\right]\|u_t\|^2 \\ & + \epsilon\left[\frac{q-2}{2q}\xi_2(t) - C_1 k^{1-p}\xi_2(t)^{\alpha(p-1)}\xi_1(t)\left(\frac{\xi_2(t)}{q}+1\right)\right]\|u\|_q^q \end{aligned} \tag{19}$$

where $C_1 = \frac{C}{pq^{\alpha(p-1)}}$. Since $\lim_{t\to\infty}\xi_1(t)\xi_2(t)^{\alpha(p-1)}$ exists, $\xi_1(t)\xi_2(t)^{\alpha(p-1)}$ is bounded for every $t > 0$. Then, we choose k large enough so that the coefficients of $H(t)$, $\|u_t\|^2$ and $\|u\|_q^q$ in (19) are strictly positive. Therefore, we arrive at

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{p-1}{p}\epsilon k\right] H^{-\alpha}(t) H'(t) \\ & + \epsilon\beta\left[H(t) + \|u_t\|_2^2 + \left(\frac{\xi_2(t)}{q}+1\right)\|u\|_q^q\right] \end{aligned} \tag{20}$$

where

$$\begin{aligned} \beta = \min & \left\{\frac{q+2}{2} - C_1 k^{1-p}\xi_2(t)^{\alpha(p-1)}\xi_1(t),\right. \\ & \frac{q+6}{4} - C_1 k^{1-p}\xi_2(t)^{\alpha(p-1)}\xi_1(t), \\ & \left.\frac{q-2}{2q}\xi_2(t) - C_1 k^{1-p}\xi_2(t)^{\alpha(p-1)}\xi_1(t)\right\} \end{aligned}$$

is the minimum of the coefficients of $H(t)$, $\|u_t\|^2$ and $\|u\|_q^q$. Once k is fixed, we can take ϵ small enough so that $1 - \alpha - \frac{p-1}{p}\epsilon k \geq 0$ and

$$L(0) = H^{1-\alpha}(0) + \epsilon\int_\Omega u_0 u_1 dx > 0. \tag{21}$$

Then (20) becomes

$$L'(t) \geq \epsilon\beta \left[H(t) + \|u_t\|_2^2 + \left(\frac{\xi_2(t)}{q} + 1 \right) \|u\|_q^q \right] \geq 0. \tag{22}$$

Then, we have

$$L(t) \geq L(0) > 0. \tag{23}$$

For the definition of $L(t)$ (see (10)) we have

$$\begin{aligned} \left| \int_\Omega u u_t dx \right| &\leq \|u\| \|u_t\| \\ &\leq C \|u\|_q \|u_t\| \end{aligned} \tag{24}$$

using Hölder's inequality and the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$ $(q > p)$. Thanks to Young's inequality, we have

$$\begin{aligned} \left| \int_\Omega u u_t dx \right|^{\frac{1}{1-\alpha}} &\leq C \|u\|_q^{\frac{1}{1-\alpha}} \|u_t\|^{\frac{1}{1-\alpha}} \\ &\leq C \left(\|u\|_q^{\frac{2}{1-2\alpha}} + \|u_t\|^2 \right) \end{aligned} \tag{25}$$

from (11), we arrive at $\frac{2}{1-2\alpha} < q$.

Combining (25) and Remark 1, we get

$$\left| \int_\Omega u u_t dx \right|^{\frac{1}{1-\alpha}} \leq C \left(H(t) + \|u_t\|_2^2 + \left(\frac{\xi_2(t)}{q} + 1 \right) \|u\|_q^q \right). \tag{26}$$

Therefore, we obtain

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= \left[H^{1-\alpha}(t) + \epsilon \int_\Omega u u_t dx \right]^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}} \left(H(t) + \left| \epsilon \int_\Omega u u_t dx \right|^{\frac{1}{1-\alpha}} \right) \\ &\leq C \left(H(t) + \|u_t\|_2^2 + \left(\frac{\xi_2(t)}{q} + 1 \right) \|u\|_q^q \right). \end{aligned} \tag{27}$$

Combining (22), (23) and (27), we have

$$L'(t) \geq \gamma L^{\frac{1}{1-\alpha}}(t) \tag{28}$$

where γ is a constant depending only on C, β and ϵ. Integrating (28), we arrive at

$$L^{\frac{1}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\gamma t} \tag{29}$$

If

$$t \to \left[\frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)} \right]^-, \quad L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\gamma t \to 0.$$

Hence, $L(t)$ blows up in finite time T^* and

$$T^* \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)},$$

which complete the proof of the Theorem. □

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Blending Type Bernstein Operators

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Abstract: Positive linear operators are used to approximate functions. Recently, certain new blending polynomials have been introduced to extend Bernstein operators and obtain more accurate and sensitive numerical results. In this study, we focus and compare some recent positive linear operators to approximate functions.

Keywords: Bersntein operators, Positive linear operators, Blending operators.

1 Introduction

The uniform approximation of continuous functions by polynomials was one of the problems that Karl Weierstrass focused on (see [24]). The classical Weierstrass approximation theorem asserts that there exists a sequence of polynomials $r_p(u)$ that converges uniformly to $r(u)$ for any continuous function $r(u)$ on the closed interval $[a, b]$. Later, Bernstein provided an alternative proof of the well-known Weierstrass approximation theorem currently called Bernstein polynomials (see [7]). The following Bernstein operators:

$$\mathcal{B}_p(r; u) = \sum_{i=0}^{p} b_{p,i}(u) r\left(\frac{i}{p}\right),$$

where,

$$b_{p,i}(u) = \binom{p}{i} u^i (1-u)^{p-i}, u \in [0, 1]$$

were given in [7] to approximate a given continuous function $r(u)$ on $[0, 1]$.

The Bernstein polynomials satisfy the following recursive formula

$$b_{m,j}(z) = (1-z)b_{m-1,j}(z) + z b_{m-1,j-1}(z).$$

There are several generalization mentioned regarding Bernstein operators, for example,

(a) λ-Bernstein operators [14] with $\tilde{b}_{n,i}(\lambda; x)$ Bézier bases and shape parameter λ (see [23]):

$$\begin{aligned}
\tilde{b}_{n,0}(\lambda; x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x),\\
\tilde{b}_{n,i}(\lambda; x) &= b_{n,i}(x) + \frac{n-2i+1}{n^2-1}\lambda b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1}\lambda b_{n+1,i+1}(x), \quad i = 1, 2 \ldots, n-1,\\
\tilde{b}_{n,n}(\lambda; x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x).
\end{aligned} \tag{1}$$

(b) Bernstein type operators by using continuously differentiable ∞ times function τ on $[0, 1]$ [15].
(c) New variant of Bernstein operators [20]
(d) (p, q)-Bernstein operators.
(e) Stancu-type λ-Bernstein operators [22].
(f) Modified U_n operators [13] and references therein.

(g) α-Bernstein operators [16, 18] $p_{m,\gamma,j}^{(\alpha)}(z)$ denotes the α-Bernstein-Schurer polynomials defined by

$$p_{1,\gamma,0}^{(\alpha)}(z) = 1 - z, \quad p_{1,\gamma,1}^{(\alpha)}(z) = z$$

and

$$\begin{aligned} p_{m,\gamma,j}^{(\alpha)}(z) &= \left[(1-\alpha) z \binom{m+\gamma-2}{j} + (1-\alpha)(1-z) \binom{m+\gamma-2}{j-2} \right. \\ &\left. + \alpha z (1-z) \binom{m+\gamma}{j} \right] z^{j-1} (1-z)^{m+\gamma-(j+1)} \qquad (m \geq 2). \end{aligned} \tag{2}$$

(h) Bivariate extension of α-Bernstein-Durrmeyer operators [19].
(i) Kantorovich modifications of α-Bernstein operators.
(j) λ-Bernstein-Schurer operators [21].
(k) Bivariate λ-Bernstein operators [34].
(l) λ-Bernstein-Kantorovich operators [17].
(m) Univariate and bivariate λ-Bernstein-Kantorovich operators [11].
(n) Genuine modified Bernstein-Durrmeyer operators.
(p) Blending type Bernstein operators [5].
(r) Blending type Bernstein-Kantorovich operators [6].

Recently, Chen et al. constructed a new family of Bernstein operators for the continuous function $r(u)$ on $[0, 1]$, which includes the shape parameter α, and named it the α-Bernstein operators [16]. They investigated certain elementary properties of these operators, such as end-point interpolation, linearity, and positivity, and obtained an upper bound for the error in terms of the usual modulus of continuity. Many variations of α-Bernstein operators have been examined (see [39]).

A new basis with shape parameter $\lambda \in [-1, 1]$ was introduced in [28] to give a practical algorithm of curve modeling. The authors studied certain important properties of the basis function and the related curves, and extended their research to the tensor product surface with two shape parameters. A new type of λ-Bernstein operators was constructed by shape parameter λ in [32]. A Korovkin-type approximation theorem was provided; a local approximation theorem was investigated; a convergence theorem for the Lipschitz continuous functions was given; a Voronovskaja-type asymptotic formula was obtained as well.

Recently, shape parameters α and λ were used to extend Bernstein operators to α-Bernstein type (see [1, 3, 16, 25, 39]) and λ-Bernstein type operators (see [28–31, 38]) in order to approximate functions better, respectively.

This paper is focused on the literature review of certain blending type Bernstein operators.

2 Recent results on blending operators

In this section, the definitions of α-Bernstein, λ-Bernstein, and blending (α, λ, s)-Bernstein operators and all needed results are provided. Let throughout the paper the binomial coefficients be given by the formula:

$$\binom{p}{i} = \begin{cases} \frac{p!}{i!(p-i)!}, & 0 \leq i \leq p, \\ 0, & \text{otherwise.} \end{cases}$$

In [1], the authors introduced generalized blending-type α-Bernstein operators by implementing a positive integer s as:

$$\begin{aligned} \mathcal{L}_p^{\alpha,s}(r;u) &= \sum_{i=0}^{p} \left\{ (1-\alpha) \binom{p-s}{i-s} u^{i-s+1} (1-u)^{p-i} \right. \\ &+ (1-\alpha) \binom{p-s}{i} u^i (1-u)^{p-s-i+1} \\ &\left. + \alpha \binom{p}{i} u^i (1-u)^{p-i} \right\} r\left(\frac{i}{p}\right), \quad \text{for} \quad p \geq s \end{aligned}$$

and:

$$\mathcal{L}_p^{\alpha,s}(r;u) = \sum_{i=0}^{p} \binom{p}{i} u^i (1-u)^{p-i} r\left(\frac{i}{p}\right), \quad \text{for} \quad p < s$$

which depend on shape parameter α, where $u, \alpha \in [0, 1]$, $r(u) \in C[0, 1]$.

Finally, blending-type (α, λ, s)-Bernstein operators were constructed in [2] as follows:

$$\mathcal{L}_{p,\lambda}^{(\alpha,s)}(r;u) = \sum_{i=0}^{p} \tilde{b}_{p,i}^{\alpha,s}(\lambda;u) r\left(\frac{i}{p}\right), \tag{3}$$

where $0 \leq \alpha \leq 1$, $-1 \leq \lambda \leq 1$ and s is a positive integer, and the blending-type (α, λ, s) basis is given as:

$$\tilde{b}_{p,i}^{\alpha,s}(\lambda;u) = \begin{cases} \tilde{b}_{p,i}(\lambda;u), & \text{if} \quad p < s \\ (1-\alpha)\left[x\tilde{b}_{p-s,i-s}(\lambda;u) + (1-u)\tilde{b}_{p-s,i}(\lambda;u)\right] & \\ +\alpha\tilde{b}_{p,i}(\lambda;u), & \text{if} \quad p \geq s \end{cases}$$

and $\tilde{b}_{p,i}(\lambda;u)$ defined in Equation (1).

In [2], the authors proposed an alternative representation for (3) as:

$$\mathcal{L}_{p,\lambda}^{(\alpha,s)}(r;u) = \begin{cases} \mathcal{B}_{p,\lambda}(r;u), & \text{if} \quad p < s \\ \\ \mathcal{B}_{p,\lambda}^{\alpha,s}(r;u), & \text{if} \quad p \geq s. \end{cases} \tag{4}$$

The operator $\mathcal{B}_{p,\lambda}^{\alpha,s}(r;u)$ is defined by:

$$\mathcal{B}_{p,\lambda}^{\alpha,s}(r;u) = (1-\alpha)\mathcal{B}_{p,\lambda}^{s,(\star)}(r;u) + \alpha\mathcal{B}_{p,\lambda}(r;u),$$

where:

$$\mathcal{B}_{p,\lambda}^{s,(\star)}(r;u) = \left[u\sum_{i=0}^{p}\tilde{b}_{p-s,i-s}(\lambda;u) + (1-u)\sum_{i=0}^{p}\tilde{b}_{p-s,i}(\lambda;u)\right] r\left(\frac{i}{p}\right).$$

In [2], the representation (4) was also explicitly written as:

$$\begin{aligned} \mathcal{B}_{p,\lambda}(r;u) \quad &= \quad \sum_{i=0}^{p} b_{p,i}(u) r\left(\frac{i}{p}\right) \\ &\quad + \lambda\sum_{i=0}^{p-1}\frac{p-2k-1}{p^2-1} b_{p+1,i+1}(u)\left[r\left(\frac{i+1}{p}\right) - r\left(\frac{i}{p}\right)\right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_{p,\lambda}^{s,(\star)}(r;u) \quad &= \quad \sum_{i=0}^{p-s} b_{p-s,i}(u)\left[ur\left(\frac{i+s}{p}\right) + (1-u)r\left(\frac{i}{p}\right)\right] \\ &\quad + \lambda u\sum_{i=0}^{p-s-1}\frac{p-s-2k-1}{(p-s)^2-1} b_{p-s+1,i+1}(u) \\ &\quad \times\left[r\left(\frac{i+s+1}{p}\right) - r\left(\frac{i+s}{p}\right)\right] \\ &\quad + \lambda(1-u)\sum_{i=0}^{p-s-1}\frac{p-s-2k-1}{(p-s)^2-1} b_{p-s+1,i+1}(u) \\ &\quad \times\left[r\left(\frac{i+1}{p}\right) - r\left(\frac{i}{p}\right)\right]. \end{aligned}$$

Moments of the operators $\mathcal{L}_{p,\lambda}^{(\alpha,s)}$ were found in the paper [[2] Theorem 2]:

If $p \geq s$, for any $0 \leq \alpha \leq 1$ and $-1 \leq \lambda \leq 1$, we have the following identities:

$$
\begin{aligned}
(i) \quad \mathcal{L}_{p,\lambda}^{(\alpha,s)}(1;u) &= 1; \\
(ii) \quad \mathcal{L}_{p,\lambda}^{(\alpha,s)}(t;u) &= u + (1-\alpha)\lambda\left[\frac{1-2u+u^{p-s+1}-(1-u)^{p-s+1}}{p(p-s-1)}\right] \\
&\quad +\alpha\lambda\left[\frac{1-2u+u^{p+1}-(1-u)^{p+1}}{p(p-1)}\right]; \\
(iii) \, \mathcal{L}_{p,\lambda}^{(\alpha,s)}(t^2;u) &= u^2 + \frac{\left[p+(1-\alpha)s(s-1)\right]u(1-u)}{p^2} \\
&\quad + \frac{\alpha\lambda}{p}\left[\frac{2u-4u^2+2u^{p+1}}{(p-1)}\right] \\
&\quad + \frac{(1-\alpha)\lambda}{p}\left[\frac{2u-4u^2+2u^{p-s+1}}{(p-s-1)}\right] \\
&\quad + \frac{\alpha\lambda}{p^2}\left[\frac{u^{p+1}+(1-u)^{p+1}-1}{(p-1)}\right] \\
&\quad + \frac{(1-\alpha)\lambda}{p^2}\left[\frac{u^{p-s+1}+(1-u)^{p-s+1}-1}{(p-s-1)}\right] \\
&\quad + \left[\frac{2su(u^{p-s+1}-(1-u)^{p-s+1})}{(p-s-1)}\right].
\end{aligned}
$$

Let $L_1[0,1]$ denote the space of all Lebesgue integrable functions on the interval $\mathcal{I}$. The following sequence of operators involving shape parameters λ and α, and a positive integer s is called it as blending (α, λ, s)-Bernstein-Kantorovich operators [6]:

$$
\mathcal{K}_{p,\lambda}^{(\alpha,s)}(r;u) = (p+1)\sum_{i=0}^{p} \tilde{b}_{p,i}^{\alpha,s}(\lambda;u)\int_{\frac{i}{p+1}}^{\frac{i+1}{p+1}} r(t)dt. \tag{5}
$$

By the following theorem we give uniform convergence of some positive linear operators.

Theorem 1. *For any $\alpha \in [0,1]$, then $L(r)$ converge uniformly to r on $[0,1]$, that is,*

$$
\lim_{m\to\infty} \|L(r) - r\| = 0,
$$

where $L = \mathcal{L}_{p,\lambda}^{(\alpha,s)}, \; \mathcal{B}_{p,\lambda}^{\alpha,s}, \; \mathcal{K}_{p,\lambda}^{\alpha,s}$.

Proof: Taking into account moments of Bernstein type operators we have

$$
L(e_0) = e_0 \text{ as } m \to \infty, \quad L(e_1;x) = e_1 \text{ as } m \to \infty
$$

and similarly $L_{m,\alpha}(e_2) = e_2$ as $m \to \infty$. Hence, by the Korovkin theorem, we obtain

$$
\lim_{m\to\infty} \|L(f) - f\| = 0,
$$

where $L = \mathcal{L}_{p,\lambda}^{(\alpha,s)}, \; \mathcal{B}_{p,\lambda}^{\alpha,s}, \; \mathcal{K}_{p,\lambda}^{\alpha,s}$. □

3 Concluding Remarks

This paper is based on the results in [1–3, 5, 6], this is why we refer these papers for further literature. We will study approximation properties of Stancu variant of blending (α, λ, s)-Bernstein operators and blending (α, λ, s)-Bernstein-Kantorovich operators in close future.

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Statistical Convergence and Certain Approximation Results

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Abstract: Our main aim in this work is to provide a literature review for certain recent extension of bivariate Bernstein type operators based on multiple shape parameters and to give an application of four-dimensional infinite matrices to approximation theory, and prove some Korovkin theorems. All the results that have been obtained about these kinds of operators can be extended to the case of n-variate functions.

Keywords: Bernstein operators, Kantorovich operators, shape parameters, approximation properties.

1 Introduction

Bernstein opened a new way [3] by giving the most well known proof of Weierstrass approximation theorem (see [35]). He constructed a sequence of approximating polynomials and many researchers have successfully extended this idea to approximate functions (see [23, 25–27]). Korovkin-type theorems provide a process to decide whether a given sequence of positive linear operators converges strongly. Using certain types of statistical convergences instead of the classical convergence in Korovkin type approximation theory gives us many advantages. Applications of Korovkin type approximation on positive linear operators can be seen in [9–12].

In this study, we construct an original extension of bivariate Bernstein type operators based on multiple shape parameters and prove some Korovkin theorems using a four-dimensional summability method, and a power series method. We obtain rate of D-statistical convergence, and rate of convergence for power series method with the help of the modulus of continuity. Finally, we demonstrate some computer graphics to numerically see the efficiency and accuracy of convergence of proposed operators, and obtain corresponding error plots.

We first provide certain notions and auxiliary results that are needed in this study.

Assume that there is $N = N(\tau) \in \mathbb{N}$ for each $\tau > 0$, so that $|\varrho_{u,v} - Q| < \tau$ whenever $u, v > N$, in this case double sequence $\varrho = (\varrho_{u,v})$ is said to be convergent to Q in Pringsheim's sense (or simply Π-convergent), and it is denoted by

$$\Pi - \lim_{u,v} \varrho_{u,v} = Q$$

(see [20]). When there is a positive number E so that $|\varrho_{u,v}| \leq E$ for all $(u, v) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, double sequence is said to be bounded. As it is well known, a convergent single sequence is bounded whereas a convergent double sequence need not to be bounded.

Assume that $D = (d_{l,o,u,v})$ is a four-dimensional summability method. Given a double sequence $\varrho = (\varrho_{u,v})$, D transform of ϱ, denoted by $D\varrho := ((D\varrho)_{l,o})$, is defined as

$$(D\varrho)_{l,o} = \sum_{u,v=1}^{\infty} d_{l,o,u,v} \varrho_{u,v},$$

and the double series is Π-convergent for $(l, o) \in \mathbb{N}^2$. When four-dimensional matrix $D = (d_{l,o,u,v})$ maps every bounded $\Pi-$convergent sequence into a $\Pi-$convergent sequence with the same $\Pi-$limit, it is called $RH-$*regular* (shortly RHR).

A four-dimensional matrix $D = (d_{l,o,u,v})$ is RHR if and only if

(a) $\Pi - \lim_{l,o} d_{l,o,u,v} = 0$,

(b) $\Pi - \lim_{l,o} \sum\limits_{u,v=1}^{\infty} d_{l,o,u,v} = 1$,

(c) $\Pi - \lim_{l,o} \sum\limits_{r=1}^{\infty} |d_{l,o,u,v}| = 0$ $(\forall s \in \mathbb{N})$,

(d) $\Pi - \lim_{l,o} \sum\limits_{s=1}^{\infty} |d_{l,o,u,v}| = 0$ $(\forall r \in \mathbb{N})$,

(e) $\sum\limits_{u,v=1}^{\infty} |d_{l,o,u,v}|$ is $\Pi-$convergent,

(f) The inequality $\sum\limits_{u,v>E_2} |d_{l,o,u,v}| < E_1$ is satisfied for finite positive integers E_1 and E_2 and for each $(l, o) \in \mathbb{N}^2$.

These conditions are called Robinson-Hamilton conditions [21]. Assume that $D = (d_{l,o,u,v})$ is a nonnegative RHR matrix, and $S \subset \mathbb{N}^2$, then $D-$density of S is defined as

$$\delta_D^2(S) := \Pi - \lim_{l,o} \sum_{(u,v)\in S} d_{l,o,u,v}.$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $\varrho = (\varrho_{u,v})$ is called $D-$*statistically convergent* to Q and denoted by

$$st_D^2 - \lim_{u,v} \varrho_{u,v} = Q$$

if, for every $\tau > 0$,

$$\delta_D^2(\left\{(u,v) \in \mathbb{N}^2 : |\varrho_{u,v} - Q| \geq \tau\right\}) = 0$$

(see also [19, 22]). A $\Pi-$convergent double sequence is $D-$statistically convergent to the same number even if converse statement may not be true. When $D = C(1,1)$, $C(1,1)-$statistical convergence becomes statistical convergence for double sequences (see also [18]), where $C(1,1) = (c_{l,o,u,v})$ is double Cesàro matrix, defined by

$c_{l,o,u,v} = 1/lo$ if $1 \leq r \leq o$, $1 \leq s \leq l$, and

$c_{l,o,u,v} = 0$ otherwise.

Suppose that $(\xi_{u,v})$ is a double sequence of nonnegative numbers with condition $\xi_{0,0} > 0$, then power series

$$\xi(a,b) := \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v$$

has radius of convergence Θ, where $\Theta \in (0,\infty]$ and $a, b \in (0,\Theta)$. When following equality is satisfied

$$\lim_{a,b\to\Theta^-} \frac{1}{\xi(a,b)} \sum_{u,v=0}^{\infty} \xi_{u,v} a^u b^v \varrho_{u,v} = Q$$

$\forall a, b \in (0,\Theta)$, then double sequence $\varrho = (\varrho_{u,v})$ is said to be convergent to Q in the sense of power series method [30]. power series method for double sequences is regular if and only if

$$\lim_{a,b\to\Theta^-} \frac{\sum_{r=0}^{\infty} \xi_{r,\upsilon} a^u}{\xi(a,b)} = 0; \qquad \lim_{a,b\to\Theta^-} \frac{\sum_{s=0}^{\infty} \xi_{\mu,s} b^v}{\xi(a,b)} = 0$$

are satisfied for any μ, υ [30]. In this work, we assume that power series method is regular.

When $\Theta = 1$ and $\xi_{u,v} = 1$ power series method becomes Abel summability method, and it becomes logarithmic summability method if

$$\xi_{u,v} = \frac{1}{(u+1)(v+1)}.$$

Also, power series method becomes Borel summability method when

$$\Theta = \infty$$

and

$$\xi_{u,v} = \frac{1}{u!v!}.$$

Some properties of modified Szász-Mirakyan, Baskakov-Schurer-Szasz and generalized Szasz operators in polynomial weight spaces were studied by power summability methods in [6–8]. We also note that applications of various statistical summability methods in approximation theory can be seen in the papers [32–34]. Finally, one can see more information about double sequences in [18, 19], and application of double sequences in approximation theory in [9, 12, 30].

2 Multivariate Operators

In this part, we give certain recent multivariate Bersntein type operators.

The following polynomial functions

$$
\begin{aligned}
a_{u,0}(\rho;x) &= (1-x)^u\,(1-\rho_1 x),\\
a_{u,i}(\rho;x) &= x^i(1-x)^{u-i}\left(\binom{u}{v}+\rho_i-\rho_i x-\rho_{i+1}x\right),\ \ i=1,2\ldots,[\frac{u}{2}]-1,\\
a_{u,[\frac{u}{2}]}(\rho;x) &= x^{[\frac{u}{2}]}(1-x)^{u-[\frac{u}{2}]}\left(\binom{u}{[\frac{u}{2}]}+\rho^{[\frac{u}{2}]}-\rho^{[\frac{u}{2}]}x+\rho^{[\frac{u}{2}]+1}x\right),\\
a_{u,i}(\rho;x) &= x^i(1-x)^{u-i}\left(\binom{u}{v}-\rho_i+\rho_i x+\rho_{i+1}x\right),\ \ i=[\frac{u}{2}]+1,\ldots,u-1,\\
a_{u,u}(\rho;x) &= x^u(1-\rho_u+\rho_u x)
\end{aligned}
\tag{1}
$$

are called generalized Bernstein polynomials of degree u ($u\geq 2$) and for $x\in[0,1]$ with shape parameters ρ_i, $i=1,2,\ldots,u$, where

$$
\begin{cases}\rho_i\in[-\binom{u}{i},\binom{u}{i-1}] & ;i=1,2,\ldots,[\frac{u}{2}]\\ \rho_i\in[-\binom{u}{i-1},\binom{u}{i}] & ;i=[\frac{u}{2}]+1,\ldots,u\end{cases}\quad \text{with}\quad \begin{cases}[\frac{u}{2}]=\frac{u}{2} & ;\text{if u is even}\\ [\frac{u}{2}]=\frac{u-1}{2} & ;\text{if u is odd.}\end{cases}
\tag{2}
$$

These polynomials were introduced by Han et al. in. [29] and they are reduced to classical Bernstein basis functions $b_{u,i}(x)$ of degree u on $x\in[0,1]$ which is defined as

$$
b_{u,i}(x)=\binom{u}{v}x^i\,(1-x)^{u-i},\qquad\qquad i=0,\ldots,u
$$

when $\rho_i=0$ ($i=1,2,\ldots,u$). Generalized Bernstein basis functions with parameters ρ_i ($i=1,2,\ldots,u$) are linearly independent (see [28]) and these basis functions are effectively and flexibly used in designing parametric curves and surfaces (see [28, 29]). These functions also have partition of unity, symmetry and nonnegativity properties (see [29]). In 2017, Hu et al. [28] have obtained the following equations to convert classical Bernstein polynomials of degree u to generalized Bernstein polynomials of degree u associated with shape parameters ρ_i:

$$
\begin{aligned}
a_{u,0}(\rho;x) &= b_{u+1,0}(x)+\frac{\binom{u}{0}-\rho_1}{\binom{u+1}{1}}b_{u+1,1}(x),\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}+\rho_i}{\binom{u+1}{i}}b_{u+1,i}(x)+\frac{\binom{u}{i}-\rho_{i+1}}{\binom{u+1}{i+1}}b_{u+1,i+1}(x),\ \ i=1,2\ldots,\big[\frac{u}{2}\big]-1,\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}+\rho_v}{\binom{u+1}{i}}b_{u+1,i}(x)+\frac{\binom{u}{i}+\rho_{i+1}}{\binom{u+1}{i+1}}b_{u+1,i+1}(x),\ i=\big[\frac{u}{2}\big],\\
a_{u,i}(\rho;x) &= \frac{\binom{u}{i}-\rho_i}{\binom{u+1}{i}}b_{u+1,i}(x)+\frac{\binom{u}{i}+\rho_{i+1}}{\binom{u+1}{i+1}}b_{u+1,i+1}(x),\ \ i=\big[\frac{u}{2}\big]+1,\ldots,u-1,\\
a_{u,u}(\rho;x) &= \frac{\binom{u}{u}-\rho_u}{\binom{u+1}{u}}b_{u+1,u}(x)+b_{u+1,u+1}(x).
\end{aligned}
\tag{3}
$$

Let $C[0,1]=\mathbf{C}$ be the space of all continuous functions on unit interval $[0,1]$ and $C\left([0,1]\times[0,1]\right)=\bar{\mathbf{C}}$. The operators $\mathcal{B}_u^{\nu},\mathcal{B}_v^{\mu}:\mathbf{C}\longrightarrow\mathbf{C}$ for any $u,v\in\mathbb{N}$ are given as follows, respectively,

$$
\mathcal{B}_u^{\nu}(f;y)=\sum_{i=0}^{u}f\left(\frac{i}{u}\right)a_{u,i}(\nu_i;y),
\tag{4}
$$

$$
\mathcal{B}_v^{\mu}(g;z)=\sum_{j=0}^{v}g\left(\frac{j}{v}\right)a_{v,j}(\mu_j;z),
\tag{5}
$$

where polynomials $a_{u,i}(\nu;y)$ and $a_{v,j}(\mu;z)$ are given in (3). The parametric extension of (4) and (5) for $u,v\in\mathbb{N}$ and $h\in\bar{\mathbf{C}}$ are the operators

$$
\mathcal{B}_u^{\nu,y},\mathcal{B}_v^{\mu,z}:\bar{\mathbf{C}}\longrightarrow\bar{\mathbf{C}},
$$

where

$$
\mathcal{B}_u^{\nu,y}(h;y,z)=\sum_{i=0}^{u}a_{u,i}(\nu_i;y)h\left(\frac{i}{u},\frac{i}{u}\right),
\tag{6}
$$

$$
\mathcal{B}_v^{\mu,z}(h;y,z)=\sum_{j=0}^{v}a_{v,j}(\mu_j;z)h\left(\frac{j}{v},\frac{j}{v}\right).
\tag{7}
$$

The parametric extension of operators defined in (6) and (7) are linear and positive.

The parametric extensions of bivariate operators commute on $\bar{\mathbf{C}}$. Their product establishes bivariate operators $\mathcal{B}_{u,v}^{\nu,\mu} : \bar{\mathbf{C}} \longrightarrow \bar{\mathbf{C}}$ defined for any $u, v \in \mathbb{N}$ and any $h \in \bar{\mathbf{C}}$ by the relation

$$\mathcal{B}_{u,v}^{\nu,\mu}(h; y, z) \quad = \quad \sum_{i=0}^{u}\sum_{j=0}^{v} a_{u,i}(\nu_i; y) a_{v,j}(\mu_j; z) h\left(\frac{i}{u}, \frac{j}{v}\right). \tag{8}$$

Let $L_p[0,1]$ denote the space of all Lebesgue integrable functions on the interval $\mathcal{I}$. For p arbitrary real values of λ_s $(s = 1, \ldots, p)$ and $z \in \mathcal{I}$ the following operator

$$\mathcal{K}_p(\vartheta; z; \lambda) = (p+1) \sum_{s=0}^{p} a_{p,s}(\lambda; z) \int_{\frac{s}{p+1}}^{\frac{s+1}{p+1}} \vartheta(t)\, dt \tag{9}$$

is called the generalized Bernstein-Kantorovich operators involving shape parameters λ_s satisfying the conditions (2).

Also, multivariate Bernstein operators were defined in [1] as:

$$\mathcal{B}_p(\vartheta; z; \lambda) = (p+1) \sum_{s=0}^{p} a_{p,s}(\lambda; z) \vartheta(s/p). \tag{10}$$

And Stancu version of [1] was defined in [2].

3 Recent results for multivariate Bernstein type operators

Theorem 1. *If ϑ is continuous on $[0,1]$, then L converges uniformly to ϑ on $[0,1]$, that is,*

$$\lim_{p\to\infty} \|L_p(\vartheta) - \vartheta\| = 0,$$

where L is $\mathcal{B}_p(\vartheta; z; \lambda)$, $\mathcal{K}_p(\vartheta; z; \lambda)$, $\mathcal{S}_p(\vartheta; z; \lambda)$.

Also, Korovkin theorem is satisfied for the bivariate cases of these operators.

Proof: Using the moments of mentioned operators we have

$$\lim_{p\to\infty} L_p(e_0) = e_0, \quad \lim_{p\to\infty} L_p(e_1; x) = e_1$$

and similarly $\lim_{p\to\infty} \|L_p(e_2) - e_2\| = 0$. Hence, by the Korovkin theorem, we obtain

$$\lim_{p\to\infty} \|L_p(\vartheta) - \vartheta\| = 0.$$

□

As a future work, we will study Stancu version of $\mathcal{B}_p(\vartheta; z; \lambda)$ and $\mathcal{K}_p(\vartheta; z; \lambda)$.

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Lacunary Invariant Convergence in Fuzzy Normed Spaces

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Abstract: In this study, we defined the notions of lacunary invariant convergence in fuzzy normed spaces. Also, we investigated some properties of lacunary invariant convergence such as uniqueness of the limit and linearity.

Keywords: Fuzzy normed spaces, Invariant convergence, Lacunary convergence.

1 Introduction

The idea of fuzzy sets initially introduced by Zadeh [34] to deal with imprecise phenomena as an alternative to classical set theory. After that, several classical concepts were reconstructed. Fuzzy topological spaces [4, 20], fuzzy metric [13, 16, 18], fuzzy norm [2, 5, 12, 17] are just some of the examples. Felbin's fuzzy norm [12], which is associated with Kaleva and Seikkala [16] type metric space by assigning a non-negative fuzzy real number to each element of a linear space, forms the basis of this study. Das and Das [6] studied fuzzy topology generated by fuzzy norm. Diamond and Kloeden [9] investigated the metric spaces of fuzzy sets-theory and applications. Fang and Huang [11] studied on the level convergence of a sequence of fuzzy numbers. Also some other authors [10, 14, 15, 22] studied the notion about of fuzzy numbers and fuzzy normed space.

Banach [3] defined the generalized limit as an application of Hahn-Banach theorem on the set of all bounded real valued sequences. It is also known as Banach limit. Later, Lorentz [19] offered that if all Banach limits of a given bounded sequence are equal, it is called almost convergent. In further studies [8, 26], invariant mean and invariant convergence are given as a more general case of Banach limit and almost convergence. Also, several authors including Schaefer [30], Mursaleen and Edely [23], Mursaleen [24, 25], Savaş [27, 28] studied on invariant convergent sequences. Additionally, Yalvaç and Dündar [32] defined invariant convergence in fuzzy normed space.

Lacunary convergent sequences space was given by Freedman et al. in their study [1] where they showed the relation between strong Cesaro convergent space and the sequence of integers (2^r). Further studies about lacunary convergence were done by several author [7, 29].

Now, we recall the basic notions and some important definitions used in our paper (See [1, 2, 5, 8, 10, 12, 14, 17, 19, 21–26, 30–34]).

A fuzzy number is a fuzzy set provided that
(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$ for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$;
(iii) u is upper semi-continuous;
(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is a compact set.
Let $L(\mathbb{R})$ be denote the set of all fuzzy number.
$\mathbb{R}$ can be embedded in $L(\mathbb{R})$ since each $r \in \mathbb{R}$ can be considered a fuzzy real number $\tilde{r}$ denoted by $\tilde{r}(t) = 1$ if $t = r$ and $\tilde{r}(t) = 0$ if $t \neq r$.
For $u \in L(\mathbb{R})$, the α-level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1], \\ cl\,\{x \in \mathbb{R} : u(x) > \alpha\}, & \text{if } \alpha = 0. \end{cases}$$

The α-level set of a fuzzy number denoted by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ is a non-empty, bounded and closed interval for each $\alpha \in [0,1]$ where $u_\alpha^- = -\infty$ and $u_\alpha^+ = \infty$ are also admissible.

If $u \in L(\mathbb{R})$ and $u(x) = 0$ for $x < 0$, then u is called a non-negative fuzzy number. Let $L^*(\mathbb{R})$ denote the set of all non-negative fuzzy number. It is easy to see $\tilde{0} \in L^*(\mathbb{R})$.

A partial ordering $\preceq$ in $L(\mathbb{R})$ is defined by for $u, v \in L(\mathbb{R})$,

$$u \preceq v \quad \text{iff} \quad u_\alpha^- \leq v_\alpha^- \quad \text{and} \quad u_\alpha^+ \leq v_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1].$$

Arithmetic equations addition, multiplication and multiplication with a scaler on $L(\mathbb{R})$ are defined by
(i) $(u \oplus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}, \quad t \in \mathbb{R}$
(ii) $(u \odot v)(t) = \sup_{s \in \mathbb{R} s \neq 0} \{u(s) \wedge v(t/s)\}, \quad t \in \mathbb{R}$
(iii) For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.

Let $u,v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$. Arithmetic equations in terms of α-level sets are defined by
(i) $[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$,
(ii) $[u \odot v]_\alpha = [u_\alpha^- . v_\alpha^-, u_\alpha^+ . v_\alpha^+]$, $u,v \in L^*(\mathbb{R})$,
(iii) $[ku]_\alpha = k[u]_\alpha = \begin{cases} [ku_\alpha^-, ku_\alpha^+], & k \geq 0, \\ [ku_\alpha^+, ku_\alpha^-], & k < 0. \end{cases}$

For $u,v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined by

$$D(u,v) = \sup_{0\leq\alpha\leq 1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

One can see that

$$D(u,\tilde{0}) = \sup_{0\leq\alpha\leq 1} \max\left\{\left|u_\alpha^-\right|, \left|u_\alpha^+\right|\right\} = \max\left\{\left|u_0^-\right|, \left|u_0^+\right|\right\}.$$

Obviously, $D(u,\tilde{0}) = u_\alpha^+$ when $u \in L^*(\mathbb{R})$.

A sequence (u_n) in $L(\mathbb{R})$ is called convergent to $u \in L(\mathbb{R})$ denoted by $D - \lim_{n\to\infty} u_n = u$ if $\lim_{n\to\infty} D(u_n,u) = 0$, i.e., for all given $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon) \in \mathbb{R}$ such that $D(u_n,u) < \varepsilon$, for $n > N$.

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and the mappings $L, R : [0,1]\times[0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (FNS) and $\|.\|$ is a fuzzy norm if the following axioms are satisfied
(i) $\|x\| = \widetilde{0}$ iff $x = \theta$,
(ii) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,
(iii) For all $x,y \in X$
(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$,
whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$,
(b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$,
whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$.

When $L = \min$ and $R = \max$ are taken in above (iii), triangle inequalities become

$$\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^- \text{ and } \|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$$

for all $\alpha \in (0,1]$ and $x,y \in X$. Since they fulfil the all conditions of norm, $\|x\|_\alpha^-$ and $\|x\|_\alpha^+$ can be seen as ordinary norms on X.

Example 1. *Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained*

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C, \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & \text{if } a\|x\|_C \leq t \leq \|x\|_C, \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \text{if } \|x\|_C \leq t \leq b\|x\|_C, \end{cases}$$

where $\|x\|_C$ is the ordinary norm of x $(\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \widetilde{0}$. Hence $(X, \|.\|)$ is a fuzzy normed linear space.

Throughout paper let $(X, \|.\|)$ be an fuzzy normed linear space (FNS).

A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\to} x$, provided that $(D) - \lim_{n\to\infty} \|x_n - x\| = \widetilde{0}$, i.e., for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D\left(\|x_n - x\|, \widetilde{0}\right) < \varepsilon$, for all $n > N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$,

$$\sup_{\alpha\in[0,1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon.$$

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean if and only if
(i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(ii) $\phi(e) = 1$, where $e = (1,1,1...)$,
(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and satisfied the condition $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Invariant mean, ϕ, is a extension of the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. The sequence is called invariant convergent when its invariant means are equal. In case $\sigma(n) = n+1$, the σ mean is often called a Banach limit and invariant convergent is almost convergent.

A bounded sequence (x_n) is σ-convergent to the number L if $\lim_{m\to\infty} t_{mn} = L$ uniformly in m, where

$$t_{mn} = \frac{x_n + x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m+1}.$$

A sequence $x = (x_n)$ in fuzzy normed space X is invariant convergent to L with respect to fuzzy norm if $(D) - \lim_{m\to\infty} \|t_{mn} - L\| = \widetilde{0}$, uniformly in n, Namely, for given $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m > m_0$,

$$D(\|t_{mn} - L\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - L\|_\alpha^+ = \|t_{mn} - L\|_0^+ < \varepsilon, \text{ for every } n \in \mathbb{N}.$$

An increasing sequence of non-negative integers $\theta = (k_r)$ with $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, is called lacunary sequence. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is given by q_r.

For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ is lacunary convergent to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} (x_i - L) = 0.$$

2 Main Results

We firstly give the concept of lacunary invariant convergence in fuzzy normed spaces. After then, we show that the uniqueness of the limit of the lacunary invariant convergent sequence in fuzzy normed spaces. Also, we investigate the linearity of this new concept.

Definition 1. *For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ is lacunary invariant convergent to L with respect to fuzzy norm and denoted by $x_n \overset{\sigma-FN_\theta}{\longrightarrow} L$, if*

$$\lim_{r\to\infty} D\left(\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|, \widetilde{0}\right) = \lim_{r\to\infty} \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ = 0,$$

unifomly in n; that is, for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$D\left(\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|, \widetilde{0}\right) = \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Theorem 1. *Let $\theta = (k_r)$ be a lacunary sequence, $(X, \|.\|)$ be a fuzzy normed space and $x = (x_n)$ be a sequence in X. If x is lacunary invariant convergent to L, then L is unique.*

Proof: Let's assume that

$$x_n \overset{\sigma-FN_\theta}{\longrightarrow} L_1 \text{ and } x_n \overset{\sigma-FN_\theta}{\longrightarrow} L_2,$$

for

$$L_1 \neq L_2.$$

Then for every $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that for all $r > r_1$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L_1\right\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$ and for the given $\varepsilon > 0$, there exists $r_2 \in \mathbb{N}$ such that for all $r > r_2$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L_2\right\|_0^+ < \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$. Take

$$r_0 = \max\{r_1, r_2\}.$$

Then for all $r > r_0$,

$$\begin{aligned}\|L_1 - L_2\|_0^+ &\leq \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L_1\right\|_0^+ + \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L_2\right\|_0^+ \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\end{aligned}$$

for all $n \in \mathbb{N}$. Since for all $\varepsilon > 0$,

$$\|L_1 - L_2\|_0^+ \leq \varepsilon$$

and so, we have

$$L_1 = L_2.$$

□

Theorem 2. *Let $\theta = (k_r)$ be a lacunary sequence, $(X, \|.\|)$ be a fuzzy normed space and $x = (x_n)$, $y = (y_n)$ be sequences in X. If x and y are lacunary invariant convergent to L_1 and L_2, respectively then the sequence $x + y$ is lacunary invariant convergent to $L_1 + L_2$.*

Proof: Let's assume that

$$x_n \overset{\sigma - FN_\theta}{\longrightarrow} L_1 \text{ and } y_n \overset{\sigma - FN_\theta}{\longrightarrow} L_2.$$

Then, for every $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that for all $r > r_1$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L_1\right\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$ and for given $\varepsilon > 0$, there exists $r_2 \in \mathbb{N}$ such that for all $r > r_2$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} y_{\sigma^i(n)} - L_2\right\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. Take

$$r_0 = \max\{r_1, r_2\}.$$

Then for all $r > r_0$,

$$\begin{aligned}&\left\|\left(\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} + \frac{1}{h_r}\sum_{i\in I_r} y_{\sigma^i(n)}\right) - (L_1 + L_2)\right\|_0^+ \\ &\leq \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L_1\right\|_0^+ + \left\|\frac{1}{h_r}\sum_{i\in I_r} y_{\sigma^i(n)} - L_2\right\|_0^+ \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\end{aligned}$$

for all $n \in \mathbb{N}$. Hence we have

$$x_n + y_n \overset{\sigma - FN_\theta}{\longrightarrow} L_1 + L_2.$$

□

Theorem 3. *Let $\theta = (k_r)$ be a lacunary sequence, $(X, \|.\|)$ be a fuzzy normed space and $x = (x_n)$ be a sequence in X. If x is lacunary invariant convergent to L and c is a scaler then the sequence $cx = (cx_n)$ is lacunary invariant convergent to cL.*

Proof: Let's assume that $x_n \stackrel{\sigma-FN_\theta}{\longrightarrow} L$ and c is a scaler. Then for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\left\| \frac{1}{h_r} \sum_{i \in I_r} x_{\sigma^i(n)} - L \right\|_0^+ < \frac{\varepsilon}{|c|},$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \left\| \frac{1}{h_r} \sum_{i \in I_r} c x_{\sigma^i(n)} - cL_1 \right\|_0^+ &= |c| \left\| \frac{1}{h_r} \sum_{i \in I_r} x_{\sigma^i(n)} - L_1 \right\|_0^+ \\ &< |c| \frac{\varepsilon}{|c|} \\ &= \varepsilon, \end{aligned}$$

for all $n \in \mathbb{N}$. So we conclude

$$(cx_n) \stackrel{\sigma-FN_\theta}{\longrightarrow} cL.$$

□

3 References

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